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Dominic Breit and Mark D. Groves

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Dominic Breit

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
Dominic.Breit@math.uni-sb.de

Mark D. Groves

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
Groves@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

1 Introduction

The problem concerns two finite, immiscible, perfect fluids separated by an interface $\{y = \eta(x,z)\}$; the fluid motion is three-dimensional and the densities of the upper and lower fluid are $\overline{\rho}$ and $\underline{\rho}$ with $\overline{\rho} < \underline{\rho}$. Furthermore we define the fluid-domains

$$\overline{S}(\eta) := \left\{(x,y,z) : \, x,y \in \mathbb{R}, \, \eta(x,z) \le y \le \overline{h}\right\},$$
$$\underline{S}(\eta) := \left\{(x,y,z) : \, x,y \in \mathbb{R}, \, -\underline{h} \le y \le \eta(x,z)\right\}.$$

Here $\overline{h}$ and $\underline{h}$ are two positive numbers describing the depth of the upper resp. lower fluid. Within each fluid domain the evolution is given by potential flow, so that

$$u = \nabla\overline{\varphi}, \quad \Delta\overline{\varphi} = 0 \quad \text{within} \quad \overline{S}(\eta), \qquad u = \nabla\underline{\varphi}, \quad \Delta\underline{\varphi} = 0 \quad \text{within} \quad \underline{S}(\eta).$$

The fluid interface obey the kinematic equations

$$\partial_t\eta = -\overline{\varphi}_y + \eta_x\overline{\varphi}_x + \eta_z\overline{\varphi}_z = \frac{\partial\overline{\varphi}}{\partial_{\overline{N}}},$$
$$\partial_t\eta = \underline{\varphi}_y - \eta_x\underline{\varphi}_x - \eta_z\underline{\varphi}_z = \frac{\partial\underline{\varphi}}{\partial_{\underline{N}}}.$$

At the bond of $\overline{S}(\eta) \cup \underline{S}(\eta)$ one imposes Neumann boundary conditions on confining vertical walls, so that

$$\overline{\varphi}_y(x,\overline{h},z) = 0 = \underline{\varphi}_y(x,-\underline{h},z)$$

for all $x, z \in \mathbb{R}$. The Bernoulli-condition reads as

$$\overline{\rho}\left(\partial_t\overline{\varphi} + \frac{1}{2}|\nabla\overline{\varphi}|^2 + g\eta - \sigma\,\mathrm{div}\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right)\right)$$
$$= \underline{\rho}\left(\partial_t\underline{\varphi} + \frac{1}{2}|\nabla\underline{\varphi}|^2 + g\eta - \sigma\,\mathrm{div}\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right)\right).$$

Here g is the acceleration due to gravity and $\sigma > 0$ is the coefficient of surface tension. The kinetic energy of the system in each fluid domain is given by the Dirichlet integrals

$$\overline{K} = \frac{1}{2}\int_{\overline{S}(\eta)} \overline{\rho}|\nabla\overline{\varphi}|^2\,dxdydz, \qquad \underline{K} = \frac{1}{2}\int_{\underline{S}(\eta)} \underline{\rho}|\nabla\underline{\varphi}|^2\,dxdydz.$$

and the potential energy of the system is

$$V = V_1 + V_2 = \frac{1}{2}\int_{\mathbb{R}^2} g(\underline{\rho} - \overline{\rho})\eta^2\,dxdz + \frac{1}{2}\int_{\mathbb{R}^2} \sigma(\underline{\rho} - \overline{\rho})\left(\sqrt{1+|\nabla\eta|^2} - 1\right)dxdz.$$

The Hamilton function is the total energy

$$H = \overline{K} + \underline{K} + V_1 + V_2.$$

In order to obtain dimensionless variables we define

$$
\begin{aligned}
(x', y', z') &:= \frac{1}{\overline{h} + \underline{h}}(x, y, z) + \frac{\underline{h}}{\overline{h} + \underline{h}}(1, 1, 1), \\
t' &:= \left(\frac{g}{\overline{h} + \underline{h}}\right)^{\frac{1}{2}} t, \\
\eta'(x', z', t') &:= \frac{1}{\overline{h} + \underline{h}}\eta(x, z, t), \\
\overline{\varphi}'(x', y', z', t') &:= \frac{1}{(\overline{h} + \underline{h})^{\frac{3}{2}} g^{\frac{1}{2}}}\overline{\varphi}(x, y, z, t), \\
\underline{\varphi}'(x', y', z', t') &:= \frac{1}{(\overline{h} + \underline{h})^{\frac{3}{2}} g^{\frac{1}{2}}}\underline{\varphi}(x, y, z, t).
\end{aligned}
$$

Hence we receive the equations (dropping the primes for notational simplicity)

$$\Delta\underline{\varphi} = 0, \qquad 0 < y < \eta + h \tag{1.1}$$

$$\Delta\overline{\varphi} = 0, \qquad \eta + h < y < 1, \tag{1.2}$$

were we have abbreviated $h := \underline{h}/(\overline{h} + \underline{h})$ with boundary conditions

$$\partial_t \eta = \underline{\varphi}_y - \eta_x \underline{\varphi}_x - \eta_z \underline{\varphi}_z, \qquad y = \eta + h, \tag{1.3}$$

$$\partial_t \eta = -\overline{\varphi}_y + \eta_x \overline{\varphi}_x + \eta_z \overline{\varphi}_z, \qquad y = \eta + h \tag{1.4}$$

$$\underline{\varphi}_y = 0, \qquad y = 0, \tag{1.5}$$

$$\overline{\varphi}_y = 0, \qquad y = 1, \tag{1.6}$$

$$
\begin{aligned}
&\rho\left(\partial_t \overline{\varphi} + \frac{1}{2}|\nabla\overline{\varphi}|^2 + \eta - \beta \operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}}\right)\right) \\
&= \partial_t \underline{\varphi} + \frac{1}{2}|\nabla\underline{\varphi}|^2 + \eta - \beta \operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}}\right), \quad y = \eta + h.
\end{aligned} \tag{1.7}
$$

Here we have $\rho := \overline{\rho}/\underline{\rho} \in (0, 1)$ and $\beta := \sigma/(g(\overline{h} + \underline{h})) > 0$. The kinetic and potential energies now reads as

$$\overline{K} = \frac{1}{2}\int_{\overline{S}(\eta)} \rho|\nabla\overline{\varphi}|^2 \, dxdydz, \tag{1.8}$$

$$\underline{K} = \frac{1}{2}\int_{\underline{S}(\eta)} |\nabla\underline{\varphi}|^2 \, dxdydz, \tag{1.9}$$

$$V_1 = \frac{1}{2}\int_{\mathbb{R}^2} (1 - \rho)\eta^2 \, dxdz, \tag{1.10}$$

$$V_2 = \frac{1}{2}\int_{\mathbb{R}^2} \beta(1-\rho)\left(\sqrt{1+|\nabla\eta|^2}-1\right)\,dxdz. \tag{1.11}$$

Here we have changed the meaning of the fluid domains, i.e.,

$$\underline{S}(\eta) := \{(x,y,z):\ x,y\in\mathbb{R},\ 0\le y\le \eta+h\},$$
$$\overline{S}(\eta) := \{(x,y,z):\ x,y\in\mathbb{R},\ \eta+h\le y\le 1\}.$$

Steady waves are water waves which travel in a distinguish horizontal direction with constant speed and without change of shape; without loss of generality we assume that the waves propagate in the x-direction with speed c, so that $\eta(x,z,t)=\eta(x-ct,z)$, $\underline{\varphi}(x,y,z,t)=\underline{\varphi}(x-ct,y,z)$ and $\overline{\varphi}(x,y,z,t)=\overline{\varphi}(x-ct,y,z)$.
Now we have to minimize the functional

$$E(\eta,\underline{\varphi},\overline{\varphi}) := \underline{K}(\eta,\underline{\varphi})+\overline{K}(\eta,\overline{\varphi})+V_1(\eta)+V_2(\eta). \tag{1.12}$$

We deenote the boundary values of the velocity potentials by $\underline{\Phi}(x,z) := \underline{\varphi}(x,z,\eta(x,z))$ and $\overline{\Phi}(x,z) := \overline{\varphi}(x,z,\eta(x,z))$. Following Benjamin and Bridges [BB] we set

$$\xi(x) := \underline{\Phi}(x) - \rho\overline{\Phi},$$

and the natural choice of canonical variables is (η,ξ) (compare [CG]). Similarly to [CG] (and [BGS], section 1.2) we define Dirichlet-Neumann operators $\underline{G}(\eta)$ and $\overline{G}(\eta)$ which maps (for a given η) Dirichlet boundary-data of solution of the Laplace-equation to the Neumann boundary-data, i.e.

$$\underline{G}(\eta)\underline{\Phi}(x,z) := (1+|\nabla\eta|^2)^{\frac{1}{2}}(\nabla\underline{\varphi}\cdot N_{\underline{S}(\eta)})(x,z),$$
$$\overline{G}(\eta)\overline{\Phi}(x,z) := (1+|\nabla\eta|^2)^{\frac{1}{2}}(\nabla\overline{\varphi}\cdot N_{\overline{S}(\eta)})(x,z).$$

If we define

$$B(\eta) := \overline{G}(\eta)+\rho\underline{G}(\eta),$$

we obtain the Hamilton (following the lines of [CG], p. 24)

$$\begin{aligned} H(\eta,\xi) &= \frac{1}{2}\int_{\mathbb{R}^2}\xi\underline{G}(\eta)B(\eta)^{-1}\overline{G}(\eta)\xi\,dxdz+\frac{1}{2}\int_{\mathbb{R}^2}(1-\rho)\eta^2\,dxdz \\ &\quad+\frac{1}{2}\int_{\mathbb{R}^2}\beta(1-\rho)\left(\sqrt{1+|\nabla\eta|^2}-1\right)\,dxdz. \end{aligned} \tag{1.13}$$

The key of our existence theory is minimizing H subject to the constraint of a fixed value for the momentum of a wave in the x-direction

$$I(\eta,\xi) := \int_{\mathbb{R}^2}\eta_x\xi\,dxdz. \tag{1.14}$$

We tackle the problem of finding minimizers of H under $I(\eta,\xi)=2\sqrt{\kappa(\rho,\mu)}\mu$, where

$$\kappa(\rho,\mu) = (1-\rho)\left(\frac{1}{h}+\frac{\rho}{1-h}\right)^{-1},$$

in two steps.

1. Fix $\eta \neq 0$ and minimize $H(\eta, \cdot)$ over $T_\mu = \left\{\xi \in H_*^{1/2}(\mathbb{R}^2) : I(\eta, \xi) = 2\sqrt{\kappa(\rho, \mu)}\mu\right\}$. This problem (of minimizing a quadratic functional over a linear manifold) admits a unique global minimizer ξ_η.

2. Minimize $\mathcal{J}(\eta) := H(\eta, \xi_\eta)$ over $\eta \in U\,\{0\}$ with $U := B_M(0) \subset H^3(\mathbb{R}^2)$. Because ξ_η minimizes $H(\eta, \cdot)$ over T_μ there exists a Lagrange multiplier λ_η such that

$$\underline{G}(\eta)B(\eta)^{-1}\overline{G}(\eta)\xi_\eta = \lambda_\eta \eta_x.$$

Hence

$$\begin{aligned}\xi_\eta &= \lambda_\eta \left[\underline{G}(\eta)B(\eta)^{-1}\overline{G}(\eta)\right]^{-1}\eta_x \\ &= \lambda_\eta \left[\underline{N}(\eta) + \rho\overline{N}(\eta)\right]\eta_x,\end{aligned}$$

where $\underline{N}(\eta) = \underline{G}(\eta)^{-1}$ and $\overline{N}(\eta) = \overline{G}(\eta)^{-1}$ are the Neumann-Dirichlet operators. Furthermore we get

$$\lambda_\eta = \frac{\sqrt{\kappa(\rho, h)}\mu}{\mathcal{L}(\eta)},\ \mathcal{L}(\eta) = \frac{1}{2}\int_{\mathbb{R}^2} \eta_x \left[\underline{N}(\eta) + \rho\overline{N}(\eta)\right]\eta_x\, dxdz. \tag{1.15}$$

For $\mathcal{J}(\eta)$ we get the representation

$$\mathcal{J}_\mu(\eta) = \mathcal{K}(\eta) + \frac{\kappa(\rho, \mu)\mu^2}{\mathcal{L}(\eta)}, \tag{1.16}$$

where

$$\mathcal{K}(\eta) = (1-\rho)\int_{\mathbb{R}^2}\left\{\frac{1}{2}\eta^2 + \beta\sqrt{1+\eta_x^2+\eta_z^2} - \beta\right\}\mathrm{d}x\,\mathrm{d}z, \tag{1.17}$$

Our paper is organized as follows: In section 2 we prove analiticity of the Neumann-Dirichlet operator. A consideration of minimizing sequences is given in section 3. In the forth section we have a look at strict sub-additivity of the infimum of $\mathcal{J}_\mu$ with respect to μ, whereas in the last section we follow the main Theorems about the existence of minimizers and the stability of the set of minimizers.

