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Abstract

We show that Korn's inequality in Orlicz spaces holds if and only if the Orlicz function satisfies the Δ_2- and the ∇_2-condition. This result applies to several types of Korn's inequality. In particular we show that Korn's inequality is false in L^1, in $L \log L$, in Exp and in L^∞.

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1 Introduction

A crucial tool in the mathematical approach to the behavior of Newtonian fluids is Korn's inequality: Given a bounded open domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary $\partial\Omega$ we have for all $u \in \mathring{W}^{1,2}(\Omega, \mathbb{R}^d)$

$$\int_\Omega |\nabla u|^2 \, dx \leq 2 \int_\Omega |\varepsilon(u)|^2 \, dx, \tag{1}$$

where $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ denotes the symmetric part of the gradient. For smooth functions with compact support the proof of (1) follows from integration by parts and the general case is treated by approximation. We note that L^2-variants of Korn's inequality go back to the works of Courant and Hilbert [CH], Friedrichs [Fr], Èidus [Ed] and Mihlin [Mi]. Many problems in the mathematical theory of Generalized Newtonian fluids and in the mechanics of solids lead to the following question (compare for example the monographs of Málek, Nečas, Rokyta and Růžička [MNRR], of Duvaut and Lions [DL] and of Zeidler [Ze]): is it possible to bound a suitable energy depending on ∇u by the corresponding one in dependence on $\varepsilon(u)$, i.e.,

$$\int_\Omega |\nabla u|^p \, dx \leq c(p, \Omega) \int_\Omega |\varepsilon(u)|^p \, dx \tag{2}$$

for functions $u \in \mathring{W}^{1,p}(\Omega, \mathbb{R}^d)$? As shown in Gobert [Go1]-[Go2], Necǎs [Ne], Mosolov and Mjasnikov [MM], Temam [Te] and later by Fuchs [Fu1] this is true for all $1 < p < \infty$ (we remark that the inequality fails in case $p = 1$, see [Or] and [CFM]). The case of the Sobolev spaces $\mathring{W}^{1,p(\cdot)}(\Omega)$, spaces with variable exponents, is considered in [DR]. This are the natural spaces for the study of electro-rheological fluids, compare [R].
In order to characterize the specific behavior of Prandtl-Eyring fluids in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, (for example lubricants) Eyring [E] suggested the following constitutive law, which relates the stress tensor $\sigma : \Omega \to \mathbb{R}^{d\times d}$ and the symmetric gradient $\varepsilon(u)$ of the velocity field $u : \Omega \to \mathbb{R}^d$ (note that we neglect the physical constants and use a potential W which is C^2-close to the original one):

$$\sigma^D = DW(\varepsilon(u)), \quad W(\varepsilon) = |\varepsilon| \log(1 + |\varepsilon|). \tag{3}$$

Hence the natural function space for weak solutions is

$$\mathring{V}_0(\Omega) := \left\{ w \in L^1(\Omega, \mathbb{R}^d) : \int_\Omega W(\varepsilon(u))\, dx < \infty,\ \operatorname{div} w = 0,\ w|_{\partial\Omega} = 0 \right\}.$$

Note that we assume no-slip boundary conditions: $u|_{\partial\Omega} = 0$ and that the fluid is incompressible, hence u has to be divergence-free. The space $\mathring{V}_0(\Omega)$ was already used in [FuS] inspired by ideas of Frehse and Seregin [FrS]. Fuchs and Seregin [FuS] prove existence and (partial) regularity of solutions to the Prandtl-Eyring fluid system under the assumption of a slow flow (thus the convective term can be neglected leading to a variational approach not applicable in general).
The existence of weak solutions to the equations of Prandtl-Eyring fluids in the space $\mathring{V}_0(\Omega)$ in 2D is proven in [BrDF]. An open question are their regularity properties. A crucial tool in the regularity approach would be Korn's inequality in $L \log L$, i.e.,

$$\int_\Omega \varphi(|\nabla u|)\, dx \leq c \int_\Omega \varphi(|\varepsilon(u)|)\, dx, \tag{4}$$

where $\varphi(t) = t \log(1 + t)$ and $u \in \mathring{W}^{1,\varphi}(\Omega)$. We have the following intuition: if Korn's inequality in $L \log L$ fails then one main tool to proving regularity of solutions to the Prandtl-Eyring model is missing. Hence the failure of (4) for $\varphi(t) = t \log(1 + t)$ is a first step in showing irregularity of solutions. Two indications give us the suggestion about this failure:

- $L \log L$ is related to L^1, in which the failure of Korn's inequality is known since many years (compare [Or] for the first approach).

