Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 292

Liouville Theorems For Entire Local Minimizers Of Energies Defined On The Class $L \log L$ And For Entire Solutions Of The Stationary Prandtl-Eyring Fluid Model

Martin Fuchs and Guo Zhang

Saarbrücken 2011

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 292
submitted: 14^{th} of February, 2011

Liouville Theorems For Entire Local Minimizers Of Energies Defined On The Class $L \log L$ And For Entire Solutions Of The Stationary Prandtl-Eyring Fluid Model

Martin Fuchs

Saarland University
Dep. of Mathematics
P.O. Box 15 11 50
D-66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Guo Zhang

University of Jyväskylä
Dept. of Mathematics and Statistics
P.O. Box 35 (MaD)
FI.-40014 University of Jyväskylä
Finland
guo.g.zhang@jyu.fi

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax:	+ 49 681 302 4443
e-Mail:	preprint@math.uni-sb.de
WWW:	http://www.math.uni-sb.de/

AMS Classification: 49 N, 35 J 50, 35 Q 30, 76 D 05.

Keywords: entire solutions, Liouville property, the space $L\log L$, Prandtl-Eyring fluids.

Abstract

If $u : \mathbb{R}^n \to \mathbb{R}^M$ locally minimizes the energy with density $|\nabla u| \ln(1 + |\nabla u|)$, then we show that the boundedness of the function u already implies its constancy. The same is true in case $n = M = 2$ for entire solutions of the equations modelling the stationary flow of a so-called Prandtl-Eyring fluid. Moreover, in the variational setting we will present various extensions of the above mentioned Liouville theorem for entire local minimizers valid in any dimensions n and M.

1 Introduction

In this paper we discuss theorems of Liouville-type for entire local minimizers of functionals defined on the space $L\log L$ and for stationary flows in 2D of the Prandtl-Eyring fluid. To be precise, let us abbreviate

$$h(t) := t\ln(1+t),\ t \geq 0\,. \tag{1.1}$$

We say that a field $u : \mathbb{R}^n \to \mathbb{R}^M$, $n \geq 2$, $M \geq 1$, is an entire local minimizer of the logarithmic energy

$$J[u,\Omega] := \int_\Omega h(|\nabla u|)\,dx,\ \ \nabla u := (\partial_\alpha u^i)_{\substack{1\leq\alpha\leq n,\\ 1\leq i\leq M}}\,, \tag{1.2}$$

it u belongs to the local Sobolev space $W^1_{1,\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^M)$ (see, e.g. [Ad] for a definition) and if u satisfies $J[u,\Omega] < \infty$ as well as $J[u,\Omega] \leq J[v,\Omega]$ for any $v \in W^1_{1,\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^M)$ such that $\mathrm{spt}(u-v) \Subset \Omega$, where Ω denotes an arbitrary bounded subdomain of $\mathbb{R}^n$. Energies of the form (1.2) with density given by (1.1) for fields $u : \mathbb{R}^n \to \mathbb{R}^n$ and with ∇u replaced by its symmetric part

$$\varepsilon(u) := \frac{1}{2}\left(\partial_\alpha u^i + \partial_i u^\alpha\right)_{1\leq\alpha,i\leq n}$$

occur for example in the setting of plasticity with logarithmic hardening as studied first by Frehse and Seregin [FrSe]. Later Seregin and the first author showed in [FuSe1] that local minimizers of (1.2) are of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$, if the case $n = 2$ is considered, whereas for $n = 3, 4$ partial $C^{1,\alpha}$-regularity was established. This partial regularity result was shown to hold in any dimension $n \geq 3$ as it is outlined in the paper [EM] of Esposito and Mingione. The most essential contribution however is a theorem of Mingione and Siepe [MS], which states that actually full interior $C^{1,\alpha}$-regularity is true in

Acknowledgement: Part of this paper was written during the first author's stay at the University of Jyväskylä in January 2011. He wants to thank the members of the Department of Mathematics for showing kind hospitality. The second author likes to thank his advisor Xiao Zhong for permanent help and support.

any dimension $n \geq 2$. This smoothness result gives rise to the hope that for entire local minimizers we have some "Liouville property", and actually on the basis of the work [MS] we will show:

THEOREM 1.1. *Suppose that $u : \mathbb{R}^n \to \mathbb{R}^M$ is an entire local minimizer of the functional J from (1.2) with density h defined in (1.1).*

a) Suppose that u satisfies

$$\lim_{|x|\to\infty} |x|^{-1}|u(x)| = 0\,. \tag{1.3}$$

Then u must be a constant vector. In particular, the only bounded entire local minimizers are the constant functions.

b) Let us replace (1.3) by the weaker assumption that $|x|^{-1}|u(x)|$ stays bounded as $|x| \to \infty$. Then u is an affine function.

REMARK 1.1. *Regularity results in the spirit of [MS] are nowadays available for a variety of variational integrals with density depending on the modulus of the Jacobian matrix of the vectorvalued function u. We refer to the papers of Marcellini et al [Ma1-3], [MP] and to the works [ABF] and [Fu1], where the reader will find further references. In the future we hope to give extensions of Theorem 1.1 for example to the class of densities studied in [Fu1].*

Let us now pass to the setting of Prandtl-Eyring fluids. Letting

$$H(\varepsilon) := h(|\varepsilon|) \tag{1.4}$$

for symmetric $(n \times n)$-matrices ε we consider a velocity field $u : \mathbb{R}^n \to \mathbb{R}^n$ and a pressure function $\pi : \mathbb{R}^n \to \mathbb{R}$ such that

$$\operatorname{div} u = 0 \tag{1.5}$$

and

$$-\operatorname{div}\,[DH(\varepsilon(u))] + u^k \partial_k u + \nabla\pi = 0 \tag{1.6}$$

hold (in the weak sense) on the whole space $\mathbb{R}^n$. (1.5) reflects the incompressibility condition, and in the equation of motion (1.6) the expression $u^k\partial_k u$ (summation with respect to k) is the so-called convective term. As explained in the book [FuSe2] (see also [FuSe3]) the equations (1.5) and (1.6) with H defined according to (1.4) and (1.1) model the stationary flow of a Prandtl-Eyring fluid. We have

THEOREM 1.2. *Let $n = 2$ and consider an entire (weak) solution $u : \mathbb{R}^2 \to \mathbb{R}^2$ of (1.5) and (1.6) with H from (1.1) and (1.4) being of class C^1. Then, if we assume that u is bounded, the velocity field u must be constant.*

REMARK 1.2. *The correct class for weak solutions to (1.5) and (1.6) is the space*

$$\left\{ v \in L^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2) : \operatorname{div} v = 0,\ h\left(|\varepsilon(v)|\right) \in L^1_{\mathrm{loc}}(\mathbb{R}^2) \right\} ,$$

where $\operatorname{div} v = 0$ *has to be understood in the sense of distributions and where it is required that the distributional symmetric gradient is generated by a tensor having the stated local integrability property. Equation (1.6) reads in its weak form*

$$0 = \int_\Omega DH\left(\varepsilon(u)\right) : \varepsilon(\varphi)\, dx - \int_\Omega u^k u^i \partial_k \varphi^i\, dx$$

valid for all $\varphi \in C^\infty_0(\Omega;\mathbb{R}^2)$*,* $\operatorname{div}\varphi = 0$*, and any bounded subdomain* $\Omega \subset \mathbb{R}^2$*. To our knowledge it is an open problem (even in 2D!), if in the presence of the convective term these weak solutions are actually regular. So our assumption that the weak solution studied in Theorem 1.2 is of class* C^1 *seems to be a severe extra hypothesis.*