2 The functional-analytic setting

2.1 The Neumann-Dirichlet operator

Our first task is to find suitable function spaces for the functionals H and I defined in equations (1.13), (1.14) and introduce rigorous definitions of the Dirichlet-Neumann operators and their inverses. Since the functional $\mathcal{J}_\mu$ to be minimised involves $\underline{G}(\eta)^{-1}$ and $\overline{G}(\eta)^{-1}$ we begin with the formal definition of this *Neumann-Dirichlet operator* $\underline{N}(\eta)$ and $\overline{N}(\eta)$: for fixed $\xi = \xi(x, z)$ we solve the boundary-value problem

$$\Delta\underline{\varphi} = 0, \qquad 0 < y < h + \eta, \tag{2.1}$$

$$\underline{\varphi}_y - \eta_x \underline{\varphi}_x - \eta_z \underline{\varphi}_z = \xi, \qquad y = h + \eta, \tag{2.2}$$
$$\underline{\varphi}_y = 0, \qquad y = 0 \tag{2.3}$$

and define

$$\underline{N}(\eta)\xi = \underline{\varphi}|_{y=h+\eta}.$$

Furthermore we solve the boundary-value problem

$$\Delta\overline{\varphi} = 0, \qquad h + \eta < y < 1, \tag{2.4}$$
$$-\overline{\varphi}_y + \eta_x \overline{\varphi}_x + \eta_z \overline{\varphi}_z = \xi, \qquad y = h + \eta, \tag{2.5}$$
$$\overline{\varphi}_y = 0, \qquad y = 1 \tag{2.6}$$

and define

$$\overline{N}(\eta)\xi = \overline{\varphi}|_{y=h+\eta}.$$

We study this boundary-value problems by transforming them to equivalent problems in fixed domains. The change of variable in the first problem

$$y' = \frac{h}{h+\eta} y, \qquad \underline{u}(x, y', z) = \underline{\varphi}(x, y, z)$$

transforms the variable domain $\{0 < y < h + \eta(x, z)\}$ into the slab $\underline{\Sigma} = \{(x, y', z) \in \mathbb{R} \times (0, h) \times \mathbb{R}\}$ and the boundary-value problem (2.1)–(2.3) into

$$\Delta\underline{u} = \partial_x \underline{F}_1 + \partial_z \underline{F}_2 + \partial_y \underline{F}_3, \qquad 0 < y < h, \tag{2.7}$$
$$\underline{u}_y = \underline{F}_3 + \xi, \qquad y = h, \tag{2.8}$$
$$\underline{u}_y = 0, \qquad y = 0, \tag{2.9}$$

where

$$\begin{aligned}
\underline{F}_1 &= -\frac{\eta}{h}\underline{u}_x + \frac{y}{h}\eta_x \underline{u}_y, \\
\underline{F}_2 &= -\frac{\eta}{h}\underline{u}_z + \frac{y}{h}\eta_z \underline{u}_y, \\
\underline{F}_3 &= \frac{\eta}{\eta + h}\underline{u}_y + \frac{y}{h}\eta_x \underline{u}_x + \frac{y}{h}\eta_z \underline{u}_z - \frac{1}{h}\frac{y^2\eta_x^2}{\eta + h}\underline{u}_y - \frac{1}{h}\frac{y^2\eta_z^2}{\eta + h}\underline{u}_y
\end{aligned}$$

and we have again dropped the primes for notational simplicity; the Neumann-Dirichlet operator is given by

$$\underline{N}(\eta)\xi = \underline{u}|_{y=h}.$$

In the second problem the transformation

$$y' = \frac{1-h}{1-h-\eta}(1-y), \qquad \overline{u}(x, y', z) = \overline{\varphi}(x, y, z)$$

converts the domain $\{h + \eta(x, z) < y < 1\}$ into $\overline{\Sigma} = \{(x, y', z) \in \mathbb{R} \times (0, 1-h) \times \mathbb{R}\}$ and the boundary-value problem (2.4)–(2.6) into

$$\Delta\overline{u} = \partial_x \overline{F}_1 + \partial_z \overline{F}_2 + \partial_y \overline{F}_3, \qquad 0 < y < 1 - h, \tag{2.10}$$

$$\overline{u}_y = \overline{F}_3 + \xi, \qquad y = 1-h, \tag{2.11}$$
$$\overline{u}_y = 0, \qquad y = 0, \tag{2.12}$$

where

$$\begin{aligned}
\overline{F}_1 &= \frac{\eta}{1-h}\overline{u}_x - \frac{y}{1-h}\eta_x\overline{u}_y, \\
\overline{F}_2 &= \frac{\eta}{1-h}\overline{u}_z - \frac{y}{1-h}\eta_z\overline{u}_y, \\
\overline{F}_3 &= \frac{\eta}{\eta+h-1}\overline{u}_y - \frac{y}{1-h}\eta_x\overline{u}_x - \frac{y}{1-h}\eta_z\overline{u}_z + \frac{1}{1-h}\frac{y^2\eta_x^2}{\eta+h-1}\overline{u}_y + \frac{1}{1-h}\frac{y^2\eta_z^2}{\eta+h-1}\overline{u}_y.
\end{aligned}$$

The Neumann-Dirichlet operator is given by

$$\overline{N}(\eta)\xi = \overline{u}|_{y=1-h}.$$

To develop a convenient theory for weak solutions of the boundary-value problems (2.7)–(2.9) and (2.10)–(2.12) we follow the lines of [BGS] (section 2.1). We define the completion $H^1_\star(\Sigma)$ of

$$\mathcal{S}(\Sigma,\mathbb{R}) = \{u \in C^\infty(\bar{\Sigma}) : |(x,z)|^m |\partial_x^{\alpha_1}\partial_z^{\alpha_2} u| \text{ is bounded for all } m, \alpha_1, \alpha_2 \in \mathbb{N}_0\}$$

with respect to the norm

$$\|u\|_\star^2 := \int_\Sigma (u_x^2 + u_y^2 + u_z^2)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z.$$

Here we have $\Sigma \in \{\underline{\Sigma}, \overline{\Sigma}\}$ and $\bar{\Sigma}$ denotes the closure of Σ. The corresponding space for the traces $\underline{u}|_{y=h}$ and $\overline{u}|_{y=1-h}$ is the completion $H_\star^{1/2}(\mathbb{R}^2)$ of the inner product space $X_\star^{1/2}(\mathbb{R}^2)$ constructed by equipping the Schwartz class $\mathcal{S}(\mathbb{R}^2,\mathbb{R})$ with the norm

$$\|\eta\|_{\star,1/2}^2 := \int_{\mathbb{R}^2} (1+|k|^2)^{-1/2}|k|^2|\hat{\eta}|^2\,\mathrm{d}k_1\,\mathrm{d}k_2;$$

its dual $(H_\star^{1/2}(\mathbb{R}^2))'$ is the space

$$(X_\star^{1/2}(\mathbb{R}^2))' = \Big\{u \in \mathcal{S}'(\mathbb{R}^2,\mathbb{R}) : \sup\{|(u,\eta)| : \eta \in X_\star^{1/2}(\mathbb{R}^2),\ \|\eta\|_{\star,1/2} < 1\} < \infty\Big\},$$

where $\mathcal{S}'(\mathbb{R}^2,\mathbb{R})$ is the class of two-dimensional, real-valued, tempered distributions. A more convenient description of $(H_\star^{1/2}(\mathbb{R}^2))'$ is proven in Prop. 2.1 in [BGS]:

Proposition 2.1. *Let $H_\star^{-1/2}(\mathbb{R}^2)$ be the completion of the inner product space $X_\star^{-1/2}(\mathbb{R}^2)$ constructed by equipping $\bar{\mathcal{S}}(\mathbb{R}^2,\mathbb{R})$ with the norm*

$$\|\eta\|_{\star,-1/2}^2 := \int_{\mathbb{R}^2} (1+|k|^2)^{1/2}|k|^{-2}|\hat{\eta}|^2\,\mathrm{d}k_1\,\mathrm{d}k_2,$$

where $\bar{\mathcal{S}}(\mathbb{R}^2,\mathbb{R})$ is the subclass of $\mathcal{S}(\mathbb{R}^2,\mathbb{R})$ consisting of functions with zero mean. The space $H_\star^{-1/2}(\mathbb{R}^2)$ can be identified with $(H_\star^{1/2}(\mathbb{R}^2))'$.

With these preparations we obtain similarly to Lemma 2.4, [BGS]

LEMMA 2.1. *For each* $\xi \in H_\star^{-1/2}(\mathbb{R}^2)$ *and* $\eta \in B_{1/2}(0) \subset W^{1,\infty}(\mathbb{R}^2)$ *the boundary-value problems (2.7)–(2.9) and (2.10)–(2.12) have unique weak solutions* $\underline{u} \in H_\star^1(\underline{\Sigma})$ *and* $\overline{u} \in H_\star^1(\overline{\Sigma})$.

Here weak solutions are defined in the sense of [BGS] (Def. 2.3).
We conclude with a rigorous definition of the Neumann-Dirichlet operators.

Definition 2.1. *a) The* <u>*Neumann-Dirichlet operator*</u> *for the boundary-value problem (2.7)–(2.9) is the bounded linear operator* $\underline{N}(\eta) : H_\star^{-1/2}(\mathbb{R}^2) \to H_\star^{1/2}(\mathbb{R}^2)$ *defined by*

$$\underline{N}(\eta)\xi = \underline{u}|_{y=h},$$

where $\underline{u} \in H_\star^1(\underline{\Sigma})$ *is the unique weak solution of (2.7)–(2.9).*

b) The <u>*Neumann-Dirichlet operator*</u> *for the boundary-value problem (2.10)–(2.12) is the bounded linear operator* $\overline{N}(\eta) : H_\star^{-1/2}(\mathbb{R}^2) \to H_\star^{1/2}(\mathbb{R}^2)$ *defined by*

$$\overline{N}(\eta)\xi = \overline{u}|_{y=1-h},$$

where $\overline{u} \in H_\star^1(\overline{\Sigma})$ *is the unique weak solution of (2.10)–(2.12).*

At the end of this section we present the following useful representations (compare [BGS], Remark 2.6, for details)

LEMMA 2.2. *We have for* $\xi \in H_\star^{-1/2}(\mathbb{R}^2)$

$$\begin{aligned}
&\int_{\mathbb{R}^2} \xi \underline{N}(\eta)\xi \,\mathrm{d}x\,\mathrm{d}z \\
&= \int_{\underline{\Sigma}} \left(\left(\underline{u}_x - \frac{y\eta_x}{h+\eta}\underline{u}_y \right)^2 + \left(\frac{h\underline{u}_y}{h+\eta} \right)^2 + \left(\underline{u}_z - \frac{y\eta_z}{\eta+h}\underline{u}_y \right)^2 \right) \frac{\eta+h}{h} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z; \\
&\int_{\mathbb{R}^2} \xi \overline{N}(\eta)\xi \,\mathrm{d}x\,\mathrm{d}z \\
&= \int_{\overline{\Sigma}} \left(\left(\overline{u}_x - \frac{y\eta_x}{\eta+h-1}\overline{u}_y \right)^2 + \left(\frac{(1-h)\overline{u}_y}{\eta+h-1} \right)^2 + \left(\overline{u}_z - \frac{y\eta_z}{\eta+h-1}\overline{u}_y \right)^2 \right) \frac{1-\eta-h}{1-h} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}$$

2.2 Analyticity of the Neumann-Dirichlet operators

In this section we establish that $\underline{N}(\eta)$ and $\overline{N}(\eta)$ are analytic functions of η in the above function spaces, which clearly implies analicity of $\underline{N}(\eta) + \rho\overline{N}(\eta)$. We start with the definition of analyticity (compare Buffoni & Toland [BT], Definition 4.3.1).

Definition 2.2. *Let X and Y be Banach spaces, U be a non-empty, open subset of X and $\mathcal{L}^k_{\mathrm{s}}(X,Y)$ be the space of bounded, k-linear symmetric operators $X^k \to Y$ with norm*

$$|||m||| := \inf\{c : \|m(\{f\}^{(k)})\|_Y \le c\|f\|_X^k \text{ for all } f \in X\}.$$

A function $F : U \to Y$ is <u>*analytic*</u> *at a point $x_0 \in U$ if there exist real numbers $\delta, r > 0$ and a sequence $\{m_k\}$, where $m_k \in \mathcal{L}^k_{\mathrm{s}}(X,Y)$, $k = 0,1,2,\ldots$, with the properties that*

$$F(x) = \sum_{k=0}^{\infty} m_k(\{x - x_0\}^{(k)}), \qquad x \in B_\delta(x_0)$$

and

$$\sup_{k\ge 0} r^k |||m_k||| < \infty.$$

Our main task is to establish the following theorem.