- One way to prove Korn's inequalities is the use of the representation formula
$$\nabla u = T(\varepsilon(u)),$$
where T is a singular integral operator. T is continuous from $L^p \to L^p$, $1 < p < \infty$, and in suitable Orlicz spaces (compare [CMP], [BrF, appendix]) but not from $L^1 \to L^1$ and not from $L \log L \to L \log L$.

Of course, these facts are no proof of the failure but motivate us to show it.

A first step in the generalization of (2) is mentioned in [AM]: Acerbi and Mingione prove (4) for the N-function

$$\varphi(t) = (1 + t^2)^{\frac{p-2}{2}} t^2.$$

Although they only consider a special case they provide tools for a much more general situation. Finally, the general case is proven in [DRS], [Fu1] and [BrF] with the result: (4) is true if φ is a N-function and satisfies the Δ_2- and ∇_2-condition (in the sense of [Ad]). A N-function φ satisfies the Δ_2-condition if and only if there is a constant $K > 0$ such that

$$\varphi(2t) \leq K\varphi(t) \quad \text{for all } t \geq 0. \tag{5}$$

The ∇_2-condition is the Δ_2-condition of the conjugate N-function φ^* defined by

$$\varphi^*(t) := \sup_{s \geq 0}(st - \varphi(s)).$$

Since we have for the conjugate N-function of $\varphi(t) = t \log(1 + t)$ the relation

$$\varphi^*(t) \approx t(\exp(t) - 1),$$

the ∇_2-condition fails in this case. Moreover, the results mentioned above lead to the question if the ∇_2-condition and the Δ_2-condition are sharp conditions for Korn-type inequalities in Orlicz spaces. The following Theorem gives the answer. In order to be a little bit more general we consider Φ-functions in place of N-functions.
A convex, left-continuous function $\varphi\colon [0, \infty) \to [0, \infty]$ with

$$\varphi(0) = 0, \ \lim_{t \to 0^+} \varphi(t) = 0, \ \lim_{t \to \infty} \varphi(t) = \infty$$

is called a Φ-function. The following property will be very useful in our proof: The convexity of φ and $\varphi(0) = 0$ implies

$$\varphi(\lambda u) \leq \lambda\varphi(u) \quad \text{for all } \lambda \in [0, 1] \tag{6}$$

and all $u \geq 0$. Now, let us state our main result.

THEOREM 1.1. *Let φ be a Φ-function. Then the following are equivalent:*

(a) *φ satisfies the Δ_2- and the ∇_2-condition.*

(b) *There exists a constant $A_1 > 0$ such that*

$$\|\nabla u\|_\varphi \leq A_1 \|\varepsilon(u)\|_\varphi$$

for all $u \in C_0^\infty(\mathbb{R}^d)$.

(c) *There exists a constant $A_2 > 0$ such that*

$$\int_{\mathbb{R}^d} \varphi(|\nabla u|)\,dx \leq \int_{\mathbb{R}^d} \varphi(A_2\,|\varepsilon(u)|)\,dx$$

for all $u \in C_0^\infty(\mathbb{R}^d)$.

(d) *There exists a constant $A_3 > 0$ such that*

$$\|\nabla u - \langle \nabla u\rangle_B\|_{L^\varphi(B)} \leq A_3 \|\varepsilon(u) - \langle \varepsilon(u)\rangle_B\|_{L^\varphi(B)}$$

for all $u \in W^{1,\varphi}(B)$ and all (one) balls $B \subset \mathbb{R}^d$.