REMARK 1.3. *We have no version of Theorem 1.2 being valid for dimensions* $n \geq 3$*.*

REMARK 1.4. *For the Prandtl-Eyring fluid we have the stress-strain relation*

$$DH(\varepsilon) = \mu(|\varepsilon|)\varepsilon$$

with strongly decreasing viscosity function

$$\mu(t) := \frac{h'(t)}{t},\ t \geq 0\,.$$

We conjecture that it is possible to replace our logarithmic density defined in (1.1) by any density with decreasing viscosity μ *(shear thinning fluids) and to prove a variant of Theorem 1.2 along similar lines.*

REMARK 1.5. *Theorems of Liouville-type for stationary flows of shear thickening fluids in 2D have been the subject of the recent paper [Fu2]. There the constancy of entire solutions is established under some asymptotic conditions. Probably the methods being presented here might enable us to get Theorem 1.2 even for shear thickening fluids.*

Our paper is organized as follows: in Section 2 we will present the proof of Theorem 1.1. Let us note that the fluid case requires completely different techniques being applicable only in 2D, which means that the arguments necessary for the verification of Theorem 1.2 are the subject of a separate Section 3.

2 Proof of Theorem 1.1

The basic ideas for obtaining the statement of Theorem 1.1 can be summarized as follows: suppose that u is an entire local J-minimizer with J defined according to (1.1) and (1.2). Suppose further that $|x|^{-1}|u(x)|$ stays bounded as $|x| \to \infty$. Based on the gradient estimates of Mingione and Siepe [MS] we show in a first step that

$|\nabla u|$ is in the space $L^\infty(\mathbb{R}^n)$. More precisely, we will combine the results of [MS] with a scaling argument and a Caccioppoli inequality to obtain the global boundedness of ∇u. In a second step we derive a differential inequality for the quantity $|\nabla u|^2$, from which $\nabla u \equiv 0$ under the assumption (1.3) will follow by applying inequalities valid for subsolutions in combination with the above mentioned Caccioppoli inequality. We refer to Remark 1.1 which suggests to extend these arguments to a wider class of densities.

Let us now pass to the details. We start with a crucial result due to Mingione and Siepe [MS], Theorem 3.1.

Lemma 2.1. *Suppose that $v \in W^1_{1,\mathrm{loc}}(B_2(0);\mathbb{R}^M)$ locally minimizes $\int_{B_2(0)} h(|\nabla v|)\,dx$ on the ball $B_2(0) := \{x \in \mathbb{R}^n : |x| < 2\}$ with h from (1.1). Then v is of class $C^{1,\alpha}$ on $B_2(0)$ and we have the gradient bound*

$$\sup_{B_{1/9}(0)} |\nabla v| \le c_1 \left[1 + \int_{B_1(0)} h\left(|\nabla v|\right)\, dx\right]^{\overline{\beta}} \tag{2.1}$$

with $c_1 = c_1(n, M)$, $\overline{\beta} = \overline{\beta}(n) > 0$ independent of v. □

From now on let us fix an entire local J-minimizer $u \in W^1_{1,\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^M)$ as explained in Section 1. Let us agree to write B_t for the (open) ball of radius t centered at the origin. For $R > 0$ we let

$$u_R(z) := \frac{1}{R} u(Rz),\ z \in \mathbb{R}^n\,, \tag{2.2}$$

and observe that for any $t > 0$ we have by the minimizing property of u

$$\int_{B_t} h\left(|\nabla u_R|\right)\, dx \le \int_{B_t} h\left(|\nabla w|\right)\, dx$$

for all w such that $w = u_R$ on ∂B_t. For this reason we can apply (2.1) to the functions u_R from (2.2) and obtain after retransformation

$$\sup_{B_{R/9}} |\nabla u| \le c_1 \left[1 + R^{-n} \int_{B_R} h\left(|\nabla u|\right)\, dx\right]^{\overline{\beta}}\,. \tag{2.3}$$

Next we claim

Lemma 2.2. *For a constant c_2 being independent of the radius r and the field u it holds*

$$\int_{B_r} h\left(|\nabla u|\right)\, dx \le c_2\, r^{-2} \int_{B_{2r}-B_r} |u|^2\, dx\,. \tag{2.4}$$

Proof of Lemma 2.2: Let $\eta \in C_0^1(B_{2r})$, $0 \le \eta \le 1$, $\eta = 1$ on B_r and $|\nabla\eta| \le c/r$. In what follows we agree to denote all constants being independent of u and the particular ball just by the same symbol c. Of course the numerical value of c may change from line to line. Letting $H(\xi) = h(|\xi|)$ for matrices $\xi \in \mathbb{R}^{nM}$, we deduce from the minimality of u

$$\begin{aligned} 0 &= \int_{B_{2r}} DH(\nabla u) : \nabla(\eta^2 u)\, dx \\ &= \int_{B_{2r}} \eta^2 DH(\nabla u) : \nabla u\, dx + 2\int_{B_{2r}} \eta DH(\nabla u) : (\nabla\eta \otimes u)\, dx\,. \end{aligned}$$

Here " : " stands for the scalar product of matrices, and " $\otimes$ " is the tensor product of vectors. Noting that

$$DH(\xi) = \frac{h'(|\xi|)}{|\xi|}\,\xi$$

and using elementary estimates for the density h, we obtain

$$\int_{B_{2r}} \eta^2 h\,(|\nabla u|)\; dx \le c\int_{B_{2r}} \eta|\nabla\eta| h'(|\nabla u|)|u|\, dx\,. \tag{2.5}$$

Let us write

$$\begin{aligned} &\int_{B_{2r}} \eta|\nabla\eta| h'(|\nabla u|)|u|\, dx \\ &\quad = \int_{B_{2r}} \eta\,(h'(|\nabla u|)|\nabla u|)^{1/2}\, |\nabla\eta||u| \left(h'(|\nabla u|)\big/|\nabla u|\right)^{1/2} dx \\ &\quad \le \delta\int_{B_{2r}} \eta^2 h'(|\nabla u|)|\nabla u|\, dx + c(\delta)\int_{B_{2r}} |\nabla\eta|^2 \frac{h'(|\nabla u|)}{|\nabla u|}|u|^2\, dx\,, \end{aligned}$$

where Young's inequality has been applied. For δ small enough we can absorb the δ-term in the left-hand side of (2.5). Observing that $\frac{h'(t)}{t} \le 2$ our claim (2.4) directly follows from the support properties of η. □