THEOREM 2.3. *The mappings from $W^{1,\infty}(\mathbb{R}^2) \to \mathcal{L}(H_\star^{-1/2}(\mathbb{R}^2), H_\star^{1/2}(\mathbb{R}^2))$ given by $\eta \mapsto (\xi \mapsto \underline{u}|_{y=h})$ and $\eta \mapsto (\xi \mapsto \overline{u}|_{y=1-h})$, where $\underline{u} \in H^1_\star(\underline{\Sigma})$ and $\overline{u} \in H^1_\star(\overline{\Sigma})$ are the unique weak solution of (2.7)–(2.9) resp. (2.10)–(2.12), are analytic at the origin.*

Proof: If we can show

$$\underline{u}(x,y,z) = \sum_{n=0}^{\infty} \underline{u}^n(x,y,z), \tag{2.13}$$

$$\overline{u}(x,y,z) = \sum_{n=0}^{\infty} \overline{u}^n(x,y,z), \tag{2.14}$$

where $\underline{u}^n$ and $\overline{u}^n$ are functions of η and ξ which are homogeneous of degree n in η and linear in ξ, then the claim of Theorem 2.3 follows by the lines of [BGS] (section 2.2). We refer to Nicholls & Reitich who developed this technique for proving analicity of Dirichlet-Neumann operators. Substituting (2.13) into the equations of the nether fluid, one finds that

$$\Delta \underline{u}^0 = 0, \qquad 0 < y < h, \tag{2.15}$$

$$\underline{u}^0_y = \xi, \qquad y = h, \tag{2.16}$$

$$\underline{u}^0_y = 0, \qquad y = 0 \tag{2.17}$$

and

$$\Delta \underline{u}^n = \partial_x \underline{F}^n_1 + \partial_z \underline{F}^n_2 + \partial_y \underline{F}^n_3, \qquad 0 < y < h, \tag{2.18}$$

$$\underline{u}^n_y = \underline{F}^n_3, \qquad y = h, \tag{2.19}$$

$$\underline{u}^n_y = 0, \qquad y = 0 \tag{2.20}$$

for $n = 1,2,3,\ldots$, where

$$\underline{F}^n_1 = -\frac{\eta}{h}\underline{u}^{n-1}_x + \frac{y}{h}\eta_x \underline{u}^{n-1}_y, \tag{2.21}$$

$$
\begin{aligned}
\underline{F}_2^n &= -\frac{\eta}{h}\underline{u}_z^{n-1} + \frac{y}{h}\eta_z \underline{u}_y^{n-1}, && (2.22)\\
\underline{F}_3^n &= \frac{\eta}{h}\sum_{\ell=0}^{n-1} h^{-\ell}(-\eta)^\ell \underline{u}_y^{n-1-\ell} + \frac{y}{h}\eta_x \underline{u}_x^{n-1} + \frac{y}{h}\eta_z \underline{u}_z^{n-1} && (2.23)\\
&- \frac{y^2}{h^2}(\eta_x^2+\eta_z^2)\sum_{\ell=0}^{n-2} h^{-\ell}(-\eta)^\ell \underline{u}_y^{n-2-\ell}.
\end{aligned}
$$

Substituting (2.14) into the equations of the upper fluid, one finds that

$$
\begin{aligned}
&\Delta \overline{u}^0 = 0, && 0 < y < 1-h, && (2.24)\\
&\overline{u}_y^0 = \xi, && y = 1-h, && (2.25)\\
&\overline{u}_y^0 = 0, && y = 0 && (2.26)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta \overline{u}^n = \partial_x \overline{F}_1^n + \partial_z \overline{F}_2^n + \partial_y \overline{F}_3^n, && 0 < y < 1-h, && (2.27)\\
&\overline{u}_y^n = \overline{F}_3^n, && y = 1-h, && (2.28)\\
&\overline{u}_y^n = 0, && y = 0 && (2.29)
\end{aligned}
$$

for $n = 1, 2, 3, \ldots$, where

$$
\begin{aligned}
\overline{F}_1^n &= \frac{\eta}{1-h}\overline{u}_x^{n-1} - \frac{y}{1-h}\eta_x \overline{u}_y^{n-1}, && (2.30)\\
\overline{F}_2^n &= \frac{\eta}{1-h}\overline{u}_z^{n-1} - \frac{y}{1-h}\eta_z \overline{u}_y^{n-1}, && (2.31)\\
\overline{F}_3^n &= -\eta\frac{1}{1-h}\sum_{\ell=0}^{n-1}(1-h)^{-\ell}\eta^\ell \overline{u}_y^{n-1-\ell} - \frac{y}{1-h}\eta_x \overline{u}_x^{n-1} - \frac{y}{1-h}\eta_z \overline{u}_z^{n-1} && (2.32)\\
&- \frac{y^2}{(1-h)^2}(\eta_x^2+\eta_z^2)\sum_{\ell=0}^{n-2}(1-h)^{-\ell}\eta^\ell \overline{u}_y^{n-2-\ell}.
\end{aligned}
$$

From this expansion we can follow the claim of Theorem 2.3 by the lines of [BGS] (section 2.1).
Observe that the formula (1.15) defining $\mathcal{L}$ may be written as

$$
\mathcal{L}(\eta) = \frac{1}{2}\int_{\mathbb{R}^2} \eta K(\eta)\eta \, \mathrm{d}x\, \mathrm{d}z,
$$

where

$$
K(\eta) = -\partial_x\left(\left[\underline{N}(\eta) + \rho\overline{N}(\eta)\right]\partial_x\right),
$$

and we obtain similarly to [BGS] (Thm. 2.19)

THEOREM 2.4. *Suppose that* $s > 1$. *The operator* $K(\cdot) : H^{s+3/2}(\mathbb{R}^2) \to \mathcal{L}(H^{s+1}(\mathbb{R}^2), H^s(\mathbb{R}^2))$ *is analytic at the origin.*

We remark that the Fourier transforms of the weak solutions $\underline{u}^0$ and $\overline{u}^0$ of (2.15)–(2.17) and (2.24)–(2.26) are given by

$$\hat{\underline{u}}^0 = \frac{\cosh|k|y}{|k|\sinh|k|h}\hat{\xi}, \qquad \hat{\overline{u}}^0 = \frac{\cosh|k|y}{|k|\sinh|k|(1-h)}\hat{\xi}. \tag{2.33}$$

Using Theorem 2.3 and the continuity of the trace operator $H^{s+1/2}(\Sigma) \to H^s(\mathbb{R}^2)$, we find that the series representations of the operators $H^{s+3/2}(\mathbb{R}^2) \to \mathcal{L}(H^{s+1}(\mathbb{R}^2), H^s(\mathbb{R}^2))$ given by $\eta \mapsto (\zeta \mapsto -\underline{u}_x|_{y=h})$ and $\eta \mapsto (\zeta \mapsto -\overline{u}_x|_{y=1-h})$ are given by

$$K(\eta) = \sum_{n=0}^{\infty} \left[\underline{K}^n(\eta) + \rho\overline{K}^n(\eta)\right],$$

where $\underline{K}^n(\eta)\zeta = -\underline{u}^n_x|_{y=h}$, $\overline{K}^n(\eta)\zeta = -\overline{u}^n_x|_{y=1-h}$ and $\xi = \zeta_x$.

2.3 The functionals $\mathcal{K}$, $\mathcal{L}$ and a special testfunction

The following lemma, whose proof is similar to the arguments in [BGS] (section 2.4), formally states the analyticity property of $\mathcal{K}$ (examine the explicit formula for $\mathcal{K}$) and $\mathcal{L}$ (see Theorem 2.4). In particular this result implies that $\mathcal{K}, \mathcal{L}$ belong to the class $C^\infty(U, \mathbb{R})$ and that equation (1.16) defines an operator $\mathcal{J}_\mu \in C^\infty(U\setminus\{0\}, \mathbb{R})$, where $U = B_M(0) \subset H^3(\mathbb{R}^2)$ and M is chosen sufficiently small.

LEMMA 2.5. *Equations (1.17), (1.15) define functionals $\mathcal{K} : H^{s+1}(\mathbb{R}^2) \to \mathbb{R}$, $\mathcal{L} : H^{s+3/2}(\mathbb{R}^2) \to \mathbb{R}$ for $s > 1$ which are analytic at the origin and satisfy $\mathcal{K}(0) = \mathcal{L}(0) = 0$.*

We have the following representation for the gradients of $\mathcal{K}$, $\underline{\mathcal{L}}$ and $\overline{\mathcal{L}}$, where the last two functionals are defined in a suitable fashion such that $\mathcal{L} = \underline{\mathcal{L}} + \rho\overline{\mathcal{L}}$.

LEMMA 2.6. *The gradients $\mathcal{K}'(\eta)$, $\underline{\mathcal{L}}'(\eta)$ and $\overline{\mathcal{L}}'(\eta)$ in $L^2(\mathbb{R}^2)$ exist for each $\eta \in U$. They are given by the formulae*

$$\mathcal{K}'(\eta) = -(1-\rho)\left(\frac{\beta\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right)_x - (1-\rho)\left(\frac{\beta\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right)_z + (1-\rho)\eta,$$

$$\begin{aligned}\underline{\mathcal{L}}'(\eta) = \int_0^h \Bigg\{ &-\frac{1}{2h}(\underline{u}_x^2 + \underline{u}_z^2) - \frac{y}{h}(\underline{u}_x\underline{u}_y)_x - \frac{y}{h}(\underline{u}_z\underline{u}_y)_z + \left(\frac{y^2\eta_x\underline{u}_y^2}{h(h+\eta)}\right)_x + \left(\frac{y^2\eta_z\underline{u}_y^2}{h(h+\eta)}\right)_z \\ &+ \frac{y^2\underline{u}_y^2}{2h(h+\eta)^2}(\eta_x^2+\eta_z^2) + h\frac{\underline{u}_y^2}{2(h+\eta)^2}\Bigg\}\,\mathrm{d}y - \underline{u}_x|_{y=h},\end{aligned}$$

$$\overline{\mathcal{L}}'(\eta) = \int_0^{1-h} \Bigg\{ \frac{1}{2(1-h)}(\overline{u}_x^2 + \overline{u}_z^2) + \frac{y}{1-h}(\overline{u}_x\overline{u}_y)_x + \frac{y}{1-h}(\overline{u}_z\overline{u}_y)_z$$

$$
\begin{aligned}
&- \left(\frac{y^2 \eta_x \overline{u}_y^2}{(1-h)(\eta+h-1)} \right)_x - \left(\frac{y^2 \eta_z \overline{u}_y^2}{(1-h)(\eta+h-1)} \right)_z \\
&- \frac{y^2 \overline{u}_y^2}{2(1-h)(\eta+h-1)^2} (\eta_x^2 + \eta_z^2) - \frac{(1-h)\overline{u}_y^2}{2(\eta+h-1)^2} \Bigg\} \, \mathrm{d}y - \overline{u}_x|_{y=1-h},
\end{aligned}
$$

and define functions $\mathcal{K}' : H^3(\mathbb{R}^2) \to H^1(\mathbb{R}^2)$, $\underline{\mathcal{L}}', \overline{\mathcal{L}}' : H^{s+3/2}(\mathbb{R}^2) \to H^s(\mathbb{R}^2)$ for $s > 1$ which are analytic at the origin and satisfy $\mathcal{K}'(0) = \underline{\mathcal{L}}'(0) = \overline{\mathcal{L}}'(0) = 0$.

Proof. The formula for $\mathcal{K}'$ is given in [BGS] (Lemma 2.27), whereas for $\underline{\mathcal{L}}'$ and $\overline{\mathcal{L}}'$ we differentiate the equations in Lemma 2.2: in the first case we have for $\xi \in H_\star^{-1/2}(\mathbb{R}^2)$

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \xi \underline{N}(\eta) \xi \, \mathrm{d}x \, \mathrm{d}z \\
&= \int_{\underline{\Sigma}} \left(\left(\underline{u}_x - \frac{y\eta_x}{h+\eta} \underline{u}_y \right)^2 + \left(\frac{h\underline{u}_y}{h+\eta} \right)^2 + \left(\underline{u}_z - \frac{y\eta_z}{\eta+h} \underline{u}_y \right)^2 \right) \frac{\eta+h}{h} . \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z
\end{aligned}
$$

We obtain abbreviating $\underline{w} = \mathrm{d}\underline{u}[\eta](\omega)$

$$
\begin{aligned}
&\mathrm{d}\underline{\mathcal{L}}[\eta](\omega) \\
&= \frac{1}{h} \int_{\underline{\Sigma}} \Bigg\{ (h+\eta)(\underline{w}_x \underline{u}_x + \underline{w}_z \underline{u}_z) - y\eta_x \underline{w}_x \underline{u}_y - y\eta_x \underline{u}_x \underline{w}_y - y\eta_z \underline{w}_z \underline{u}_y - y\eta_z \underline{u}_z \underline{w}_y \\
&\qquad + \frac{y^2 \underline{u}_y \underline{w}_y}{h+\eta} (\eta_x^2 + \eta_z^2) + \frac{h^2 \underline{u}_y \underline{w}_y}{h+\eta} + \frac{\omega}{2} (\underline{u}_x^2 + \underline{u}_z^2) - y\omega_x \underline{u}_x \underline{u}_y - y\omega_z \underline{u}_z \underline{u}_y \\
&\qquad + \frac{y^2 \underline{u}_y^2}{h+\eta} (\eta_x \omega_x + \eta_z \omega_z) - \frac{\omega y^2 \underline{u}_y^2}{2(h+\eta)^2} (\eta_x^2 + \eta_z^2) - \frac{\omega \underline{u}_y^2 h^2}{2(h+\eta)^2} \Bigg\} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \quad (2.34)
\end{aligned}
$$