(e) *There exists a constant $A_4 > 0$ such that*

$$\int_B \varphi(|\nabla u - \langle \nabla u\rangle_B|)\,dx \leq \int_B \varphi(A_4\,|\varepsilon(u) - \langle \varepsilon(u)\rangle_B|)\,dx$$

for all $u \in W^{1,\varphi}(B)$ and all (one) balls $B \subset \mathbb{R}^d$.

If (a) *is satisfied then the constants in the other parts only depend on the Δ_2- and the ∇_2-constants of φ.*

Many of the implications in Theorem 1.1 are obvious. The implications (c)⇒(b) and (e)⇒(d) follow from the fact that modular estimates are always stronger than norm estimates. The implications (e)⇒(c) and (d)⇒(b) follow from the fact that $\langle \nabla u\rangle_B = \langle \varepsilon(u)\rangle_B = 0$ for every ball that contains the support of $u \in C_0^\infty(\mathbb{R}^d)$. These considerations show that (e) is the strongest one and (b) is the weakest one among (b)–(e).
Therefore, for the proof of Theorem 1.1 it suffices to prove that (a)⇒(e) and (b)⇒(a). The implication (a)⇒(e) is non-trivial but has already been shown in [DRS, Theorem 6.13]. Let us mention that the implication (a)⇒(c) has also been shown in [Fu2].
Hence, it only remains to prove the implication (b)⇒(a). We will split this implication into to steps. Firstly, we show in Section 2, Lemma 2.1, that (b)

implies that φ satisfies the Δ_2-condition. This step is based on the construction a sequence of functions concentrating at the origin. Secondly, we show in Section 3, Lemma 3.1, that (b) and the Δ_2-condition of φ proves that φ also satisfies the ∇_2-condition. This step is based on the technique of laminates developed in [CFM] for counter examples to Korn's inequality in L^1. These two steps conclude the proof of Theorem 1.1.

Remark 1.2. *A simple scaling and translation argument shows that if* (d) *and* (e) *holds for one ball then it stays true for all balls. From the results in [DRS] if follows that we can even replace the ball B by an arbitrary bounded John domain (see [DRS] for the definition of a John domain), which includes in particular all bounded domains with Lipschitz boundary.*

Remark 1.3. *It follows in particular from Theorem 1.1 that Korn's inequality fails on L^1, $L \log L$, Exp and L^∞. Indeed, the corresponding Φ-functions are $\varphi_1(t)$, $\varphi_{L\log L}(t) = t\ln(1+t)$, $\varphi_{\mathrm{Exp}}(t) = \exp(t) - t - 1$, $\varphi_\infty(t) = \infty \cdot \chi_{(1,\infty)}(t)$. Now φ_1 and $\varphi_{L\log L}$ do not satisfy the ∇_2-condition and φ_{Exp} and φ_∞ do not satisfy the Δ_2-condition.*

Remark 1.4. *We can conclude from Theorem 1.1 that the Δ_2- and the ∇_2-condition are also necessary and sufficient for solving Poisson's equation in an Orlicz space generated by a Φ-function φ: For $f \in L^\varphi$ find a weak solution w with zero boundary data satisfying*

$$\Delta u = \operatorname{div} f$$

in the sense of distributions such that $\nabla w \in L^\varphi$. This is possible if φ fulfils the Δ_2- and the ∇_2 condition (see [JLW]). If φ does not, by Theorem 1.1, we can find a function w with $\varepsilon(w) \in L^\varphi$ and $\nabla w \notin L^\varphi$ (and $w = 0$ at the boundary). But since $\varepsilon(w) \in L^1$, w is a weak solution (which is unique) of

$$\Delta u = \operatorname{div} V(u), \quad V_{ij}(u) = 2\varepsilon^D(u)_{i,j} - \left(\frac{1}{2} - \frac{1}{d}\right)(\operatorname{div} u)\delta_{ij}, \quad i, j = 1, ..., d,$$

where $\varepsilon^D = \varepsilon - \frac{1}{d}\operatorname{tr}\varepsilon I$. On account of $|V(w)| \leq c|\varepsilon(w)|$ we have $V(w) \in L^\varphi$ and $\nabla w \notin L^\varphi$.