Lemma 2.3. *Suppose in addition that our entire local J-minimizer u has the property that $|x|^{-1}|u(x)|$ stays bounded as $|x| \to \infty$. Then it holds*

$$\sup_{\mathbb{R}^n} |\nabla u| < \infty\,. \tag{2.6}$$

Proof of Lemma 2.3: From (2.3) and (2.4) it follows for any $R > 0$

$$\begin{aligned} \sup_{B_{R/9}} |\nabla u| &\le c_1\left[1 + R^{-n-2}c_2\int_{B_R - B_{R/2}} |u|^2\, dx\right]^{\overline{\beta}} \\ &\le c_1\left[1 + c_3 R^{-2}\sup_{B_R - B_{R/2}} |u|^2\right]^{\overline{\beta}}\,, \end{aligned}$$

and since the constants c_i, $i = 1, 2, 3$, do not depend on R, our claim (2.6) follows. $\square$

Let us consider an entire solution u as in Lemma 2.3. Then the scaled functions u_R from (2.2) are also entire local minimizers, and from the discussion after (3.19) in [MS] we deduce

$$(2.7) \qquad |\nabla u_R(x) - \nabla u_R(y)| \le c_4(m(R))|x-y|^\alpha,\ x, y \in B_1\,,$$

where α is a positive exponent being independent of u_R and where $m(R) := \sup_{B_2} |\nabla u_R|$. But according to (2.6) the quantities $m(R)$ stay bounded independent of R, hence (2.7) implies

$$(2.8) \qquad |\nabla u(Rx) - \nabla u(Ry)| \le c_4|x-y|^\alpha,\ x, y \in B_1\,.$$

Suppose now that $\widetilde{x}, \widetilde{y} \in \mathbb{R}^n$ are given. For all $R \gg 1$ the points $x := \widetilde{x}/R$, $y := \widetilde{y}/R$ belong to B_1, hence by (2.8)

$$|\nabla u(\widetilde{x}) - \nabla u(\widetilde{y})| \le c_4 R^{-\alpha}|\widetilde{x} - \widetilde{y}| \longrightarrow 0,\ R \to \infty\,,$$

so that ∇u is a constant matrix. This proves Theorem 1.1 b). Due to its importance let us add a further comment concerning the Hölder estimate (2.7) of the gradient. For simplicity consider an entire local minimizer v such that

$$L := \sup_{\mathbb{R}^n} |\nabla v| < \infty\,.$$

Following the construction in [MS], proof of Theorem 2.1, we replace our density h by a smooth function $\widetilde{h}$ of quadratic growth such that $h = \widetilde{h}$ on $[0, 2L]$. Then v is an (entire) local minimizer of the corresponding energy and we obtain (2.7) for v with $m(R)$ replaced by L from inequality (1.15) of Tolksdorf's paper [To] with the choice $p = 2$. In fact, the paper [To] contains a much stronger local estimate for the oscillation of the gradient of a local solution.
Applying this result to the entire local minimizers u_R, for which "L" is uniformly bounded, we arrive at (2.8).

In order to continue we just observe that a) of the theorem directly follows from b): in fact, it is obvious that the only affine functions satisfying (1.3) are the constants.

For future applications we like to give a separate proof of Theorem 1.1 a), which does not make use of estimates for the oscillation of the gradients of entire local minimizers, but only exploits the information (2.6) in combination with assumption (1.3). So let u as described in Lemma 2.3 and observe that from (2.6) it easily follows

$$(2.9) \qquad u \in W^2_{2,\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^M),\ |\nabla u|^2 \in W^1_{2,\mathrm{loc}}(\mathbb{R}^n)\,.$$

In fact, the second statement of (2.9) is a direct consequence of the first one in combination with (2.6), and we will add some comments on the first part of (2.9) at the end of this section. Using (2.9) and the minimality of u it is easy to show that

$$0 = \int_{\mathbb{R}^n} D^2H(\nabla u)(\partial_\alpha \nabla u, \nabla\varphi)\, dx \tag{2.10}$$

holds for $\alpha = 1, \ldots, n$ and $\varphi \in C_0^1(\mathbb{R}^n; \mathbb{R}^M)$. We select $\varphi = \eta^2 \partial_\alpha u$, $\eta \in C_0^1(\mathbb{R}^n)$, and obtain from (2.10)

$$\int_{\mathbb{R}^n} D^2H(\nabla u)\left(\partial_\alpha \nabla u, \nabla\eta^2 \otimes \partial_\alpha u\right)\, dx \le 0\,, \tag{2.11}$$

where the convention of summation with respect to indices repeated twice is used again. We have the formula

$$D^2H(X)(Y,Z) = \frac{h'(|X|)}{|X|}\left[Y : Z - \frac{(X:Y)(X:Z)}{|X|^2}\right] + h''(|X|)\frac{(X:Y)(X:Z)}{|X|^2}$$

together with the ellipticity estimate

$$\frac{|Y|^2}{1+|X|} \le D^2H(X)(Y,Y) \le 2\,\frac{\ln(1+|X|)}{|X|}\,|Y|^2 \tag{2.12}$$

valid for all matrices X, Y, $Z \in \mathbb{R}^{nM}$. If we introduce

$$a_{\alpha\beta} := \frac{1}{2}\,\delta_{\alpha\beta}\frac{h'(|\nabla u|)}{|\nabla u|} + \frac{1}{2}\left[h''(|\nabla u|) - \frac{h'(|\nabla u|)}{|\nabla u|}\,\frac{\partial_\alpha u\cdot\partial_\beta u}{|\nabla u|^2}\right]\,,$$

then by (2.12) and (2.6) these functions are bounded generating a uniformly elliptic matrix, moreover, inequality (2.11) implies that $w := |\nabla u|^2 \in W^1_{2,\mathrm{loc}}(\mathbb{R}^n)$ (recall (2.9)) satisfies

$$\int_{\mathbb{R}^n} a_{\alpha\beta}\partial_\alpha w \partial_\beta \eta^2\, dx \le 0\,. \tag{2.13}$$

Let us select some exponent $p > 1$. From Theorem 8.17 in [GT] applied to (2.13) we deduce the existence of a constant c_5 not depending on R and u such that

$$\sup_{B_R} w \le c_5\left[R^{-n}\int_{B_{2R}} w^p\, dx\right]^{1/p}\,, \tag{2.14}$$

and inequality (2.14) is valid for any radius $R > 0$ (even with the choice $p = 1$ as suggested in Theorem 1.1, Chapter 4, of [HL]). We have

$$|\nabla u|^{2p} \le c(p)h(|\nabla u|)$$

on the set $[|\nabla u| \le 1]$, whereas on $[|\nabla u| \ge 1]$ it holds

$$|\nabla u|^{2p} \le \frac{1}{\ln 2}\left(\sup_{\mathbb{R}^n} |\nabla u|\right)^{2p-1} h(|\nabla u|)\,.$$