Since $\underline{u}$ is a weak solution of (2.1)-(2.3), letting $\xi = \eta_x$, we get

$$
\begin{aligned}
&\int_{\underline{\Sigma}} \Bigg\{ \frac{h+\eta}{h} (\underline{u}_x v_x + \underline{u}_z v_z) - \frac{y}{h} \eta_x v_x \underline{u}_y - \frac{y}{h} \eta_x \underline{u}_x v_y - \frac{y}{h} \eta_z v_z \underline{u}_y - \frac{y}{h} \eta_z \underline{u}_z v_y \\
&\qquad + \frac{y^2 \underline{u}_y v_y}{h(h+\eta)} (\eta_x^2 + \eta_z^2) + \frac{h \underline{u}_y v_y}{h+\eta} \Bigg\} \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}z \\
&= \int_{\mathbb{R}^2} \eta_x v|_{y=h} \, \mathrm{d}x \, \mathrm{d}z
\end{aligned}
$$

for every $v \in H_\star^1(\underline{\Sigma})$. Differentiating this equation with respect to η, we find that

$$
\begin{aligned}
&\int_{\Sigma} \Bigg\{ \frac{h+\eta}{h} (\underline{w}_x v_x + \underline{w}_z v_z) - \frac{y}{h} \eta_x \underline{w}_x v_y - \frac{y}{h} \eta_x v_x \underline{w}_y - \frac{y}{h} \eta_z \underline{w}_z v_y - \frac{y}{h} \eta_z v_z \underline{w}_y \\
&\qquad + \frac{y^2 v_y \underline{w}_y}{h(h+\eta)} (\eta_x^2 + \eta_z^2) + h \frac{v_y \underline{w}_y}{h+\eta} + \frac{\omega}{h} (\underline{u}_x v_x + \underline{u}_z v_z) - \frac{y}{h} \omega_x v_x \underline{u}_y - \frac{y}{h} \omega_x \underline{u}_x v_y
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{y}{h}\omega_z v_z \underline{u}_y - \frac{y}{h}\omega_z \underline{u}_z v_y + 2\frac{y^2 \underline{u}_y v_y}{h(h+\eta)}(\eta_x \omega_x + \eta_z \omega_z) \\
&\left. - \frac{y^2 \underline{u}_y v_y}{h(h+\eta)^2}(\eta_x^2 + \eta_z^2)\omega - h\frac{\underline{u}_y v_y}{(h+\eta)^2}\omega \right\} \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z \\
= &\int_{\mathbb{R}^2} \omega_x v|_{y=h}\,\mathrm{d}x\,\mathrm{d}z
\end{aligned}
$$

for every $v \in H^1_\star(\Sigma)$; subtracting this equation with $v = \underline{u}$ from (2.34) yields

$$
\begin{aligned}
&\mathrm{d}\underline{\mathcal{L}}[\eta](\omega) \\
&= \int_\Sigma \left\{ -\frac{\omega}{2h}(\underline{u}_x^2 + \underline{u}_z^2) + \frac{y}{h}\omega_x \underline{u}_x \underline{u}_y + \frac{y}{h}\omega_z \underline{u}_z \underline{u}_y - \frac{y^2 \underline{u}_y^2}{h(h+\eta)}(\eta_x \omega_x + \eta_z \omega_z) \right. \\
&\qquad \left. + \frac{y^2 \underline{u}_y^2}{2h(h+\eta)^2}(\eta_x^2 + \eta_z^2)\omega + h\frac{\omega \underline{u}_y^2}{2(h+\eta)^2} \right\} \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z + \int_{\mathbb{R}^2} \omega_x \underline{u}|_{y=h}\,\mathrm{d}x\,\mathrm{d}z \\
&= \int_\Sigma \left\{ -\frac{1}{2h}(\underline{u}_x^2 + \underline{u}_z^2) - \frac{y}{h}(\underline{u}_x \underline{u}_y)_x - \frac{y}{h}(\underline{u}_z \underline{u}_y)_z + \left(\frac{y^2 \eta_x \underline{u}_y^2}{h(h+\eta)}\right)_x + \left(\frac{y^2 \eta_z \underline{u}_y^2}{h(h+\eta)}\right)_z \right. \\
&\qquad \left. + \frac{y^2 \underline{u}_y^2}{2h(h+\eta)^2}(\eta_x^2 + \eta_z^2) + h\frac{\underline{u}_y^2}{2(h+\eta)^2} \right\} \omega\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z - \int_{\mathbb{R}^2} \omega \underline{u}_x|_{y=h}\,\mathrm{d}x\,\mathrm{d}z.
\end{aligned}
$$

Furthermore we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \xi \overline{N}(\eta)\xi\,\mathrm{d}x\,\mathrm{d}z \\
&= \int_{\overline{\Sigma}} \left(\left(\overline{u}_x - \frac{y\eta_x}{\eta+h-1}\overline{u}_y\right)^2 + \left(\frac{(1-h)\overline{u}_y}{\eta+h-1}\right)^2 \right. \\
&\qquad \left. + \left(\overline{u}_z - \frac{y\eta_z}{\eta+h-1}\overline{u}_y\right)^2 \right) \frac{1-\eta-h}{1-h}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,
\end{aligned}
$$

hence

$$
\begin{aligned}
&\mathrm{d}\overline{\mathcal{L}}[\eta](\omega) \\
&= \int_\Sigma \left\{ \frac{1-\eta+h}{1-h}(w_x \overline{u}_x + w_z \overline{u}_z) + \frac{y}{1-h}\eta_x w_x \overline{u}_y + \frac{y}{1-h}\eta_x \overline{u}_x w_y + \frac{y}{1-h}\eta_z w_z \overline{u}_y \right. \\
&\qquad + \frac{y}{1-h}\eta_z \overline{u}_z w_y + \frac{y^2 \overline{u}_y w_y}{(1-h)(\eta+h-1)}(\eta_x^2 + \eta_z^2) \\
&\qquad - (1-h)\frac{\overline{u}_y w_y}{\eta+h-1} - \frac{\omega}{2(1-h)}(\overline{u}_x^2 - \overline{u}_z^2) \\
&\qquad + \frac{y}{1-h}\omega_x \overline{u}_x \overline{u}_y + \frac{y}{1-h}\omega_z \overline{u}_z \overline{u}_y - \frac{y^2 \overline{u}_y^2}{(1-h)(\eta+h-1)}(\eta_x \omega_x + \eta_z \omega_z)
\end{aligned}
$$

$$+\frac{y^2\overline{u}_y^2}{2(1-h)(\eta+h-1)^2}(\eta_x^2+\eta_z^2)\omega+(1-h)\frac{\omega\overline{u}_y^2}{2(\eta+h-1)^2}\Bigg\}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z, \quad (2.35)$$

where $\overline{w}=\mathrm{d}\overline{u}[\eta](\omega)$. Since $\overline{u}$ is a weak solution of (2.4)-(2.6), we get

$$\int_{\underline{\Sigma}}\Bigg\{\frac{\eta+h-1}{1-h}(\overline{u}_xv_x+\overline{u}_zv_z)-\frac{y}{1-h}\eta_xv_x\overline{u}_y-\frac{y}{1-h}\eta_x\overline{u}_xv_y-\frac{y}{1-h}\eta_zv_z\overline{u}_y-\frac{y}{1-h}\eta_z\overline{u}_zv_y$$

$$\frac{y^2\overline{u}_yv_y}{(1-h)(\eta+h-1)}(\eta_x^2+\eta_z^2)+\frac{(1-h)\overline{u}_yv_y}{\eta+h-1}\Bigg\}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$

$$=-\int_{\mathbb{R}^2}\eta_xv|_{y=1-h}\,\mathrm{d}x\,\mathrm{d}z$$

for every $v\in H^1_\star(\underline{\Sigma})$. Differentiating this equation with respect to η, we find that

$$\int_{\Sigma}\Bigg\{\frac{\eta+h-1}{1-h}(\overline{w}_xv_x+\overline{w}_zv_z)-\frac{y}{1-h}\eta_x\overline{w}_xv_y-\frac{y}{1-h}\eta_xv_x\overline{w}_y-\frac{y}{1-h}\eta_z\overline{w}_zv_y$$

$$-\frac{y}{1-h}\eta_zv_z\overline{w}_y+\frac{y^2v_y\overline{w}_y}{(1-h)(\eta+h-1)}(\eta_x^2+\eta_z^2)+\frac{(1-h)v_y\overline{w}_y}{\eta+h-1}+\frac{\omega}{1-h}(u_xv_x+u_zv_z)$$

$$-\frac{y}{1-h}\omega_xv_x\overline{u}_y-\frac{y}{1-h}\omega_x\overline{u}_xv_y\quad-\frac{y}{1-h}\omega_zv_z\overline{u}_y-\frac{y}{1-h}\omega_z\overline{u}_zv_y+2\frac{y^2\overline{u}_yv_y\eta_x\omega_x}{(1-h)(\eta+h-1)}$$

$$2\frac{y^2\overline{u}_yv_y\eta_z\omega_z}{(1-h)(\eta+h-1)}-\frac{y^2\overline{u}_yv_y}{(1-h)(\eta+h-1)^2}(\eta_x^2+\eta_z^2)\omega-\frac{(1-h)\overline{u}_yv_y}{(\eta+h-1)^2}\omega\Bigg\}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$

$$=-\int_{\mathbb{R}^2}\omega_xv|_{y=1-h}\,\mathrm{d}x\,\mathrm{d}z$$

for every $v\in H^1_\star(\Sigma)$; adding this equation with $v=\overline{u}$ to (2.35) yields

$$\mathrm{d}\overline{\mathcal{L}}[\eta](\omega)$$

$$=\int_{\Sigma}\Bigg\{\frac{\omega}{2(1-h)}(u_x^2+u_z^2)-\frac{y}{1-h}\omega_x\overline{u}_x\overline{u}_y-\frac{y}{1-h}\omega_z\overline{u}_z\overline{u}_y+\frac{y^2\overline{u}_y^2\eta_x\omega_x}{(1-h)(\eta+h-1)}$$

$$+\frac{y^2\overline{u}_y^2\eta_z\omega_z}{(1-h)(\eta+h-1)}-\frac{y^2\overline{u}_y^2(\eta_x^2+\eta_z^2)\omega}{2(1-h)(\eta+h-1)}-\frac{(1-h)\omega\overline{u}_y^2}{2(\eta+h-1)^2}\Bigg\}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$

$$+\int_{\mathbb{R}^2}\omega_x\overline{u}|_{y=1-h}\,\mathrm{d}x\,\mathrm{d}z$$

$$=\int_{\Sigma}\Bigg\{\frac{1}{2(1-h)}(\overline{u}_x^2+\overline{u}_z^2)+\frac{y}{1-h}(\overline{u}_x\overline{u}_y)_x+\frac{y}{1-h}(\overline{u}_z\overline{u}_y)_z$$

$$-\left(\frac{y^2\eta_x\overline{u}_y^2}{(1-h)(\eta+h-1)}\right)_x-\left(\frac{y^2\eta_z\overline{u}_y^2}{(1-h)(\eta+h-1)}\right)_z+\frac{y^2\overline{u}_y^2(\eta_x^2+\eta_z^2)}{2(1-h)(\eta+h-1)^2}$$

$$-\frac{(1-h)\overline{u}_y^2}{2(\eta+h-1)^2}\Bigg\}\omega\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z-\int_{\mathbb{R}^2}\omega\overline{u}_x|_{y=1-h}\,\mathrm{d}x\,\mathrm{d}z.\qquad\square$$

Corollary 2.7. *The gradient $\mathcal{J}_\mu'(\eta)$ in $L^2(\mathbb{R}^2)$ exists for each $\eta \in U$ and defines a function $\mathcal{J}' \in C^\infty(H^3(\mathbb{R}^2), H^1(\mathbb{R}^2))$.*

Our final results are useful *a priori* estimates. Lemma 2.8 shows in particular that

$$\inf_{\eta\in U\setminus\{0\}} \mathcal{J}_\mu(\eta) < 2\kappa(\rho,h)\mu, \quad \kappa(\rho,h) := (1-\rho)\left(\frac{1}{h}+\frac{\rho}{1-h}\right)^{-1}.$$

LEMMA 2.8. *There exists $\eta_\mu^\star \in U\setminus\{0\}$ with compact support and a positive constant $c^\star$ such that $\|\eta_\mu^\star\|_3 \le c^\star\mu^{1/2}$ and $\mathcal{J}_\mu(\eta_\mu^\star) < 2\kappa(\rho,h)\mu - c\mu^3$.*

Proof: We follow the ideas of [BGS] (Lemma 2.29) and consider

$$\eta_\mu^\star(x,z) = \alpha^2\Psi(\alpha x, \alpha^2 z), \qquad 0<\alpha\ll 1$$

with an appropriate choice of $\Psi \in C_0^\infty([-\frac{1}{2},\frac{1}{2}]^2)$ and $\alpha=\alpha(\mu)$. We choose

$$\Psi(x,z) := \psi_x(x,z),$$

where ψ also belongs to $C_0^\infty([-\frac{1}{2},\frac{1}{2}]^2)$.
We begin by computing the leading-order terms in the asymptotic expansions of $\mathcal{K}(\eta^\star)$ and $\mathcal{L}(\eta^\star)$ in powers of α. We quote from [BGS], (60),

$$\mathcal{K}(\eta^\star) = (1-\rho)\frac{\alpha}{2}\int_{\mathbb{R}^2}\Psi^2\,\mathrm{d}x\,\mathrm{d}z + (1-\rho)\frac{\alpha^3\beta}{2}\int_{\mathbb{R}^2}\Psi_x^2\,\mathrm{d}x\,\mathrm{d}z + O(\alpha^5). \tag{2.36}$$

Furthermore we see

$$\begin{aligned}\mathcal{L}(\eta^\star) &= \mathcal{L}_2(\eta^\star)+\mathcal{L}_3(\eta^\star)+O(\|\eta^\star\|_{1,\infty}^2\|\eta^\star\|_3^2) \;=\; \mathcal{L}_2(\eta^\star)+\mathcal{L}_3(\eta^\star)+O(\alpha^5)\\ &= \underline{\mathcal{L}}_2(\eta^\star)+\overline{\mathcal{L}}_2(\eta^\star)+\underline{\mathcal{L}}_3(\eta^\star)+\overline{\mathcal{L}}_3(\eta^\star)+O(\alpha^5).\end{aligned}$$

Applying the calculations of [BGS] (Appendix B) and noting (2.33) we conclude that

$$\begin{aligned}&\underline{\mathcal{L}}_2(\eta^\star)\\ &\quad= \frac{\alpha}{2h}\int_{\mathbb{R}^2}\Psi^2\,\mathrm{d}x\,\mathrm{d}z + \frac{h\alpha^3}{6}\int_{\mathbb{R}^2}\Psi_x^2\,\mathrm{d}x\,\mathrm{d}z - \frac{\alpha^3}{2h}\int_{\mathbb{R}^2}\frac{k_1^2}{k_1^2+\alpha^2k_2^2}|\hat{\psi}_z|^2\,\mathrm{d}k_1\,\mathrm{d}k_2 + O(\alpha^5)\end{aligned}$$

as well as (have a look at Lemma 2.6)

$$\underline{\mathcal{L}}_3(\eta^\star) = -\frac{\alpha^3}{2h}\int_{\mathbb{R}^2}\Psi^3\,\mathrm{d}x\,\mathrm{d}z + O(\alpha^4).$$