2 The Δ_2-condition

In this section we show that Korn's inequality in the form of (b) of Theorem 1.1 implies the Δ_2-condition. In particular, the Korn's inequality must fail on L^∞. This was firstly shown in [LM] in the case of a more general operator than $u \mapsto \varepsilon(u)$. However, let us present a simplified argument for the failure of Korn's inequality on L^∞. Indeed, define $v \in W_0^{1,1}(B_1(0))$ by

$$v(x) := Qx \ln |x|$$

for any anti-symmetric matrix $Q \in \mathbb{R}^{d\times d}$ with $Q \neq 0$. Then

$$\begin{aligned}\varepsilon(u)(x) &= \frac{1}{|x|^2} x \otimes^{\mathrm{sym}} (Qx),\\ \nabla u(x) &= \frac{1}{|x|^2} x \otimes (Qx) + Q \ln |x|.\end{aligned}$$

This shows $\|\varepsilon(u)\|_\infty \leq |Q|$ but $\nabla u \notin L^\infty$.
This counterexample suggests that the ∇_2-condition is needed for the validity of Korn's inequality. The proof is based on a slightly modified but similar function, which is additionally in C_0^∞.

Lemma 2.1. *Let φ be a Φ-function such that there exists a constant $A > 0$ such that*

$$\|\nabla u\|_\varphi \leq A \|\varepsilon(u)\|_\varphi$$

for all $u \in C_0^\infty(\mathbb{R}^d)$. Then φ satisfies the Δ_2-condition, where the Δ_2-constant only depends on A.

Proof. We will first show that there exists a constant $K > 0$ such that for all balls B and all $t \geq 0$ the following implication holds

$$|B|\varphi(t) \leq 1 \qquad \Rightarrow \qquad |B|\varphi(2t) \leq K. \tag{7}$$

We will show later by iteration that this implies the Δ_2-condition of φ.
So let B be a ball and $t \geq 0$ such that $|B|\varphi(t) \leq 1$. Let r_B denote the radius of B. Choose $\eta_k \in C_0^\infty(2^{-k}B)$ with $\chi_{2^{-k-1}B} \leq \eta_k \leq \chi_{2^{-k}B}$ and $|\nabla \eta_k| \leq 2^{k+2} r_B^{-1}$. Take $Q \in \mathbb{R}^{n\times n}$ antisymmetric with $|Q| = \frac{1}{4}t$ and fix $m \in \mathbb{N}$ with $m \geq 8A$. We consider the $C_0^\infty(B)$ function

$$u(x) := \sum_{k=1}^m \eta_k(x) Qx.$$

Then

$$\begin{aligned}\varepsilon(u)(x) &= \sum_{k=1}^m \nabla \eta_k(x) \otimes^{\mathrm{sym}} (Qx),\\ \nabla u(x) &= \sum_{k=1}^m \nabla \eta_k(x) \otimes (Qx) + Q \sum_{k=1}^m \eta_k.\end{aligned}$$

The estimate

$$|\varepsilon(u)| \le \sum_{k=1}^{m} \chi_{2^{-k}B\setminus 2^{-k-1}B} 2^{k+2} r_B^{-1} |Q| 2^{-k} r_B \le t$$

shows that $\varepsilon(u)$ is bounded and

$$\int \varphi(|\varepsilon(u)|)\, dx \le |B|\varphi(t) \le 1.$$

This implies that $\|\varepsilon(u)\|_\varphi \le 1$, so by assumption $\|\nabla u\|_\varphi \le A$, which gives $\int \varphi(|\nabla u|/A)\, dx \le 1$.
On the other hand we see that

$$|\nabla u(x)| = m|Q| \qquad \text{for all } x \in 2^{-m-1}B$$

using the fact, that $\eta_1 = \cdots = \eta_m = 1$ on $2^{-m-1}B$. Hence,

$$1 \ge \int \varphi\bigg(\frac{|\nabla u|}{A}\bigg)\, dx \ge |2^{-m-1}B|\varphi\bigg(\frac{m}{4A}t\bigg) \ge 2^{-(m+1)d}|B|\varphi(2t).$$