Therefore (2.14) implies

$$\sup_{B_R} |\nabla u|^2 \le c\left[R^{-n}\int_{B_{2R}} h\left(|\nabla u|\right)\,dx\right]^{1/p} \tag{2.15}$$

for any $R > 0$ with c independent of R. On the right-hand side of (2.15) we apply (2.4) and get

$$\sup_{B_R} |\nabla u|^2 \le c\left[R^{-n-2}\int_{B_{4R}-B_{2R}} |u|^2\,dx\right]^{1/p} \le c\left\{\sup_{|x|\ge R} |x|^{-2}|u(x)|^2\right\}^{1/p}.$$

Now, if we impose condition (1.3) on the growth of u, we immediately end up with $\nabla u = 0$, which completes the separate proof of Theorem 1.1 a). □

Finally we briefly comment on (2.9) keeping in mind that we have (2.6). Using the boundedness of ∇u it is easy to see by testing the system of Euler equations valid for u with $\Delta_{-h}(\eta^2\Delta_h u)$, where η is a cut-off function and where $\Delta_{\pm h}u$ denote difference quotients of u (in a fixed direction), that $\Delta_h(\nabla u)$ is uniformly bounded in the space $L^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^{nM})$, which implies (2.9).
Alternatively we can argue via regularisation as done for example in [FuSe1]. Completely using the notation from [FuSe1] we quote inequality (3.13) from this paper, i.e.

$$\int_{\omega_1} \frac{1}{1+|\nabla u_m|}|\nabla^2 u_m|^2\,dx \le c\|\nabla\eta\|^2_{L^\infty(\omega_2)} J_m(u_m,\omega_2)\,, \tag{2.16}$$

where $J_m(u_m,\omega_2) \to J(u,\omega_2)$ as $m \to \infty$ according to (3.11) of [FuSe1]. We also observe (see [FuSe1], end of the proof of Lemma 3.1) that $\nabla u_m \to \nabla u$ in L^r for any $r < \frac{n}{n-2}$. This yields ($p > 1$)

$$\begin{aligned}
\int_{\omega_1} |\nabla^2 u_m|^p\,dx &= \int_{\omega_1} (1+|\nabla u_m|)^{-p/2}\,|\nabla^2 u_m|^p\,(1+|\nabla u_m|)^{p/2}\,dx\\
&\le c\left[\int_{\omega_1} \frac{1}{1+|\nabla u_m|}|\nabla^2 u_m|^2\,dx + \int_{\omega_1} (1+|\nabla u_m|)^{\frac{p}{2-p}}\,dx\right]\\
&\le c\left[\int_{\omega_1} \frac{1}{1+|\nabla u_m|}|\nabla^2 u_m|^2\,dx + 1\right],
\end{aligned}$$

provided p is sufficiently close to 1. Now (2.16) implies

$$\sup_m \int_{\omega_1} |\nabla^2 u_m|^p\,dx < \infty\,,$$

so that at least $u \in W^2_p(\omega_1;\mathbb{R}^M)$ together with $\nabla^2 u_m \rightharpoondown \nabla^2 u$ weakly in L^p. On the left-hand-side of (2.16) we can apply De Giorgi's theorem on lower-semicontinuity, hence we obtain (2.16) "without index m". But then (2.6) immediately shows the validity of (2.9).
□

3 Proof of Theorem 1.2

We start with a collection of auxiliary results. The first lemma is a slight extension of a contribution due to Giaquinta and Modica (compare Lemma 0.5 in [GM]). In this lemma and also during this section we abbreviate

$$Q_R(z) := \left\{ x \in \mathbb{R}^2 : |x_i - z_i| < R,\ i = 1, 2 \right\},\ z \in \mathbb{R}^2, R > 0\,.$$

Lemma 3.1. *Let $f,\ f_1, \dots, f_\ell$ denote non-negative functions from the space $L^1_{\mathrm{loc}}(\mathbb{R}^2)$. Suppose further that we are given exponents $\alpha_1, \dots, \alpha_\ell > 0$. Then we can find a number $\delta_0 > 0$ depending on $\alpha_1, \dots, \alpha_\ell$ as follows: if for $\delta \in (0, \delta_0)$ it is possible to calculate a constant $c(\delta) > 0$ such that the inequality*

$$\int_{Q_R(z)} f\,dx \le \delta \int_{Q_{2R}(z)} f\,dx + c(\delta) \sum_{j=1}^{\ell} R^{-\alpha_j} \int_{Q_{2R}(z)} f_j\,dx$$

holds for any choice of $Q_R(z) \subset \mathbb{R}^2$, then there is a constant c with the property

$$\int_{Q_R(z)} f\,dx \le c \sum_{j=1}^{\ell} R^{-\alpha_j} \int_{Q_{2R}(z)} f_j\,dx$$

for all squares $Q_R(z)$.

REMARK 3.1. *Of course Lemma 3.1 extends to $\mathbb{R}^n$, $n \ge 3$, replacing squares by cubes.*

Proof of Lemma 3.1: see Appendix. □

Next we recall a standard result concerning the "divergence equation", see e.g. [Ga] or [La].

Lemma 3.2. *Consider a function $f \in L^2(Q_R(z))$ such that $\int_{Q_R(z)} f\,dx = 0$. Then there exists a field $v \in \overset{\circ}{W}{}^1_2(Q_R(z); \mathbb{R}^2)$ and a constant C independent of $Q_R(z)$ such that we have* $\operatorname{div} v = f$ *on $Q_R(z)$ together with the estimate*

$$\int_{Q_R(z)} |\nabla v|^2\,dx \le C \int_{Q_R(z)} f^2\,dx\,.$$

We further will make use of the classical L^2-variant of Korn's inequality.

Lemma 3.3. *There is a constant C independent of $Q_R(z)$ such that for all $v \in \overset{\circ}{W}{}^1_2(Q_R(z); \mathbb{R}^2)$ it holds*

$$\int_{Q_R(z)} |\nabla v|^2\,dx \le C \int_{Q_R(z)} |\varepsilon(v)|^2\,dx\,.$$

REMARK 3.2. *Clearly Lemma 3.2 and 3.3 hold in* $\mathbb{R}^n$, $n \geq 3$. *Moreover we have the same statements, if we replace* $Q_R(z)$ *by* $Q_{2R}(z) - \overline{Q}_R(z)$.