On the other hand we have

$$\begin{aligned}&\overline{\mathcal{L}}_2(\eta^\star)\\ &\quad= \frac{\alpha}{2(1-h)}\int_{\mathbb{R}^2}\Psi^2\,\mathrm{d}x\,\mathrm{d}z + \frac{(1-h)\alpha^3}{6}\int_{\mathbb{R}^2}\Psi_x^2\,\mathrm{d}x\,\mathrm{d}z\end{aligned}$$

$$-\frac{\alpha^3}{2(1-h)}\int_{\mathbb{R}^2}\frac{k_1^2}{k_1^2+\alpha^2k_2^2}|\hat{\psi}_z|^2\,\mathrm{d}k_1\,\mathrm{d}k_2+O(\alpha^5)$$

and

$$\overline{\mathcal{L}}_3(\eta^\star)=\frac{\alpha^3}{2(1-h)}\int_{\mathbb{R}^2}\Psi^3\,\mathrm{d}x\,\mathrm{d}z+O(\alpha^4).$$

Combining the above results shows that

$$\begin{aligned}\mathcal{L}(\eta^\star) &= \frac{\alpha}{2}\left(\frac{1}{h}+\frac{\rho}{1-h}\right)\int_{\mathbb{R}^2}\Psi^2\,\mathrm{d}x\,\mathrm{d}z+\frac{\alpha^3}{6}(h+\rho(1-h))\int_{\mathbb{R}^2}\Psi_x^2\,\mathrm{d}x\,\mathrm{d}z\\ &- \frac{\alpha^3}{2}\left(\frac{1}{h}-\frac{\rho}{1-h}\right)\int_{\mathbb{R}^2}\Psi^3\,\mathrm{d}x\,\mathrm{d}z\\ &- \frac{\alpha^3}{2}\left(\frac{1}{h}+\frac{\rho}{1-h}\right)\int_{\mathbb{R}^2}\frac{k_1^2}{k_1^2+\alpha^2k_2^2}|\hat{\psi}_z|^2\,\mathrm{d}k_1\,\mathrm{d}k_2+O(\alpha^4).\end{aligned}\tag{2.37}$$

Let α be the solution of the equation $\mu=\mathcal{L}(\eta^\star)$ then

$$\frac{\kappa(\rho,h)\mu^2}{\mathcal{L}(\eta^\star)}-2\kappa(\rho,h)\mu=-\kappa(\rho,h)\mathcal{L}(\eta^\star)=-\left(\frac{1}{h}+\frac{\rho}{1-h}\right)^{-1}(1-\rho)\mathcal{L}(\eta^\star).$$

This means we have $\alpha=c(h,\rho)\mu/\|\Psi\|_0^2+o(\mu)$. Hence one finds abbreviating

$$\begin{aligned}C_1(\rho,h)&:=\left(\frac{1}{h}+\frac{\rho}{1-h}\right)^{-1}(h+\rho(1-h))\\ C_2(\rho,h)&:=\left(\frac{1}{h}+\frac{\rho}{1-h}\right)^{-1}\left(\frac{1}{h}-\frac{\rho}{1-h}\right)\end{aligned}$$

that

$$\begin{aligned}&\mathcal{J}(\eta_\mu^\star)-2\kappa(\rho,h)\mu\\ &\quad=\mathcal{K}(\eta_\mu^\star)-\left(\frac{1}{h}+\frac{\rho}{1-h}\right)^{-1}(1-\rho)\mathcal{L}(\eta^\star)\\ &\quad=\frac{\alpha^3(1-\rho)}{2}\int_{\mathbb{R}^2}\left((\beta-\tfrac{C_1(\rho,h)}{3})\Psi_x^2+C_2(\rho,h)\Psi^3\right)\mathrm{d}x\,\mathrm{d}z\\ &\quad\quad+\frac{\alpha^3(1-\rho)}{2}\int_{\mathbb{R}^2}\frac{k_1^2}{k_1^2+\alpha^2k_2^2}|\hat{\psi}_z|^2\,\mathrm{d}k_1\,\mathrm{d}k_2+O(\alpha^4).\end{aligned}$$

Finally, let us choose $\tilde{\psi}\in C_0^\infty([-\frac{1}{2},\frac{1}{2}]^2)$ such that

$$\int_{\mathbb{R}^2}\tilde{\psi}_x^3\,\mathrm{d}x\,\mathrm{d}z<0$$

and set $\psi=A\tilde{\psi}$; it follows that

$$\mathcal{J}_\mu(\eta_\mu^\star)-2\kappa(\rho,h)\mu$$

$$= \frac{\alpha^3(1-\rho)}{2}\left[A^2\int_{\mathbb{R}^2}\left(\beta - \tfrac{C_1(\rho,h)}{3}\right)\tilde{\psi}_{xx}^2\,\mathrm{d}x\,\mathrm{d}z + A^2\int_{\mathbb{R}^2}\frac{k_1^2}{k_1^2+\alpha^2k_2^2}|\hat{\tilde{\psi}}_z|^2\,\mathrm{d}k_1\,\mathrm{d}k_2 \right.$$
$$\left. + A^3C_2(\rho,h)\int_{\mathbb{R}^2}\tilde{\psi}_x^3\,\mathrm{d}x\,\mathrm{d}z\right] + O(\alpha^4) < 0$$

for sufficiently large values of A. □

3 Minimising sequences

3.1 The penalised minimisation problem

In this section we study the functional $\mathcal{J}_{\rho,\mu} : H^3(\mathbb{R}^2) \to \mathbb{R}\cup\{\infty\}$ defined by

$$\mathcal{J}_{\rho,\mu}(\eta) = \begin{cases} \mathcal{K}(\eta) + \dfrac{\mu^2}{\mathcal{L}(\eta)} + \rho(\|\eta\|_3^2), & u \in U\backslash\{0\}, \\ \\ \infty, & \eta \notin U\backslash\{0\}, \end{cases}$$

in which $\rho : [0, M^2) \to \mathbb{R}$ is a smooth, increasing 'penalisation' function such that $\rho(t) = 0$ for $0 \le t \le \tilde{M}^2$ and $\rho(t) \to \infty$ as $t \uparrow M^2$. The number $\tilde{M} \in (0, M)$ is chosen so that

$$\tilde{M}^2 > (c^\star + 2\kappa(\rho,h)D)\mu$$

(see below), and the following analysis is valid for every such choice of $\tilde{M}$, which in particular may be chosen arbitrarily close to M.
The following two lemma collects some properties of minimizing sequences of $\mathcal{J}_{\rho,\mu}$

LEMMA 3.1. *Every minimising sequence $\{\eta_n\}$ for $\mathcal{J}_{\rho,\mu}$ has the properties that*

$$\mathcal{J}_{\rho,\mu}(\eta_n) < 2\kappa(\rho,h)\mu, \quad \mathcal{L}(\eta_n) > \frac{\mu}{2}, \quad \mathcal{L}_2(\eta_n) \ge c\mu, \quad \mathcal{M}_\mu(\eta_n) \le -c\mu^3, \quad \|\eta_n\|_{1,\infty} \ge c\mu^3$$

for each $n \in \mathbb{N}$, where

$$\begin{aligned} \mathcal{M}_\mu(\eta) &= \mathcal{J}_{\rho,\mu}(\eta) - \mathcal{K}_2(\eta) - \frac{\kappa(\rho,h)\mu^2}{\mathcal{L}_2(\eta)}, \\ \mathcal{K}_2(\eta) &= (1-\rho)\int_{\mathbb{R}^2}\left\{\frac{\beta}{2}\eta_x^2 + \frac{\beta}{2}\eta_z^2 + \frac{\eta^2}{2}\right\}\mathrm{d}x\,\mathrm{d}z, \\ \mathcal{L}_2(\eta) &= \underline{\mathcal{L}}_2(\eta) + \rho\overline{\mathcal{L}}_2(\eta). \end{aligned}$$

The last two terms are the quadratic parts in the expansions of $\underline{\mathcal{L}}$ und $\overline{\mathcal{L}}$.

Proof. Only part four needs a comment, for the rest we refer to [BGS] (Lemma 3.2). Observe that (remember $h \in (0,1)$)

$$2\underline{\mathcal{L}}_2(\eta) = -\int_{\mathbb{R}^2}\eta\underline{u}_x^0|_{y=h}\,\mathrm{d}x\,\mathrm{d}z$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} \frac{k_1^2}{|k|} \coth|hk||\hat{\eta}|^2 \, \mathrm{d}k_1 \, \mathrm{d}k_2 \\
&\le \frac{1}{h} \int_{\mathbb{R}^2} \frac{k_1^2}{|k|^2} |\hat{\eta}|^2 \, \mathrm{d}k_1 \, \mathrm{d}k_2 + \frac{1}{3h} \int_{\mathbb{R}^2} k_1^2 |\hat{\eta}|^2 \, \mathrm{d}k_1 \, \mathrm{d}k_2 \\
&\qquad + \frac{1}{h} \int_{\mathbb{R}^2} \frac{k_1^2}{|k|^2} (|hk| \coth|hk| - 1 - \tfrac{1}{3}|hk|^2) |\hat{\eta}|^2 \, \mathrm{d}k_1 \, \mathrm{d}k_2 \\
&\le \frac{1}{h} \int_{\mathbb{R}^2} |\hat{\eta}|^2 \, \mathrm{d}k_1 \, \mathrm{d}k_2 + \frac{1}{h} \beta \int_{\mathbb{R}^2} |k|^2 |\hat{\eta}|^2 \, \mathrm{d}k_1 \, \mathrm{d}k_2 \\
&= \frac{2}{h(1-\rho)} \mathcal{K}_2(\eta),
\end{aligned}
$$

in which we have used (2.33) and estimated $\beta > 1/3$, $|k| \coth|k| - 1 - \frac{1}{3}|k|^2 \le 0$. Analogously we receive

$$
2\overline{\mathcal{L}}_2(\eta) \;\le\; \frac{2}{(1-h)(1-\rho)} \mathcal{K}_2(\eta).
$$

It follows that

$$
\mathcal{K}_2(\eta) \ge (1-\rho) \left(\frac{1}{h} + \frac{\rho}{1-h} \right)^{-1} \mathcal{L}_2(\eta) = \kappa(h,\rho) \mathcal{L}_2(\eta).
$$

Hence we obtain

$$
\mathcal{K}_2(\eta) + \frac{\kappa(\rho,\mu)\mu^2}{\mathcal{L}_2(\eta)} \;\ge\; 2\mu \sqrt{\frac{\kappa(\rho,\mu)\mathcal{K}_2(\eta)}{\mathcal{L}_2(\eta)}} \;\ge\; 2\kappa(h,\rho)\mu
$$

and

$$
M_\mu(\eta_n) \;\le\; \mathcal{J}_{\rho,\mu}(\eta_n) - 2\kappa(\rho,h)\mu \;\le\; -c\mu^3
$$

using the arguments from [BGS] (proof of Lemma 3.4). □

3.2 Application of the concentration-compactness principle

The next step is to apply the concentration-compactness principle (Lions [Li1,2]) in order to show strong convergence of a subsequence to a minimizer of the functional $\mathcal{J}_{\rho,\mu}$ which do not touch the boundary of U.

THEOREM 3.2. *Any sequence $\{u_n\} \subset L^1(\mathbb{R}^2)$ of non-negative functions with the property that*

$$
\lim_{n\to\infty} \int_{\mathbb{R}^2} u_n(x,z) \, \mathrm{d}x \, \mathrm{d}z = \ell > 0
$$

admits a subsequence for which one of the following phenomena occurs.

Vanishing: For each $R > 0$ one has that

$$
\lim_{n\to\infty} \left(\sup_{(\tilde{x},\tilde{z})\in\mathbb{R}^2} \int_{B_R(\tilde{x},\tilde{z})} u_n(x,z) \, \mathrm{d}x \, \mathrm{d}z \right) = 0.
$$

Concentration: There is a sequence $\{(x_n, z_n)\} \subset \mathbb{R}^2$ with the property that for each $\varepsilon > 0$ there exists a positive real number R with

$$\int_{B_R(0)} u_n(x + x_n, z + z_n)\,\mathrm{d}x\,\mathrm{d}z \geq \ell - \varepsilon$$

for each $n \in \mathbb{N}$.

Dichotomy: There are sequences $\{(x_n, z_n)\} \subset \mathbb{R}^2$, $\{M_n\}, \{N_n\} \subset \mathbb{R}$ and a real number $\lambda \in (0, \ell)$ with the properties that M_n, $N_n \to \infty$, $M_n/N_n \to 0$,

$$\int_{B_{M_n}(0)} u_n(x + x_n, z + z_n)\,\mathrm{d}x\,\mathrm{d}z \to \lambda,$$

$$\int_{B_{N_n}(0)} u_n(x + x_n, z + z_n)\,\mathrm{d}x\,\mathrm{d}z \to \lambda,$$

as $n \to \infty$. Furthermore

$$\lim_{n\to\infty} \left(\sup_{(\tilde{x},\tilde{z})\in\mathbb{R}^2} \int_{B_r(\tilde{x},\tilde{z})} u_n(x, z)\,\mathrm{d}x\,\mathrm{d}z \right) \leq \lambda$$

for each $r > 0$, and for each $\varepsilon > 0$ there is a positive, real number R such that

$$\int_{B_R(0)} u_n(x + x_n, z + z_n)\,\mathrm{d}x\,\mathrm{d}z \geq \lambda - \varepsilon$$

for each $n \in \mathbb{N}$.

We apply Theorem 3.2 to the sequence $\{u_n\}$ defined by

$$u_n = \eta_{nxx}^2 + 2\eta_{nxz}^2 + \eta_{nzz}^2 + 2\eta_{nx}^2 + 2\eta_{nz}^2 + \eta_n^2, \tag{3.1}$$

so that $\|u_n\|_{L^1(\mathbb{R}^2)} = \|\eta_n\|_3^2$. Quoting the arguments from [BGS] (section 3.2) on can easily deduce from Lemma 3.1 part 5

LEMMA 3.3. *The sequence $\{u_n\}$ does not have the 'vanishing' property.*

Let us now investigate the consequences of 'concentration' and 'dichotomy', replacing $\{u_n\}$ by the subsequence identified by the relevant clause in Theorem 3.2 and, with a slight abuse of notation, abbreviating the sequences $\{u_n(\cdot + x_n, \cdot + z_n)\}$ and $\{\eta_n(\cdot + x_n, \cdot + z_n)\}$ to respectively $\{u_n\}$ and $\{\eta_n\}$. The fact that $\{\|\eta_n\|_3\}$ is bounded implies that $\{\eta_n\}$ is weakly convergent in $H^3(\mathbb{R}^2)$; we denote its weak limit by $\eta^{(1)}$.
Lemma 3.4 deals with the 'concentration' case; which is proved by an argument given by Groves & Sun [GS]. We refer to [BGS] (Prop. 3.7 and Lemma 3.8) for details, note that in our situation the constants depends on ρ and h, too.