This proves (7) with $K := 2^{(m+1)d}$.
If $\varphi(t) = 0$ for some $t > 0$, then it follows by iteration of (7) (and the convexity of φ) that $\varphi(t) = 0$ for all $t \ge 0$, which contradicts $\lim_{t\to\infty} \varphi(t) = \infty$. Hence, we have $\varphi(t) > 0$ for all $t > 0$.
On the other hand $\lim_{t\to 0} \varphi(t) = 0$ implies that there exists a $t_0 > 0$ such that $\varphi(t_0) < \infty$. Now the repeated use of (7) shows that $\varphi(t) < \infty$ for all $t > 0$. Now the convexity of φ enforces that φ is continuous.
Let $t > 0$, then $\varphi(t) > 0$ implies that we can choose B in (7) such that $|B|\varphi(t) = 1$ and therefore

$$\varphi(2t) \le K\frac{1}{|B|} = K\varphi(t).$$

This proves that φ satisfies the Δ_2-condition. □

3 The ∇_2-condition

In this section we show that Korn's inequality implies the ∇_2-condition. Our approach is based on the technique of laminates as in [CFM]. In general a first order laminate is a probability measure ν on $\mathbb{R}^{d\times d}$ given by

$$\nu = \lambda\delta_A + (1-\lambda)\delta_B$$

where $\lambda \in (0,1)$ and $\operatorname{rank}(A-B)=1$. Here δ_F denotes the Dirac measure supported on the matrix F. We say ν has average C if $\lambda A + (1-\lambda)B = C$. We obtain a second order laminate if we replace δ_A (resp. δ_B) by a first order laminate with average A (resp. B). Iteratively we can define laminates of arbitrary order with a given average. For a detailed discussion we refer to the work of Müller and his cooperators in [KMS], [M] and [MS].

Lemma 3.1. *Let ν be a laminate with average C, then there is a sequence of uniformly Lipschitz continuous functions $u_i : (0,r)^2 \to \mathbb{R}^2$ with boundary data Cx such that*

$$\int_{(0,r)^2} \Phi(\nabla u_i)\,dx \longrightarrow r^{-2} \int_{\mathbb{R}^{2\times 2}} \Phi(F)\,d\nu(F),$$

for every continuous function $\Phi : \mathbb{R}^{2\times 2} \to \mathbb{R}$.

Proof. The result for $r=1$ is from [CFM, pg. 293, (5)]. The other case follows by scaling. □

Lemma 3.1. *Let φ be a Φ-function such that there exists a constant $A>0$ such that*

$$\|\nabla u\|_\varphi \le A\|\varepsilon(u)\|_\varphi$$

for all $u \in C_0^\infty(\mathbb{R}^d)$. Then φ satisfies the ∇_2-condition, where the ∇_2-constant only depends on A.

Proof. We already know from Lemma 2.1 that φ satisfies the Δ_2-condition and is therefore continuous, everywhere finite and positive on $(0,\infty)$. Now the space $\mathring{W}^{1,\varphi}(\Omega)$ can be defined in the usual way (see [Ad]) and if Korn's inequality holds for all $u \in C_0^\infty(\mathbb{R}^d)$ it also holds for all $u \in \mathring{W}^{1,\varphi}(\mathbb{R}^d)$, especially for all $u \in \mathring{W}^{1,\infty}((0,r)^d, \mathbb{R}^d)$. For simplification we argue in case $d=2$ but everything extends to higher dimension (see [CFM], especially Lemma 3). We define $\mu_{a,b} := \delta_{G_{a,b}}$. Now, we define a sequence $\mu^{(n)}$ of laminates of order $2n$ by