From now on we assume that the velocity field $u : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the hypothesis of Theorem 1.2. Then, using the information $u \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ we may argue as done at the end of Section 2 - now applying an inequality like the one stated after (3.20) in [FuSe3] - to deduce that u is in $W^2_{2,\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$. Note that due to the C^1- regularity of u the convective term causes no difficulties. As a matter of fact we may also apply difference quotients. Recalling the definitions (1.1) and (1.4) we first claim the existence of a constant $c = c(\|u\|_{L^\infty(\mathbb{R}^2)})$ such that for all $Q_R(x_0)$

$$\int_{Q_R(x_0)} H\left(\varepsilon(u)\right)\, dx \leq c(R+1)\,. \tag{3.1}$$

For proving (3.1) we let $\eta \in C^1_0(Q_{2R}(x_0))$ such that $\eta = 1$ on $Q_R(x_0)$, $0 \leq \eta \leq 1$ and $|\nabla\eta| \leq c/R$. The weak form of equation (1.6) states that

$$0 = \int_{Q_{2R}(x_0)} DH\left(\varepsilon(u)\right) : \varepsilon(\varphi)\, dx + \int_{Q_{2R}(x_0)} u^k \partial_k u^i \varphi^i\, dx \tag{3.2}$$

for all φ such that $\operatorname{div}\varphi = 0$ on $Q_{2R}(x_0)$ and $\varphi = 0$ on $\partial Q_{2R}(x_0)$. We choose $\varphi := \eta^2 u - w$ with w defined according to Lemma 3.2 for the choice $f = \operatorname{div}(\eta^2 u)$ and with $Q_R(z)$ replaced by $Q_{2R}(x_0)$. From (3.2) we obtain

$$\begin{aligned}
&\int_{Q_{2R}(x_0)} DH\left(\varepsilon(u)\right) : \varepsilon(u)\eta^2\, dx \\
&\quad + \int_{Q_{2R}(x_0)} 2\frac{\partial H}{\partial \varepsilon_{i\alpha}}\left(\varepsilon(u)\right) \partial_\alpha \eta u^i \eta\, dx \\
&\quad - \int_{Q_{2R}(x_0)} DH\left(\varepsilon(u)\right) : \varepsilon(w)\, dx + \int_{Q_{2R}(x_0)} u^k \partial_k u^i u^i \eta^2\, dx \\
&\quad - \int_{Q_{2R}(x_0)} u^k \partial_k u^i w^i\, dx = 0\,.
\end{aligned}$$

We have

$$\int_{Q_{2R}(x_0)} DH\left(\varepsilon(u)\right) : \varepsilon(u)\eta^2\, dx \geq \int_{Q_{2R}(x_0)} \eta^2 H\left(\varepsilon(u)\right)\, dx\,,$$

$$\begin{aligned}
&\left| 2\int_{Q_{2R}(x_0)} \frac{\partial H}{\partial \varepsilon_{i\alpha}}\left(\varepsilon(u)\right) \partial_\alpha \eta u^i \eta\, dx \right| \\
&\quad \leq\ c \int_{Q_{2R}(x_0)} h'\left(|\varepsilon(u)|\right) |\nabla\eta||u|\eta\, dx \\
&\quad \leq\ \delta \int_{Q_{2R}(x_0)} h'\left(|\varepsilon(u)|\right) |\varepsilon(u)|\eta^2\, dx + c(\delta) \int_{Q_{2R}(x_0)} \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} |u|^2 |\nabla\eta|^2\, dx\,,
\end{aligned}$$

and if δ is chosen small enough and if we take into account the inequality $\frac{h'(t)}{t} \leq 2$ it follows

$$\int_{Q_{2R}(x_0)} \eta^2 H(\varepsilon(u))\, dx \leq c \left[\int_{Q_{2R}(x_0)} |u|^2 |\nabla \eta|^2\, dx \right. \tag{3.3}$$
$$+ \left| \int_{Q_{2R}(x_0)} DH(\varepsilon(u)) : \varepsilon(w)\, dx \right|$$
$$\left. + \left| \int_{Q_{2R}(x_0)} u^k \partial_k u^i u^i \eta^2\, dx \right| + \left| \int_{Q_{2R}(x_0)} u^k \partial_k u^i w^i\, dx \right| \right]$$
$$=: \quad c\left[T_1 + T_2 + T_3 + T_4\right].$$

The quantities T_i are estimated as follows: it clearly holds

$$T_1 \leq cR^{-2} \int_{Q_{2R}(x_0)} |u|^2\, dx\,. \tag{3.4}$$

We have by Young's inequality for any $\delta > 0$

$$\begin{aligned} T_2 &\leq \delta \int_{Q_{2R}(x_0)} h'(|\varepsilon(u)|)^2\, dx + c(\delta) \int_{Q_{2R}(x_0)} |\nabla w|^2\, dx \\ &\leq \delta \int_{Q_{2R}(x_0)} h'(|\varepsilon(u)|)^2\, dx + c(\delta) \int_{Q_{2R}(x_0)} |\operatorname{div}(\eta^2 u)|^2\, dx \\ &\leq \delta \int_{Q_{2R}(x_0)} h'(|\varepsilon(u)|)^2\, dx + c(\delta) R^{-2} \int_{Q_{2R}(x_0)} |u|^2\, dx\,, \end{aligned}$$

where the estimate from Lemma 3.2 has been applied. Now it is easy to see the validity of

$$h'(t) \leq 2\ln(1+t),\ t \geq 0\,,$$

and since $\ln(1+t)^2 \leq t\ln(1+t)$ is true for all $t \geq 0$, we find $h'(t)^2 \leq 4h(t)$, hence (replacing δ by $\delta/4$)

$$T_2 \leq \delta \int_{Q_{2R}(x_0)} H(\varepsilon(u))\, dx + c(\delta) R^{-2} \int_{Q_{2R}(x_0)} |u|^2\, dx\,. \tag{3.5}$$

Next we observe

$$\begin{aligned} T_3 &= \frac{1}{2} \left| \int_{Q_{2R}(x_0)} u^k \partial_k |u|^2 \eta^2\, dx \right| \\ &= \frac{1}{2} \left| \int_{Q_{2R}(x_0)} u^k |u|^2 \partial_k \eta^2\, dx \right| \leq cR^{-1} \int_{Q_{2R}(x_0)} |u|^3\, dx\,, \end{aligned} \tag{3.6}$$

and from

$$\int_{Q_{2R}(x_0)} u^k \partial_k u^i w^i\, dx = - \int_{Q_{2R}(x_0)} u^k u^i \partial_k w^i\, dx$$

it follows using Lemma 3.2 and Hölder's inequality

$$
\begin{aligned}
(3.7)\qquad T_4 &\le \left(\int_{Q_{2R}(x_0)} |u|^4\,dx\right)^{1/2}\left(\int_{Q_{2R}(x_0)} |\nabla w|^2\,dx\right)^{1/2}\\
&\le c\left(\int_{Q_{2R}(x_0)} |u|^4\,dx\right)^{1/2}\left(R^{-2}\int_{Q_{2R}(x_0)} |u|^2\,dx\right)^{1/2}\\
&= cR^{-1}\left[\int_{Q_{2R}(x_0)} |u|^4\,dx\int_{Q_{2R}(x_0)} |u|^2\,dx\right]^{1/2}\\
&\le cR^{-1}\left[\int_{Q_{2R}(x_0)} |u|^4\,dx+\int_{Q_{2R}(x_0)} |u|^2\,dx\right].
\end{aligned}
$$

Inserting (3.4) - (3.7) into (3.3) we get

$$
\begin{aligned}
\int_{Q_R(x_0)} H\left(\varepsilon(u)\right)\,dx \le \delta\int_{Q_{2R}(x_0)} H\left(\varepsilon(u)\right)\,dx\\
+c(\delta)\left[R^{-1}\int_{Q_{2R}(x_0)}\left(|u|^2+|u|^3+|u|^4\right)\,dx+R^{-2}\int_{Q_{2R}(x_0)}|u|^2\,dx\right]
\end{aligned}
$$

for any $\delta > 0$ and all $Q_R(x_0)$. Lemma 3.1 then yields

$$
\int_{Q_R(x_0)} H\left(\varepsilon(u)\right)\,dx \le c\left[R^{-1}\int_{Q_{2R}(x_0)}\left(|u|^2+|u|^3+|u|^4\right)\,dx+R^{-2}\int_{Q_{2R}(x_0)}|u|^2\,dx\right],
$$

and since u is bounded we have established (3.1).