LEMMA 3.4. *Suppose that* $\{u_n\}$ *has the 'concentration' property. The sequence* $\{\eta_n\}$ *admits a subsequence which satisfies*

$$\lim_{n\to\infty} \|\eta_n\|_3 \le \tilde{M}$$

and converges in $H^r(\mathbb{R}^2)$ *for* $r \in [0,3)$ *to* $\eta^{(1)}$*. The function* $\eta^{(1)}$ *satisfies the estimate*

$$\|\eta^{(1)}\|_3^2 \ \le\ D\mathcal{K}(\eta^{(1)}) \ <\ 2D\mu,$$

minimises $\mathcal{J}_{\rho,\mu}$ *and minimises* $\mathcal{J}_\mu$ *over* $\tilde{U}\backslash\{0\}$*, where* $\tilde{U} = \{\eta \in H^3(\mathbb{R}^2) : \|\eta\|_3 < \tilde{M}\}$.

Now we have to exclude the 'dichotomy'-case. Therefore we can follow the ideas of [BGS]. A cruical tool are the pseudo-local properties of the operator $\mathcal{L}$, which means we have

LEMMA 3.5. *Consider two sequences* $\{v_m^{(1)}\}$, $\{v_m^{(2)}\}$ *with* $\sup \|v_m^{(1)} + v_m^{(1)}\|_3 < M$ *and* $\operatorname{supp} v_m^{(1)} \subset B_{2R}(0)$, $\operatorname{supp} v_m^{(2)} \subset \mathbb{R}^2 \setminus B_{S_m}(0)$*, where* $R > 0$ *and* $\{S_m\}$ *is an increasing, unbounded sequence of positive real numbers. Clearly*

$$\begin{aligned}
&\mathcal{L}(v_m^{(1)} + v_m^{(2)}) - \mathcal{L}(v_m^{(1)}) - \mathcal{L}(v_m^{(2)}) \to 0,\\
&\mathcal{L}'(v_m^{(1)} + v_m^{(2)}) - \mathcal{L}'(v_m^{(1)}) - \mathcal{L}'(v_m^{(2)}) \to 0,\\
&\langle \mathcal{L}'(v_m^{(2)}), v_m^{(1)} \rangle_0 \to 0
\end{aligned}$$

as $m \to \infty$.

For the proof we refer the reader to [BGS] (Appendix D). The arguements which are presented there have to be applied seperately to the funcitonals $\underline{\mathcal{L}}$ and $\overline{\mathcal{L}}$. Note that Lemma 3.5 clearly stays true, if we replace $\mathcal{L}$ by $\mathcal{K}$, since $\mathcal{K}$ and $\mathcal{K}'$ are local operators. This finally shows

LEMMA 3.6. *The sequence* $\{u_n\}$ *does not have the 'dichotomy' property.*

4 Strict sub-additivity

The goal of this section is to establish that the quantity

$$c_\mu = \inf_{\eta \in U\backslash\{0\}} \mathcal{J}_\mu(\eta)$$

is a *strictly sub-homogeneous* function of μ, that is

$$c_{a\mu} < a c_\mu, \qquad a > 1.$$

Its strict sub-homogeneity implies that c_μ also has the *strict sub-additivity* property that

$$c_{\mu_1+\mu_2} < c_{\mu_1} + c_{\mu_2}, \qquad \mu_1, \mu_2 > 0 \tag{4.1}$$

(see Buffoni [B]); inequality 4.1 plays a crucial role in the variational theory for the stability theory below. Applying the arguments from [BGS] (Thm. 4.1) to our problem we obtain

THEOREM 4.1. *There exists a minimising sequence* $\{\tilde{\eta}_n\}$ *for* $\mathcal{J}_\mu$ *over* $U\backslash\{0\}$ *with the properties that* $\|\tilde{\eta}_n\|_3^2 \leq c\mu$ *for each* $n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty} J_\mu(\tilde{\eta}_n) = c_{\rho,\mu}, \qquad \lim_{n\to\infty} \|J'_\mu(\tilde{\eta}_n)\|_1 = 0.$$

The strict sub-homogeneity of c_μ follows from the existence of a minimising sequence $\{\eta_n\}$ for $\mathcal{J}$ on $U\backslash\{0\}$ with the property that the function

$$a \mapsto a^{-5/2}\mathcal{M}_{a^2\mu}(a\eta_n), \qquad a \in [1,2] \tag{4.2}$$

is decreasing and strictly negative (see Lemma 4.5). Suppose that $\mathcal{L}_2(\eta) > c\mu$, $\mathcal{L}(\eta) > c\mu$ and observe that

$$\begin{aligned}
\mathcal{M}_\mu(\eta) &= \mathcal{K}_{\mathrm{nl}}(\eta) + \mu^2\left(\frac{1}{\mathcal{L}(\eta)} - \frac{1}{\mathcal{L}_2(\eta)}\right) \\
&= \mathcal{K}_4(\eta) + \mathcal{K}_6(\eta) + \ldots + \mathcal{K}_{2m_1-2}(\eta) + \mathcal{R}_{m_1}(\eta) \\
&\qquad + \mu^2\Big(\mathcal{N}_{-1}(\eta) + \mathcal{N}_0(\eta) + \mathcal{N}_1(\eta) + \ldots + \mathcal{N}_{m_2-1}(\eta) + \mathcal{S}_{m_2}(\eta)\Big),
\end{aligned} \tag{4.3}$$

where $\mathcal{N}_j(\eta)$ and $\mathcal{J}_j(\eta)$ are homogeneous of degree j and

$$\mathcal{R}_{m_1}(\eta),\ \langle\mathcal{R}'_{m_1}(\eta),\eta\rangle_0 = O(\|\eta\|_3^{2m_1}), \qquad \mathcal{S}_{m_2}(\eta),\ \langle\mathcal{S}'_{m_2}(\eta),\eta\rangle_0 = O(\|\eta\|_3^{m_2})$$

for integers $m_1 \geq 2$ and $m_2 \geq 0$.
For this purpose we construct a wighted norm on $H^3(\mathbb{R}^2)$. Due to the structure of our problem with parameters $\rho, h \in (0,1)$ it is not possible to use the approach from [BGS], hence we proceed as in [GW]. The idea mentioned there is more natural to the problem itself. Firstly we define

$$\begin{aligned}
g(k) = g(k_1,k_2) &:= (1-\rho)(1+\beta|k|^2) \\
&\quad - \kappa(h,\rho)\left(\frac{k_1^2}{|k|^2}|k|\coth|hk| + \rho\frac{k_1^2}{|k|^2}|k|\coth|(1-h)k|\right) \\
&= (1-\rho)(1+\beta|k|^2) - \kappa(h,\rho)\left(|k|\coth|hk| + \rho|k|\coth|(1-h)k|\right) \\
&\quad + \kappa(h,\rho)\left(\frac{k_2^2}{|k|^2}|k|\coth|hk| + \rho\frac{k_2^2}{|k|^2}|k|\coth|(1-h)k|\right) \\
&=: g_1(k) + g_2(k)
\end{aligned}$$

and for $\mu > 0$ and $\alpha \in (-\infty, 1)$

$$|||\eta|||_\alpha^2 := \int_{\mathbb{R}^2} (1 + \mu^{-\frac{11}{2}\alpha} g^{\frac{11}{4}}(k))|\widehat{\eta}|^2 dk$$

a norm on $H^3(\mathbb{R}^2)$. For the norm $|||\cdot|||_\alpha$ we obtain the following properties.

Proposition 4.1. *a) The function* g *behaves like* $|k|^2 + \frac{k_2^2}{|k|^2} = r^2 + \sin^2\theta$.

b) There is a constant c, independent of the value of μ, such that we have for all $\eta \in H^3(\mathbb{R}^2)$

i) $\|\eta\|_\infty^2 \leq c\mu^{3\alpha} |||\eta|||_\alpha^2$;

ii) $\|\nabla\eta\|_\infty^2 \leq c\mu^{5\alpha} |||\eta|||_\alpha^2$.

Proof: For the lower bound we have

$$\begin{aligned} g_1(k) &\geq (1-\rho)(1+\beta|k|^2) - \kappa(h,\rho)\left(\frac{1}{h}\left[1+\frac{|hk|^2}{3}\right] + \frac{\rho}{1-h}\left[1+\frac{|(1-h)k|^2}{3}\right]\right) \\ &= (1-\rho) - \kappa(h,\rho)\left(\frac{1}{h}+\frac{\rho}{1-h}\right) + \left((1-\rho)\beta - \frac{\kappa(h,\rho)}{3}\left[h+\rho(1-h)\right]\right)|k|^2 \\ &= (1-\rho)\left(\beta - \frac{1}{3}\frac{\kappa(h,\rho)}{(1-\rho)}\left[h+\rho(1-h)\right]\right)|k|^2, \end{aligned}$$

where the term in brackets is strictly positive since

$$\frac{\kappa(h,\rho)}{(1-\rho)}\left[h+\rho(1-h)\right] = \left(\frac{1}{h}+\frac{\rho}{1-h}\right)^{-1}\left[h+\rho(1-h)\right] < 1$$

and $\beta > \frac{1}{3}$. Furthermore we have

$$g_2(k) \geq \kappa(h,\rho)\left(\frac{1}{h}\frac{k_2^2}{|k|^2} + \frac{\rho}{1-h}\frac{k_2^2}{|k|^2}\right) = (1-\rho)\frac{k_2^2}{|k|^2},$$

where we used $t\coth t \geq 1$ for $t \geq 0$. Both together proves the lower bound, whereas the upper bound is a consequence of

$$g_1(k) \leq (1-\rho)(1+\beta|k|^2) - \kappa(h,\rho)\left(\frac{1}{h}+\frac{\rho}{1-h}\right) = (1-\rho)\beta|k|^2$$

and

$$\begin{aligned} g_2(k) &\leq \kappa(h,\rho)\left(\frac{1}{h}\frac{k_2^2}{|k|^2}\left[1+\frac{|hk|^2}{3}\right] + \frac{\rho}{1-h}\frac{k_2^2}{|k|^2}\left[1+\frac{|(1-h)k|^2}{3}\right]\right) \\ &\leq (1-\rho)\frac{k_2^2}{|k|^2} + \frac{\kappa(h,\rho)}{3}(h+\rho(1-h))|k|^2. \end{aligned}$$

Let $P(\nabla)$ be a Fourier-multiplier-operator. Then

$$\|P(\nabla)\eta\|_\infty^2 \leq \|P(k)\widehat{\eta}\|_{L^1}^2 \leq \left(\int_{\mathbb{R}^2} \frac{|P(k)|^2}{1+\mu^{-\frac{11}{2}\alpha}g(k)^{\frac{11}{4}}}\,dk\right)|||\eta|||_\alpha^2.$$

In the case $P(k)=1$ we can bound the term in brackets in the following way using part a)

$$\int_{\mathbb{R}^2} \frac{|P(k)|^2}{1+\mu^{-\frac{11}{2}\alpha}g(k)^{\frac{11}{4}}}\,dk \leq C' \int_{\mathbb{R}^2} \frac{1}{1+\mu^{-\frac{11}{4}\alpha}|k|^{\frac{11}{2}} + \mu^{-\frac{11}{2}\alpha}\frac{k_2^{\frac{11}{2}}}{|k|^{\frac{11}{2}}}}\,dk$$

$$
\begin{aligned}
&= 2C' \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r}{1 + \mu^{-\frac{11}{2}\alpha} r^{\frac{11}{2}} + \mu^{-\frac{11}{2}\alpha} \sin^{\frac{11}{2}} \theta} \, d\theta \, dr \\
&= C \int_0^\infty \int_{\mathbb{R}} \frac{r}{1 + \mu^{-\frac{11}{2}\alpha} r^{\frac{11}{2}} + \mu^{-\frac{11}{2}\alpha} \theta^{\frac{11}{2}}} \, d\theta \, dr \\
&\leq C\mu^{3\alpha} \int_0^\infty \int_{\mathbb{R}} \frac{s}{1 + s^{\frac{11}{2}} + \phi^{\frac{11}{2}}} \, d\phi \, ds
\end{aligned}
$$

and the claim of b) i) follows since the remaining integral is finite. If $P(k) = ik$ we obtain with the same arguments

$$
\begin{aligned}
\int_{\mathbb{R}^2} \frac{|P(k)|^2}{1 + \mu^{-\frac{11}{2}\alpha} g(k)^{\frac{11}{4}}} \, dk &\leq C' \int_0^\infty \int_{\mathbb{R}} \frac{r^3}{1 + \mu^{-\frac{11}{2}\alpha} r^{\frac{11}{2}} + \mu^{-\frac{11}{2}\alpha} \theta^{\frac{11}{2}}} \, d\theta \, dr \\
&\leq C\mu^{5\alpha}.
\end{aligned}
$$

Now we come to the main tool in this section.