$$\begin{aligned}
\mu^{(0)} &:= \delta_{t,t},\\
\mu^{(n)} &:= \tfrac{1}{3}\delta_{2^{-n}t,-2^{-n}t} + \tfrac{2}{3}\big(\tfrac{1}{4}\delta_{-2^{1-n}t,2^{1-n}t} + \tfrac{3}{4}\mu^{(n-1)}\big)\\
&= \tfrac{1}{3}\delta_{2^{-n}t,-2^{-n}t} + \tfrac{1}{6}\delta_{-2^{1-n}t,2^{1-n}t} + \tfrac{1}{2}\mu^{(n-1)}
\end{aligned}$$

for $n \in \mathbb{N}$. Firstly we want to clarify why $\mu^{(n)}$ is a laminate for $n \ge 1$. Since we have $\operatorname{rank}(G_{-t,t} - G_{t,t}) = 1$ the term in brackets is for $n=1$ a laminate with mean value $G_{2^{-1}t,t}$. Hence $\mu^{(1)}$ is a laminate with mean value $G_{2^{-1}t,2^{-1}t}$.

It follows by mathematical induction that $\mu^{(n)}$ is a laminate with mean value $G_{2^{-n}t,2^{-n}t}$ for every $n \in \mathbb{N}$. Moreover, we get the representation

$$\mu^{(n)} = 2^{-n}\delta_{t,t} + \sum_{k=1}^{n} \left(\tfrac{1}{3}2^{k-n}\delta_{2^{-k}t,-2^{-k}t} + \tfrac{1}{6}2^{k-n}\delta_{-2^{1-k}t,2^{1-k}t}\right) \tag{8}$$

for $n \in \mathbb{N}$.
Define

$$\begin{aligned} \Phi_1(F) &:= \varphi(|F^{\mathrm{sym}} - G_{2^{-n}t,2^{-n}t}|), \\ \Phi_2(F) &:= \varphi(A^{-1}|F - G_{2^{-n}t,2^{-n}t}|). \end{aligned}$$

It follows from Lemma 3.1 for $\mu^{(n)}$ that there exists (for fixed n) a sequence $u_i : (0,r)^2 \to \mathbb{R}^2$ with boundary values $G_{2^{-n}t,2^{-n}t}x$ such that

$$\lim_{i\to\infty} \int_{(0,r)^2} \Phi_j(\nabla u_i)\,dx = r^{-2} \int_{\mathbb{R}^{2\times 2}} \Phi_j(F)\,d\nu(F), \qquad j = 1,2. \tag{9}$$

Let us define $v_i(x) := u_i(x) - G_{2^{-n}t,2^{-n}t}x$, then $v_i \in W_0^{1,\infty}((0,r)^2)$ and by (9) it follows that

$$\lim_{i\to\infty} \int_{(0,r)^2} \varphi(|\varepsilon(v_i)|)\,dx = r^{-2} \int_{\mathbb{R}^{2\times 2}} \varphi(|(F^{\mathrm{sym}} - G_{2^{-n}t,2^{-n}t})|)\,d\nu(F), \tag{10}$$

$$\lim_{i\to\infty} \int_{(0,r)^2} \varphi(A^{-1}|\nabla v_i|)\,dx = r^{-2} \int_{\mathbb{R}^{2\times 2}} \varphi(A^{-1}|F - G_{2^{-n}t,2^{-n}t}|)\,d\nu(F). \tag{11}$$

We calculate

$$\begin{aligned} &\int \varphi\big(|F^{\mathrm{sym}} - G_{2^{-n}t,2^{-n}t}|\big)\,d\mu^n(F) \\ &\qquad \le \frac{1}{2}\int \varphi\big(2|F^{\mathrm{sym}}|\big)\,d\mu^n(F) + \frac{1}{2}\int \varphi\big(2|G_{2^{-n}t,2^{-n}t}|\big)\,d\mu^n(F) \\ &\qquad = \frac{1}{2}2^{-n}\varphi(2|G_{t,t}|) + \frac{1}{2}\varphi(2|G_{2^{-n}t,2^{-n}t}|) \\ &\qquad \le 2^{-n}\varphi(2|G_{t,t}|), \end{aligned}$$

where we used the convexity of φ in first and last step (see (6)) and in the second step (8) and that the total mass of a laminate is one. So with (10) follows

$$\lim_{i\to\infty} \int_{(0,r)^2} \varphi(|\varepsilon(v_i)|)\,dx \le r^{-2}2^{-n}\varphi(2|G_{t,t}|). \tag{12}$$