Next we like to prove the validity of

$$
(3.8)\qquad \int_{\mathbb{R}^2} D^2H\left(\varepsilon(u)\right)\left(\partial_k\varepsilon(u),\partial_k\varepsilon(u)\right)dx < \infty\,.
$$

Note that from (3.8) we immediately get

$$
(3.9)\qquad \int_{\mathbb{R}^2}\frac{1}{1+|\varepsilon(u)|}|\nabla\varepsilon(u)|^2\,dx < \infty\,.
$$

For the discussion of (3.8) we return to equation (3.2). Replacing φ by $\partial_\alpha\varphi$ for $\varphi \in C_0^\infty(Q_{\frac{3}{2}R}(x_0);\mathbb{R}^2)$ with $\operatorname{div}\varphi = 0$ we obtain by partial integration

$$
\begin{aligned}
(3.10)\qquad 0 &= \int_{Q_{\frac{3}{2}R}(x_0)} D^2H\left(\varepsilon(u)\right)\left(\partial_\alpha\varepsilon(u),\varepsilon(\varphi)\right)\,dx\\
&\quad-\int_{Q_{\frac{3}{2}R}(x_0)} u^k\partial_k u^i\partial_\alpha\varphi^i\,dx,\ \ \alpha = 1,2\,.
\end{aligned}
$$

Let $\eta \in C_0^\infty(Q_{\frac{3}{2}R}(x_0))$ such that $\eta = 1$ on $Q_R(x_0)$, $0 \leq \eta \leq 1$ and $|\nabla\eta| \leq c/R$. Let $f_\alpha := \operatorname{div}(\partial_\alpha u \eta^2) = \partial_\alpha u \cdot \nabla\eta^2$ and select w_α according to Lemma 3.2 from the space $\overset{\circ}{W}{}^1_2\left(Q_{\frac{3}{2}R}(x_0);\mathbb{R}^2\right)$ such that

$$\begin{aligned}
&\operatorname{div} w_\alpha = f_\alpha \text{ on } Q_{\frac{3}{2}R}(x_0)\,,\\
&\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla w_\alpha|^2\,dx \leq c\int_{Q_{\frac{3}{2}R}(x_0)} |\partial_\alpha u \cdot \nabla\eta|^2\,dx\,.
\end{aligned} \tag{3.11}$$

Finally we choose $\varphi := \eta^2\partial_\alpha u - w_\alpha$ in (3.10) and agree from now on to take the sum also with respect to α. Equation (3.10) then yields

$$\begin{aligned}
&\int_{Q_{\frac{3}{2}R}(x_0)} D^2H\left(\varepsilon(u)\right)\left(\partial_\alpha\varepsilon(u),\partial_\alpha\varepsilon(u)\right)\eta^2\,dx\\
&= -\int_{Q_{\frac{3}{2}R}(x_0)} D^2H\left(\varepsilon(u)\right)\left(\partial_\alpha\varepsilon(u),\nabla\eta^2\odot\partial_\alpha u\right)\,dx\\
&\quad +\int_{Q_{\frac{3}{2}R}(x_0)} D^2H\left(\varepsilon(u)\right)\left(\partial_\alpha\varepsilon(u),\varepsilon(w_\alpha)\right)\,dx\\
&\quad +\int_{Q_{\frac{3}{2}R}(x_0)} u^k\partial_k u^i\partial_\alpha\left(\eta^2\partial_\alpha u^i\right)\,dx - \int_{Q_{\frac{3}{2}R}(x_0)} u^k\partial_k u^i\partial_\alpha w_\alpha^i\,dx\\
&=: -S_1 + S_2 + S_3 - S_4\,,
\end{aligned} \tag{3.12}$$

where "$\odot$" is the symmetric product of vectors. Using the Cauchy–Schwarz inequality for the bilinear form $D^2H(\varepsilon(u))$ in combination with Young's inequality we obtain for any $\delta > 0$

$$\begin{aligned}
|S_2| &\leq \delta\int_{Q_{\frac{3}{2}R}(x_0)} D^2H\left(\varepsilon(u)\right)\left(\partial_\alpha\varepsilon(u),\partial_\alpha\varepsilon(u)\right)\,dx\\
&\quad +\frac{1}{\delta}\int_{Q_{\frac{3}{2}R}(x_0)} D^2H\left(\varepsilon(u)\right)\left(\varepsilon(w_\alpha),\varepsilon(w_\alpha)\right)\,dx\\
&\leq \delta\int_{Q_{\frac{3}{2}R}(x_0)} D^2H\left(\varepsilon(u)\right)\left(\partial_\alpha\varepsilon(u),\partial_\alpha\varepsilon(u)\right)\,dx\\
&\quad +\frac{2}{\delta}\int_{Q_{\frac{3}{2}R}(x_0)} \nabla w_\alpha : \nabla w_\alpha\,dx\\
&\overset{(3.11)}{\leq} \delta\int_{Q_{\frac{3}{2}R}(x_0)} D^2H\left(\varepsilon(u)\right)\left(\partial_\alpha\varepsilon(u),\partial_\alpha\varepsilon(u)\right)\,dx + c(\delta)\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2|\nabla\eta|^2\,dx\,,
\end{aligned}$$

hence

$$|S_2| \leq \delta\int_{Q_{\frac{3}{2}R}(x_0)} \omega\,dx + c(\delta)\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2|\nabla\eta|^2\,dx\,. \tag{3.13}$$

Here we have abbreviated

$$\omega := D^2H\left(\varepsilon(u)\right)\left(\partial_\alpha\varepsilon(u), \partial_\alpha\varepsilon(u)\right) . \tag{3.14}$$

By applying exactly the same arguments to S_1 we see

$$|S_1| \le \delta \int_{Q_{\frac{3}{2}R}(x_0)} \omega \, dx + c(\delta) \int_{Q_{\frac{3}{2}R}(x_0)} |\nabla\eta|^2 |\nabla u|^2 \, dx \, , \tag{3.15}$$

and (3.15) is valid for any choice of $\delta > 0$.