THEOREM 4.2. *The minimising sequence $\{\tilde{\eta}_n\}$ for $\mathcal{J}$ over $U \backslash \{0\}$ has the property that $|||\tilde{\eta}_n|||_\alpha \leq c\mu^{1/2}$ for each $n \in \mathbb{N}$ provided we have $\alpha < \frac{2}{5}$.*

Proof: We start with the following equality

$$
\begin{aligned}
&\kappa(h,\rho)\mathcal{L}_2'(\tilde{\eta}_n) - \mathcal{K}_2'(\tilde{\eta}_n) \\
&= \mathcal{K}'(\tilde{\eta}_n) - \mathcal{K}_2'(\tilde{\eta}_n) - \mathcal{J}_\mu'(\tilde{\eta}_n) - \frac{\kappa(h,\rho)\mu^2}{\mathcal{L}(\tilde{\eta}_n)^2}\mathcal{L}'(\tilde{\eta}_n) + \kappa(h,\rho)\left\{\mathcal{L}_2'(\tilde{\eta}_n) - \mathcal{L}'(\tilde{\eta}_n)\right\} + \kappa(h,\rho)\mathcal{L}'(\tilde{\eta}_n) \\
&= \mathcal{K}_{nl}'(\tilde{\eta}_n) - \kappa(h,\rho)\mathcal{L}_{nl}'(\tilde{\eta}_n) + \kappa(h,\rho)S(\eta)\mathcal{L}'(\eta) - \mathcal{J}_\mu'(\tilde{\eta}_n) =: RHS, \qquad (4.4)
\end{aligned}
$$

where we abbreviate

$$
S(\tilde{\eta}_n) := \left\{1 - \frac{\mu^2}{\mathcal{L}(\tilde{\eta}_n)^2}\right\}.
$$

From the calculation (4.4) we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^2} g^{\frac{11}{4}}(k)|\widehat{\tilde{\eta}}_n|^2 dk = \langle g(k)^{\frac{3}{4}} g(k)\widehat{\tilde{\eta}}_n, g(k)\widehat{\tilde{\eta}}_n \rangle_0 &= \langle g(k)^{\frac{3}{4}} \widehat{RHS}, \widehat{RHS} \rangle_0 \\
&\leq c\langle |k|^{\frac{3}{2}} \widehat{RHS}, \widehat{RHS} \rangle_0 + c\langle \widehat{RHS}, \widehat{RHS} \rangle_0.
\end{aligned}
$$

By Young's inequality we obtain

$$
\begin{aligned}
\langle |k|^{\frac{3}{2}} \widehat{RHS}, \widehat{RHS} \rangle_0 &= \langle |k|\widehat{RHS}, |k|^{\frac{1}{2}}\widehat{RHS} \rangle_0 \\
&\leq \langle |k|\widehat{RHS}, |k|\widehat{RHS} \rangle_0 + \langle |k|^{\frac{1}{2}}\widehat{RHS}, |k|^{\frac{1}{2}}\widehat{RHS} \rangle_0; \\
\langle |k|^{\frac{1}{2}} \widehat{RHS}, |k|^{\frac{1}{2}}\widehat{RHS} \rangle_0 &= \langle |k|\widehat{RHS}, \widehat{RHS} \rangle_0 \\
&\leq \langle |k|\widehat{RHS}, |k|\widehat{RHS} \rangle_0 + \langle \widehat{RHS}, \widehat{RHS} \rangle_0
\end{aligned}
$$

and consequently

$$\int_{\mathbb{R}^2} g^{\frac{11}{4}}(k)|\widehat{\tilde{\eta}_n}|^2 dk \le c\|RHS\|_1^2. \tag{4.5}$$

Hence we have to estimate the r.h.s. of (4.4). A Taylor expansion for $\mathcal{K}'_{nl}$ shows that the leading term is (where O is to understand in terms of $\|\cdot\|_0^2$)

$$\begin{aligned} \mathcal{K}'_4(\tilde{\eta}_n) &= \frac{(1-\rho)\beta}{2}\left([(\tilde{\eta}_n)_x^2 + (\tilde{\eta}_n)_z^2)(\tilde{\eta}_n)_x]_x + [(\tilde{\eta}_n)_x^2 + (\tilde{\eta}_n)_z^2)(\tilde{\eta}_n)_z]_z\right) \\ &= O(\|\nabla\tilde{\eta}_n\|_\infty^4 \|\nabla^2\tilde{\eta}_n\|_0^2). \end{aligned}$$

It follows

$$|\nabla\mathcal{K}'_4(\tilde{\eta}_n)| \le c\left(|\nabla\tilde{\eta}_n|^2|\nabla^2\tilde{\eta}_n| + |\tilde{\eta}_n||\nabla\tilde{\eta}_n||\nabla^3\tilde{\eta}_n| + |\tilde{\eta}_n|^2|\nabla^3\tilde{\eta}_n|\right)$$

and by Proposition 4.1 b) ii)

$$\|\nabla\mathcal{K}'_{nl}(\tilde{\eta}_n)\|_0^2 \le c(\|\nabla\tilde{\eta}_n\|_\infty^4 + \|\tilde{\eta}_n\|_\infty^4)\|\tilde{\eta}_n\|_3^2 \le c\mu^{6\alpha}|||\tilde{\eta}_n|||_\alpha^4\|\tilde{\eta}_n\|_3^2 \le c\mu^{6\alpha+1}|||\tilde{\eta}_n|||_\alpha^4,$$

hence

$$\|\mathcal{K}'_{nl}(\tilde{\eta}_n)\|_1^2 \le c\mu^{6\alpha+1}|||\tilde{\eta}_n|||_\alpha^4. \tag{4.6}$$

In the estimation of $\mathcal{L}'_{nl}$ we just have to calculate $\mathcal{L}'_3$ using the following formulae (compare the calculations in [BGS], (62) and (63) and Lemma 2.6)

$$\begin{aligned} \underline{\mathcal{L}}'_3(\tilde{\eta}_n) &= -\frac{1}{2}(\underline{u}_x^0)^2 - \frac{1}{2}(\underline{u}_z^0)^2 + \frac{1}{2}(\underline{u}_y^0)^2\Big|_{y=h} + \underline{K}_1(\tilde{\eta}_n)\tilde{\eta}_n; \\ \overline{\mathcal{L}}'_3(\tilde{\eta}_n) &= \frac{1}{2}(\overline{u}_x^0)^2 + \frac{1}{2}(\overline{u}_z^0)^2 - \frac{1}{2}(\overline{u}_y^0)^2\Big|_{y=1-h} + \overline{K}_1(\tilde{\eta}_n)\tilde{\eta}_n. \end{aligned}$$

We have

$$\|(\underline{u}_x^0)^2\big|_{y=h}\|_0^2 \le \|\underline{u}_x^0\big|_{y=h}\|_\infty^2\|\underline{u}_x^0\big|_{y=h}\|_0^2.$$

For the first norm we get on account of Proposition 4.1 b)

$$\|\underline{u}_x^0\big|_{y=h}\|_\infty^2 \le c\left(\|\tilde{\eta}_n\|_\infty^2 + \|\nabla\tilde{\eta}_n\|_\infty^2\right) \le c\mu^{3\alpha}|||\tilde{\eta}_n|||^2.$$

Moreover we obtain

$$\begin{aligned} &\left\|\underline{u}_x^0\big|_{y=h}\right\|_0^2 = \int_{\mathbb{R}^2}\left(\mathcal{F}^{-1}\left[\frac{k_1^2}{|k|}\coth|hk|\widehat{\tilde{\eta}_n}\right]\right)^2 dx\,dz \\ &\le c\int_{\mathbb{R}^2}\left(\frac{k_1^2}{|k|^2}\widehat{\tilde{\eta}_n}\right)^2 \mathrm{d}x\,\mathrm{d}z + c\int_{\mathbb{R}^2}\left(\frac{k_1^2}{|k|^2}(|hk|\coth|hk|-1)\widehat{\tilde{\eta}_n}\right)^2 \mathrm{d}x\,\mathrm{d}z \end{aligned}$$

from which we deduce on account of $t \coth t - 1 = O(t)$

$$\|(\underline{u}_x^0)^2|_{y=h}\|_0^2 \le c\mu^{3\alpha}|||\tilde{\eta}_n|||^2 \|\tilde{\eta}_n\|_3^2 \le c\mu^{3\alpha+1}|||\tilde{\eta}_n|||^2.$$

Furthermore we have to calculate

$$\left\|\nabla(\underline{u}_x^0)^2|_{y=h}\right\|_0^2 = 4\left\|\underline{u}_x^0 \nabla \underline{u}_x^0|_{y=h}\right\|_0^2 \le 4\left\|\underline{u}_x^0|_{y=h}\right\|_\infty^2 \left\|\nabla \underline{u}_x^0|_{y=h}\right\|_0^2.$$

We observe

$$\begin{aligned}
&\left\|\nabla \underline{u}_x^0|_{y=h}\right\|_0^2 = \int_{\mathbb{R}^2} \left(\mathcal{F}^{-1}\left[k_1^2 \coth|hk|\widehat{\tilde{\eta}_n}\right]\right)^2 \, dx\, dz \\
&\le c\int_{\mathbb{R}^2}\left(\frac{k_1^2}{|k|}\widehat{\tilde{\eta}_n}\right)^2 \mathrm{d}x\,\mathrm{d}z + c\int_{\mathbb{R}^2}\left(\frac{k_1^2}{|k|}(|hk|\coth|hk|-1)\widehat{\tilde{\eta}_n}\right)^2 \mathrm{d}x\,\mathrm{d}z \\
&\le c\|\tilde{\eta}_n\|_3^2 \le c\mu.
\end{aligned}$$

Plugging all together we have shown

$$\left\|(\underline{u}_x^0)^2|_{y=h}\right\|_1^2 \le c\mu^{3\alpha+1}|||\tilde{\eta}_n|||_\alpha^2.$$

Using the ame arguments we obtain

$$\left\|(\underline{u}_z^0)^2|_{y=h}\right\|_1^2 \le c\mu^{3\alpha+1}|||\tilde{\eta}_n|||_\alpha^2.$$

Furthermore we obtain

$$\left\|\nabla(\underline{u}_y^0)^2|_{y=h}\right\|_0^2 = 4\,\|(\tilde{\eta}_n)_x \nabla(\tilde{\eta}_n)_x\|_0^2 \le c\|\nabla\tilde{\eta}_n\|_\infty^2 \|\tilde{\eta}_n\|_3^2 \le c\mu^{5\alpha+1}|||\tilde{\eta}_n|||_\alpha^2$$

using again Proposition 4.1 b) ii), as well as

$$\left\|(\underline{u}_y^0)^2|_{y=h}\right\|_0^2 = \left\|(\tilde{\eta}_n)_x^2\right\|_0^2 \le c\|\nabla\tilde{\eta}_n\|_\infty^2 \|\tilde{\eta}_n\|_3^2 \le c\mu^{5\alpha+1}|||\tilde{\eta}_n|||_\alpha^2.$$

Finally we get

$$\|\underline{K}_1(\tilde{\eta}_n)\tilde{\eta}_n\|_1^2 = \left\|\underline{u}_x^1|_{y=h}\right\|_1^2 \le c\left(\left\|\underline{F}_1^1\right\|_1^2 + \left\|\underline{F}_2^1\right\|_1^2 + \left\|\underline{F}_3^1\right\|_1^2\right).$$

This follows exactly as in [BGS], Lemma A.6. We only show how to bound $\left\|\underline{F}_3^1\right\|_1^2$, the other norms can be calculated in the same fashion:

$$\underline{F}_3^1 = -\tilde{\eta}_n \underline{u}_y^0 + (\tilde{\eta}_n)_x \underline{u}_x^0 + (\tilde{\eta}_n)_z \underline{u}_z^0,$$

hence we have to estimate

$$\left(\|\tilde{\eta}_n\|_\infty^2 + \|\nabla\tilde{\eta}_n\|_\infty^2\right)\left(\|\underline{u}_x^0\|_1^2 + \|\underline{u}_z^0\|_1^2 + \|\underline{u}_y^0\|_1^2\right).$$

If we can show

$$\|\underline{u}_y^0\|_1^2 \le \|\tilde{\eta}_n\|_3^2 \tag{4.7}$$

we receive the final estimation (not that the calculations for $\|\underline{u}_x^0\|_1^2$ and $\|\underline{u}_z^0\|_1^2$ follows similar)

$$\|\underline{K}_1(\tilde{\eta}_n)\tilde{\eta}_n\|_1^2 \le c\mu^{3\alpha+1}|||\tilde{\eta}_n|||_\alpha^2.$$

Now we have a look at (4.7): from (2.33) it follows

$$\begin{aligned}\|\underline{u}_y^0\|_0^2 &= \int_0^h \int_{\mathbb{R}^2} \left(\mathcal{F}^{-1}\left[k_1 \frac{\cosh|k|y}{\sinh|k|h}\widehat{\tilde{\eta}_n}\right]\right)^2 dx\,dz\,dy \le h\int_{\mathbb{R}^2}\left(k_1\coth(|k|h)\widehat{\tilde{\eta}_n}\right)^2 dx\,dz\\ &\le c\int_{\mathbb{R}^2}\left(\frac{k_1}{|k|}\widehat{\tilde{\eta}_n}\right)^2 \mathrm{d}x\,\mathrm{d}z + c\int_{\mathbb{R}^2}\left(\frac{k_1}{|k|}(|hk|\coth|hk|-1)\widehat{\tilde{\eta}_n}\right)^2 \mathrm{d}x\,\mathrm{d}z \le c\|\tilde{\eta}_n\|_3^2.\end{aligned}$$

Plugging all together we arrive at

$$\|\underline{\mathcal{L}}'_{nl}(\tilde{\eta}_n)\|_1^2 \le c\mu^{3\alpha+1}|||\tilde{\eta}_n|||_\alpha^2.$$

Since exactly the same arguments are applicable for estimating $\nabla\overline{\mathcal{L}}'_{nl}(\tilde{\eta}_n)$ we receive

$$\|\mathcal{L}'_{nl}(\tilde{\eta}_n)\|_1^2 \le c\mu^{3\alpha+1}|||\tilde{\eta}_n|||_\alpha^2. \tag{4.8}$$