For fixed $t > 0$ choose r such that

$$r^{-2} 2^{-n} \varphi(2|G_{t,t}|) = \frac{1}{2}. \tag{13}$$

So by neglecting the first elements of the sequence v_i we can assume that

$$\int_{(0,r)^2} \varphi(|\varepsilon(v_i)|)\, dx \le 1$$

for all $i \in \mathbb{N}$, which implies $\|\varepsilon(v_i)\|_\varphi \le 1$. Hence by assumption $\|\nabla v_i\|_\varphi \le A$ and therefore

$$\int_{(0,r)^2} \varphi(A^{-1}|\nabla v_i|)\, dx \le 1$$

for all $i \in \mathbb{N}$. This and (11) proves

$$r^{-2} \int_{\mathbb{R}^{2\times 2}} \varphi(A^{-1}|F - G_{2^{-n}t,2^{-n}t}|)\, d\nu(F) \le 1.$$

With the help of (8) we estimate

$$\begin{aligned}
r^2 &\ge \int \varphi(A^{-1}|F - G_{2^{-n}t,2^{-n}t}|)\, d\mu^{(n)}(F) \\
&= 2^{-n}\varphi\big(A^{-1}(1-2^{-n})|G_{t,t}|\big) + \sum_{k=1}^{n} \Big(\tfrac{1}{3} 2^{k-n} \varphi\big(A^{-1}(2^{-k} - 2^{-n})|G_{t,t}|\big)\Big) \\
&\quad + \sum_{k=1}^{n} \Big(\tfrac{1}{6} 2^{k-n} \varphi\big(A^{-1}(2^{1-k} - 2^{-n})|G_{t,t}|\big)\Big) \\
&\ge \sum_{k=1}^{n-1} \tfrac{1}{3} 2^{k-n} \varphi\big(A^{-1}(2^{-k} - 2^{-n})|G_{t,t}|\big) \\
&\ge \sum_{k=1}^{n-1} \tfrac{1}{3} 2^{k-n} \varphi\big(A^{-1} 2^{-k-1}|G_{t,t}|\big).
\end{aligned}$$

Using $\sum_{k=1}^{n} \frac{1}{3} 2^{k-n} \le 1$ and the convexity of φ, we get

$$\begin{aligned}
r^2 &\ge \varphi\bigg(A^{-1} \sum_{k=1}^{n-1} \tfrac{1}{3} 2^{k-n} 2^{-k-1} |G_{t,t}|\bigg) \ge \varphi\big(A^{-1} n 2^{-n-3} |G_{t,t}|\big) \\
&\ge n\varphi\big(A^{-1} 2^{-n-3} |G_{t,t}|\big),
\end{aligned}$$

where we used in the second step that $n \geq 4$ and (6) in the last step. So by the choice of r in (13) we get

$$n\varphi\big(A^{-1}2^{-n-3}|G_{t,t}|\big) \leq 2^{1-n}\varphi(2|G_{t,t}|).$$

for all $t > 0$. Substituting $u = 2|G_{t,t}|$ we get

$$n2^{n-2}\varphi\big(A^{-1}2^{-n-4}u\big) \leq \frac{1}{2}\varphi(u)$$

for all $u \geq 0$. Now, choosing that $n \geq 64A$ we get

$$\frac{\varphi\big(A^{-1}2^{-n-4}u\big)}{A^{-1}2^{-n-4}} \leq \frac{1}{2}\varphi(u).$$

Now, using the fact that the conjugate function of $t \mapsto \alpha\psi(\beta t)$ (for a Φ-function ψ) is $\alpha\psi(t/(\alpha\beta))$, we obtain by conjugation

$$2^{n+4}A\varphi^*(u) \geq \frac{1}{2}\varphi^*(2u).$$

In particular, φ^* satisfies the Δ_2-condition, so φ satisfies the ∇_2-condition. □

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