Next we look at S_3 : it holds

$$\begin{aligned} S_3 &= \int_{Q_{\frac{3}{2}R}(x_0)} u^k \partial_k u^i \partial_\alpha \left(\eta^2 \partial_\alpha u^i\right) \, dx = -\int_{Q_{\frac{3}{2}R}(x_0)} \partial_\alpha (u^k \partial_k u^i) \eta^2 \partial_\alpha u^i \, dx \\ &= -\int_{Q_{\frac{3}{2}R}(x_0)} \partial_\alpha u^k \partial_k u^i \partial_\alpha u^i \eta^2 \, dx - \int_{Q_{\frac{3}{2}R}(x_0)} u^k \partial_\alpha \partial_k u^i \partial_\alpha u^i \eta^2 \, dx \, , \end{aligned}$$

and since we are in the 2D-case, the first integral on the right-hand side vanishes. This shows

$$|S_3| = \frac{1}{2} \left| \int_{Q_{\frac{3}{2}R}(x_0)} u^k \partial_k |\nabla u|^2 \eta^2 \, dx \right| = \frac{1}{2} \left| \int_{Q_{\frac{3}{2}R}(x_0)} u \cdot \nabla\eta^2 |\nabla u|^2 \, dx \right| \, ,$$

and we obtain

$$|S_3| \le cR^{-1} \int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx \tag{3.16}$$

for a constant c depending on $\|u\|_{L^\infty(\mathbb{R}^2)}$. Finally we discuss S_4 again using the boundedness of the velocity field:

$$\begin{aligned} |S_4| &\le c \int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u| |\partial_\alpha w_\alpha| \, dx \\ &\overset{(3.11)}{\le} c \left(\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx \right)^{1/2} \left(\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla\eta|^2 |\nabla u|^2 \, dx \right)^{1/2} \, , \end{aligned}$$

thus

$$|S_4| \le cR^{-1} \int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx \, . \tag{3.17}$$

Putting together our estimates (3.13) - (3.17) and returning to (3.12) we have shown for any $\delta > 0$ the validity of the inequality

$$\begin{aligned}(3.18)\qquad \int_{Q_{\frac{3}{2}R}(x_0)} \eta^2\omega\, dx \le \delta \int_{Q_{\frac{3}{2}R}(x_0)} \omega\, dx \\ +c(\delta)\left[R^{-2}\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\, dx + R^{-1}\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\, dx\right]\end{aligned}$$

with $c(\delta)$ also depending on $\|u\|_{L^\infty(\mathbb{R}^2)}$. In order to control Dirichlet's integral on the right-hand side of (3.18) in an appropriate way, let us select $\varphi \in C_0^\infty(Q_{2R}(x_0))$, $0 \le \varphi \le 1$, $\varphi = 1$ on $Q_{\frac{3}{2}R}(x_0)$, $|\nabla\varphi| \le c/R$. We have by Lemma 3.3

$$\begin{aligned}\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\, dx &\le \int_{Q_{2R}(x_0)} \varphi^2|\nabla u|^2\, dx \\ &\le c\left[\int_{Q_{2R}(x_0)} |\nabla(\varphi u)|^2\, dx + \int_{Q_{2R}(x_0)} |\nabla\varphi|^2|u|^2\, dx\right] \\ &\le c\left[\int_{Q_{2R}(x_0)} |\varepsilon(\varphi u)|^2\, dx + \int_{Q_{2R}(x_0)} |\nabla\varphi|^2|u|^2\, dx\right] \\ &\le c\left[\int_{Q_{2R}(x_0)} \varphi^2|\varepsilon(u)|^2\, dx + R^{-2}\int_{Q_{2R}(x_0)} |u|^2\, dx\right],\end{aligned}$$

and if we recall the support property of η, inequality (3.18) in combination with the above estimates implies

$$\begin{aligned}(3.19)\qquad \int_{Q_R(x_0)} \omega\, dx \le \delta \int_{Q_{2R}(x_0)} \omega\, dx \\ +c(\delta)\Bigg[R^{-4}\int_{Q_{2R}(x_0)} |u|^2\, dx + R^{-3}\int_{Q_{2R}(x_0)} |u|^2\, dx \\ +R^{-2}\int_{Q_{2R}(x_0)} \varphi^2|\varepsilon(u)|^2\, dx + R^{-1}\int_{Q_{2R}(x_0)} \varphi^2|\varepsilon(u)|^2\, dx\Bigg].\end{aligned}$$

We have by Hölder's and Young's inequality

$$\begin{aligned}
\int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 \, dx &= \int_{Q_{2R}(x_0)} \varepsilon_{ij}(u)\varepsilon_{ij}(u)\varphi^2 \, dx \\
&= -\int_{Q_{2R}(x_0)} u^i \partial_j \left(\varepsilon_{ij}(u)\varphi^2\right) \, dx \\
&= -\int_{Q_{2R}(x_0)} u^i \partial_j \varepsilon_{ij}(u)\varphi^2 \, dx - \int_{Q_{2R}(x_0)} u^i \varepsilon_{ij}(u)\partial_j \varphi^2 \, dx \\
&\leq c\left[\int_{Q_{2R}(x_0)} |\nabla\varepsilon(u)| \, dx + R^{-1}\int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx\right] \\
&= c\left[\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)^{-1/2} |\nabla\varepsilon(u)| (1+|\varepsilon(u)|)^{1/2} \, dx\right. \\
&\quad \left. +R^{-1}\int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx\right] \\
&\leq \left[\left(\int_{Q_{2R}(x_0)} \frac{|\nabla\varepsilon(u)|^2}{1+|\varepsilon(u)|} \, dx\right)^{1/2} \left(\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|) \, dx\right)^{1/2}\right. \\
&\quad \left. +R^{-1}\int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx\right] \\
&\leq \tau \int_{Q_{2R}(x_0)} \omega \, dx + c\tau^{-1}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|) \, dx \\
&\quad +cR^{-1}\int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx \, ,
\end{aligned}$$

where τ is any positive number. Choosing

$$\tau = \delta c(\delta)^{-1} R^2, \ c(\delta) \text{ from } (3.19)\, ,$$

we get with a new constant $\widetilde{c}(\delta)$ recalling also (3.14)

$$\begin{aligned}
(3.20) \qquad c(\delta)R^{-2}\int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 \, dx &\leq \delta \int_{Q_{2R}(x_0)} \omega \, dx \\
+\widetilde{c}(\delta)\left[R^{-4}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|) \, dx + R^{-3}\int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx\right]&,
\end{aligned}$$

whereas the choice

$$\tau := \delta c(\delta)^{-1} R$$

leads to

$$\begin{aligned}
(3.21) \qquad c(\delta)R^{-1}\int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 \, dx &\leq \delta \int_{Q_{2R}(x_0)} \omega \, dx \\
+\widetilde{c}(\delta)\left[R^{-2}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|) \, dx + R^{-2}\int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx\right]&.
\end{aligned}$$

With (3.20) and (3.21) we return to (3.19) writing again $c(\delta)$ for constants depending on δ (and $\|u\|_{L^\infty(\mathbb{R}^2)}$) and replacing the parameter δ by $\delta/3$. We obtain:

$$\begin{aligned}(3.22)\qquad \int_{Q_R(x_0)} \omega\, dx \le \delta \int_{Q_{2R}(x_0)} \omega\, dx + c(\delta)\Bigg[R^{-4}\int_{Q_{2R}(x_0)} |u|^2\, dx \\ +R^{-3}\int_{Q_{2R}(x_0)} |u|^2\, dx + R^{-4}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)\, dx \\ +R^{-3}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)\, dx + R^{-2}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)\, dx\Bigg]\,.\end{aligned}$$