Replacing $(\tilde{\eta}_n)$ by a subsequence if necessary, we may assume

$$\|\mathcal{J}'_\mu(\tilde{\eta}_n)\|_1^2 \le c\mu^{2N} \tag{4.9}$$

for a N such that $2N - \frac{11}{2}\alpha \ge 1$ (compare [BGS], Thm. 4.1, for details). In order to estimate $S(\tilde{\eta}_n)$ we apply the arguments from [BGS], (73) and (74), to obtain

$$\frac{\mu}{\mathcal{L}(\tilde{\eta}_n)} \le 1 + \mathcal{R}(\tilde{\eta}_n),$$

in which

$$\begin{aligned}\mathcal{R}(\eta) \;=\; &-\frac{\langle \mathcal{J}'_\mu(\eta),\eta\rangle_0}{2\kappa(h,\rho)\mu} - \frac{\mathcal{K}_{\rm nl}(\eta)}{2\kappa(h,\rho)\mu} + \frac{\langle \mathcal{K}'_{\rm nl}(\eta),\eta\rangle_0}{4\kappa(h,\rho)\mu} - \frac{\mu\langle \mathcal{L}'_{\rm nl}(\eta),\eta\rangle_0}{4\mathcal{L}(\eta)\mathcal{L}_2(\eta)}\\ &+\frac{\mu\mathcal{L}_{\rm nl}(\eta)}{\mathcal{L}(\eta)\mathcal{L}_2(\eta)} + \frac{\mu\mathcal{L}_{\rm nl}(\eta)}{2\mathcal{L}(\eta)^2} + \frac{\mu\mathcal{L}_{\rm nl}(\eta)\langle\mathcal{L}'_{\rm nl}(\eta),\eta\rangle_0}{4\mathcal{L}(\eta)^2\mathcal{L}_2(\eta)}.\end{aligned}$$

Using the arguments already mentioned, as well as $\mathcal{L}(\tilde{\eta}_n) > \frac{\mu}{2}$ and $\mathcal{L}_2(\tilde{\eta}_n) > c\mu$, we can conclude

$$\left|\frac{\langle \mathcal{J}'_\mu(\eta),\eta\rangle_0}{2\kappa(h,\rho)\mu}\right| \le c\mu^{N-\frac{1}{2}}; \quad \left|\frac{\mathcal{K}_{\rm nl}(\eta)}{2\kappa(h,\rho)\mu}\right|, \left|\frac{\langle \mathcal{K}'_{\rm nl}(\eta),\eta\rangle_0}{4\kappa(h,\rho)\mu}\right| \le c\mu^{5\alpha+1}|||\tilde{\eta}_n|||_\alpha^2$$

$$\left|\frac{\mu\langle\mathcal{L}'_{\mathrm{nl}}(\eta),\eta\rangle_0}{4\mathcal{L}(\eta)\mathcal{L}_2(\eta)}\right|,\ \left|\frac{\mu\mathcal{L}_{\mathrm{nl}}(\eta)}{\mathcal{L}(\eta)\mathcal{L}_2(\eta)}\right|,\ \left|\frac{\mu\mathcal{L}_{\mathrm{nl}}(\eta)}{2\mathcal{L}(\eta)^2}\right| \le c\mu^{\frac{3\alpha}{2}}|||\tilde{\eta}_n|||_\alpha;$$

$$\left|\frac{\mu\mathcal{L}_{\mathrm{nl}}(\eta)\langle\mathcal{L}'_{\mathrm{nl}}(\eta),\eta\rangle_0}{4\mathcal{L}(\eta)^2\mathcal{L}_2(\eta)}\right| \le c\mu^{3\alpha}|||\tilde{\eta}_n|||_\alpha^2.$$

Combining this inequalities shows

$$\|S(\tilde{\eta}_n)\mathcal{L}'(\tilde{\eta}_n)\|_0^2 \le c\mu^{3\alpha+1}|||\tilde{\eta}_n|||_\alpha^2. \tag{4.10}$$

Finally from (4.4)-(4.10) we can follow

$$|||\tilde{\eta}_n|||_\alpha^2 \le c(\mu^{1-\frac{5}{2}\alpha}|||\tilde{\eta}_n|||_\alpha^2 + \mu^{2N-\frac{11}{2}\alpha} + \mu), \tag{4.11}$$

and consequently the claim, choosing μ small enough to satisfy $\mu^{1-\frac{5}{2}\alpha} \le \frac{1}{2c}$ which is possible on account of $\alpha < \frac{2}{5}$. □

The following proposition shows how to estimate the terms in (4.3).

Proposition 4.3. *Every function $\eta \in H^3(\mathbb{R}^2)$ with $|||\eta|||_\alpha \le c\mu^{1/2}$, $\|\eta\|_3 \le c\mu^{1/2}$ and $\mathcal{L}_2(\eta) > c\mu$ satisfies the inequalities*

$$\mathcal{K}_4(\eta),\ \mathcal{K}_6(\eta) = O(\mu^{5\alpha+2})$$

and

$$\mathcal{N}_{-1}(\eta) = O(\mu^{\frac{3}{2}\alpha-\frac{1}{2}});\ \mathcal{N}_0(\eta),\ \mathcal{N}_1(\eta),\ \mathcal{N}_2(\eta) = O(\mu^{3\alpha}).$$

If $\alpha > \frac{1}{3}$ we obtain for an $\delta > 0$

$$\mathcal{K}_4(\eta),\ \mathcal{K}_6(\eta) = O(\mu^{3+\delta})$$

and

$$\mathcal{N}_{-1}(\eta) = O(\mu^{\delta});\ \mathcal{N}_0(\eta),\ \mathcal{N}_1(\eta),\ \mathcal{N}_2(\eta) = O(\mu^{1+\delta}).$$

Proof. Suppose that $|||\eta|||_\alpha \le c\mu^{1/2}$. It follows from the formulae

$$\mathcal{K}_4(\eta) = -\frac{\beta}{8}\int_{\mathbb{R}^2}(\eta_x^2+\eta_z^2)^2\,\mathrm{d}x\,\mathrm{d}z, \qquad \mathcal{K}_6(\eta) = \frac{\beta}{16}\int_{\mathbb{R}^2}(\eta_x^2+\eta_z^2)^3\,\mathrm{d}x\,\mathrm{d}z$$

and Proposition 4.1 b) ii) that

$$|\mathcal{K}_4(\eta)| \le c\|\nabla\eta\|_\infty^2\|\eta\|_3^2 \le c\mu^{5\alpha}|||\eta|||_\alpha^2\|\eta\|_3^2 \le c\mu^{5\alpha+2},$$

$$|\mathcal{K}_6(\eta)| \le c\|\nabla\eta\|_\infty^4\|\eta\|_3^2 \le c\mu^{10\alpha}|||\eta|||_\alpha^4\|\eta\|_3^2 \le c\mu^{10\alpha+3},$$

from which the first inequalities are a direct consequence.
The calculation

$$\mathcal{L}_j(\eta) \le c\|\eta\|_{1,\infty}^{j-2}\|\eta\|_3^2 \le c\mu^{\frac{3}{2}\alpha(j-2)}|||\eta|||_\alpha^{j-2}\|\eta\|_3^2 \le c\mu^{\frac{3}{2}\alpha(j-2)+\frac{1}{2}(j-2)+1}$$

shows that in particular

$$\mathcal{L}_3(\eta) = O(\mu^{\frac{3}{2}\alpha+\frac{3}{2}}), \qquad \mathcal{L}_4(\eta),\ \mathcal{L}_5(\eta),\ \mathcal{L}_6(\eta) = O(\mu^{3\alpha+2}),$$

and combining these estimates with the explicit formulae

$$\begin{aligned}
\mathcal{N}_{-2}(\eta) &= \frac{1}{\mathcal{L}_2(\eta)}, \\
\mathcal{N}_{-1}(\eta) &= -\frac{\mathcal{L}_3(\eta)}{\mathcal{L}_2(\eta)^2}, \\
\mathcal{N}_0(\eta) &= -\frac{\mathcal{L}_4(\eta)}{\mathcal{L}_2(\eta)^2} + \frac{\mathcal{L}_3(\eta)^2}{\mathcal{L}_2(\eta)^3}, \\
\mathcal{N}_1(\eta) &= -\frac{\mathcal{L}_5(\eta)}{\mathcal{L}_2(\eta)^2} + \frac{2\mathcal{L}_3(\eta)\mathcal{L}_4(\eta)}{\mathcal{L}_2(\eta)^3} - \frac{\mathcal{L}_3(\eta)^3}{\mathcal{L}_2(\eta)^4}, \\
\mathcal{N}_2(\eta) &= -\frac{\mathcal{L}_6(\eta)}{\mathcal{L}_2(\eta)^2} + \frac{\mathcal{L}_4(\eta}{\mathcal{L}_2(\eta)^3} - \frac{3\mathcal{L}_3(\eta)^2\mathcal{L}_4(\eta)}{\mathcal{L}_2(\eta)^4} + \frac{\mathcal{L}_3(\eta)^4}{\mathcal{L}_2(\eta)^5},
\end{aligned}$$

we find that

$$\mathcal{N}_0(\eta),\ \mathcal{N}_1(\eta),\ \mathcal{N}_2(\eta) = O(\mu^{3\alpha}). \qquad \square$$

Now, we find that

Proposition 4.4. *The function*

$$a \mapsto a^{-5/2}\mathcal{M}_{a^2\mu}(a\tilde{\eta}_n), \qquad a \in [1,2]$$

is decreasing and strictly negative, if $\alpha > \frac{1}{3}$.

By Proposition 4.3 above we can quote the claim from [BGS], Proposition 4.9.

The final result in this section follows directly from Proposition 4.4 (compare [BGS], Lemma 4.10)

LEMMA 4.5. *The strict sub-homogeneity property*

$$c_{a\mu} < ac_\mu$$

holds for each $a > 1$.

5 Conclusion

In this section we present the results of the paper, the existence and stability theorem. The first one is a consequece of the theory from section 3 and 4, whereas the stability theorem follows from this theorem (details are given in [BGS], section 5).
The following theorem, which is proved using the results of Sections 3 and 4, is our final result concerning the set of minimisers of $\mathcal{J}_\mu$ over $U\backslash\{0\}$.

THEOREM 5.1.

i) The set C_μ of minimisers of $\mathcal{J}_\mu$ over $U\backslash\{0\}$ is non-empty.

ii) Suppose that $\{\eta_n\}$ is a minimising sequence for $\mathcal{J}_\mu$ on $U\backslash\{0\}$ which satisfies

$$\sup_{n\in\mathbb{N}} \|\eta_n\|_3 < M. \tag{5.1}$$

There exists a sequence $\{(x_n, z_n)\} \subset \mathbb{R}^2$ with the property that a subsequence of $\{\eta_n(x_n + \cdot, z_n + \cdot)\}$ converges in $H^r(\mathbb{R}^2)$, $0 \le r < 3$ to a function $\eta \in C_\mu$.

The next step is to relate the above result to our original problem finding minimisers of $E(\eta, \Phi)$ subject to the constraint $I(\eta, \Phi) = 2\kappa(\rho, h)\mu$, where E and I are defined in equations (1.12) and (1.14).

THEOREM 5.2.

i) The set D_μ of minimisers of E on the set

$$S_\mu = \{(\eta, \Phi) \in U \times H_\star^{1/2}(\mathbb{R}^2) : I(\eta, \Phi) = 2\kappa(\rho, h)\mu\}$$

is non-empty.

ii) Suppose that $\{(\eta_n, \Phi_n)\} \subset S_\mu$ is a minimising sequence for E with the property that

$$\sup_{k\in\mathbb{N}} \|\eta_n\|_3 < M.$$

There exists a sequence $\{(x_n, z_n)\} \subset \mathbb{R}^2$ with the property that a subsequence of $\{\eta_n(x_n + \cdot, z_n + \cdot), \Phi_n(x_n + \cdot, z_n + \cdot)\}$ converges in $H^r(\mathbb{R}^2) \times H_\star^{1/2}(\mathbb{R}^2)$, $0 \le r < 3$ to a function in D_μ.

It is also possible to obtain a bound on the speed of the waves described by functions in D_μ.

LEMMA 5.3. *The fully localised solitary wave corresponding to $(\eta, \Phi) \in D_\mu$ is subcritical, that is its dimensionless speed is less than unity.*

Our stability result (Theorem 5.4 below) is obtained from Theorem 5.2 under the following assumption concerning the well-posedness of the hydrodynamic problem with small initial data.

(Well-posedness assumption) There exists a subset $\mathcal{S}$ of $U \times H_\star^{1/2}(\mathbb{R}^2)$ with the following properties.

i) The closure of $S\backslash D_\mu$ in $L^2(\mathbb{R}^2)$ has a non-empty intersection with D_μ.

ii) For each $(\eta_0, \Phi_0) \in \mathcal{S}$ there exists $T > 0$ and a continuous function $t \mapsto (\eta(t), \Phi(t)) \in U \times H_\star^{1/2}(\mathbb{R}^2)$, $t \in [0,T]$ such that $(\eta(0), \Phi(0)) = (\eta_0, \Phi_0)$,

$$E(\eta(t), \Phi(t)) = E(\eta_0, \Phi_0), \;\; I(\eta(t), \Phi(t)) = I(\eta_0, \Phi_0), \qquad t \in [0,T]$$

and

$$\sup_{t \in [0,T]} \|\eta(t)\|_3 < M.$$

THEOREM 5.4. *Choose $r \in [0,3)$. For each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$(\eta_0, \Phi_0) \in \mathcal{S}, \;\; \mathrm{dist}((\eta_0, \Phi_0), D_\mu) < \delta \quad \Rightarrow \quad \mathrm{dist}((\eta(t), \Phi(t)), D_\mu) < \varepsilon,$$

for $t \in [0,T]$, where 'dist' denotes the distance in $H^r(\mathbb{R}^2) \times H_\star^{1/2}(\mathbb{R}^2)$.

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