To estimate (3.22) we can apply Lemma 3.1 and get for all squares $Q_R(x_0)$ with $c = c(\|u\|_{L^\infty(\mathbb{R}^2)})$

$$\begin{aligned}(3.23)\qquad \int_{Q_R(x_0)} \omega\, dx \le c\Bigg[R^{-4}\int_{Q_{2R}(x_0)} |u|^2\, dx \\ +R^{-3}\int_{Q_{2R}(x_0)} |u|^2\, dx + R^{-4}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)\, dx \\ +R^{-3}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)\, dx + R^{-2}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)\, dx\Bigg]\,.\end{aligned}$$

Now, if the case $R \ge 1$ is considered, inequality (3.23) implies the bound

$$(3.24)\qquad \int_{Q_R(x_0)} \omega\, dx \le c\left[1 + R^{-2}\int_{Q_{2R}(x_0)} |\varepsilon(u)|\, dx\right]\,.$$

Clearly we have ($Q^\pm := Q_{2R}(x_0) \cap \left[|\varepsilon(u)| \gtrless 1\right]$)

$$\begin{aligned}\int_{Q_{2R}(x_0)} |\varepsilon(u)|\, dx &= \int_{Q^-} |\varepsilon(u)|\, dx + \int_{Q^+} |\varepsilon(u)|\, dx \\ &\le \left(\int_{Q^-} 1\, dx\right)^{1/2}\left(\int_{Q^-} |\varepsilon(u)|^2\, dx\right)^{1/2} + \frac{1}{\ln 2}\int_{Q^+} H(\varepsilon(u))\, dx \\ &\le cR\left(\int_{Q_{2R}(x_0)} H(\varepsilon(u))\, dx\right)^{1/2} + \frac{1}{\ln 2}\int_{Q_{2R}(x_0)} H(\varepsilon(u))\, dx\end{aligned}$$

and since we still assume that $R \ge 1$, we get from (3.1) the bound

$$(3.25)\qquad \int_{Q_{2R}(x_0)} |\varepsilon(u)|\, dx \le cR^{3/2}\,.$$

Now, if we insert (3.25) into (3.24), our claims (3.8) and (3.9) are clearly established.

In a final step we show

$$(3.26)\qquad \int_{\mathbb{R}^2} \omega\, dx = 0\,.$$

Obviously (recall (3.14)) equation (3.26) gives $\nabla\varepsilon(u) = 0$, hence $\nabla^2 u = 0$ so that u must be affine. But since we assume $u \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$, the claim of Theorem 1.2 clearly follows. It remains to prove (3.26): let

$$\omega_\infty := \int_{\mathbb{R}^2} \omega\, dx\,.$$

Going through the calculations leading to (3.18) with the choice $x_0 = 0$, a closer look at the quantities $S_i, i = 1, \dots, 4$, implies the inequality

$$\begin{aligned}(3.27)\quad \int_{Q_R} \omega\, dx \le \delta \int_{Q_{\frac{3}{2}R}} \omega\, dx + c(\delta) \Bigg[R^{-2} \int_{T_{\frac{3}{2}R}} |\nabla u|^2\, dx \\ + R^{-1} \int_{T_{\frac{3}{2}R}} |\nabla u|^2\, dx + R^{-1} \left(\int_{Q_{\frac{3}{2}R}} |\nabla u|^2\, dx \right)^{1/2} \left(\int_{T_{\frac{3}{2}R}} |\nabla u|^2\, dx \right)^{1/2} \Bigg]\,,\end{aligned}$$

where we have abbreviated $T_{\frac{3}{2}R} := Q_{\frac{3}{2}R} - \overline{Q}_R$ and where on the right-hand side of (3.27) the integration over $T_{\frac{3}{2}R}$ has to be performed in appropriate places due to the support properties of $\nabla\eta$. In the calculations after (3.18) we estimated $\int_{Q_{\frac{3}{2}R}} |\nabla u|^2\, dx$, but of course we can bound $\int_{T_{\frac{3}{2}R}} |\nabla u|^2\, dx$ in the same way with the help of Lemma 3.3 and Remark 3.2 by choosing $\varphi \equiv 1$ on $T_{\frac{3}{2}R}$, $0 \le \varphi \le 1$, $|\nabla\varphi| \le c/R$ and $\operatorname{spt}\varphi \subset Q_{2R} - \overline{Q}_{R/2} =: T_{2R}$. This yields

$$(3.28)\qquad \int_{T_{\frac{3}{2}R}} |\nabla u|^2\, dx \le c \left[\int_{T_{2R}} \varphi^2 |\varepsilon(u)|^2\, dx + R^{-2} \int_{T_{2R}} |u|^2\, dx \right]$$

and from the arguments used after (3.19) we deduce

$$\begin{aligned}(3.29)\qquad \int_{T_{2R}} \varphi^2 |\varepsilon(u)|^2\, dx \;\le\; c \Bigg[\left(\int_{T_{2R}} \omega\, dx \right)^{1/2} \left(\int_{T_{2R}} (1 + |\varepsilon(u)|)\, dx \right)^{1/2} \\ + R^{-1} \int_{T_{2R}} |\varepsilon(u)|\, dx \Bigg] =: \Phi(R)\,.\end{aligned}$$

Putting together (3.28) and (3.29) and going back to (3.27) we obtain choosing $\delta = 1/2$

$$\begin{aligned}(3.30)\quad \int_{Q_R} \omega\, dx \le \frac{1}{2}\omega_\infty \;+\; c \Bigg\{ R^{-4} \int_{Q_{2R}} |u|^2\, dx + R^{-3} \int_{Q_{2R}} |u|^2\, dx \\ + R^{-2}\Phi(R) + R^{-1}\Phi(R) \\ + R^{-1} \left(\int_{Q_{\frac{3}{2}R}} |\nabla u|^2\, dx \right)^{1/2} \left(\int_{T_{\frac{3}{2}R}} |\nabla u|^2\, dx \right)^{1/2} \Bigg\}\,.\end{aligned}$$

Clearly $R^{-4}\int_{Q_{2R}}|u|^2\,dx + R^{-3}\int_{Q_{2R}}|u|^2\,dx \longrightarrow 0$ as $R\to\infty$, and from (3.25) we obtain ($R\ge 1$)

$$\Phi(R)\le c\left[R\left(\int_{T_{2R}}\omega\,dx\right)^{1/2}+R^{1/2}\right],$$

hence $R^{-2}\Phi(R)+R^{-1}\Phi(R)\longrightarrow 0$ as $R\to\infty$ on account of (3.8). At this stage we like to remark that here it is essential to integrate ω just over the set T_{2R}. Let us finally look at the quantity

$$\Psi(R):=R^{-1}\left(\int_{Q_{\frac{3}{2}R}}|\nabla u|^2\,dx\right)^{1/2}\left(\int_{T_{\frac{3}{2}R}}|\nabla u|^2\,dx\right)^{1/2}.$$

By (3.28) and (3.29) we have

$$\int_{T_{\frac{3}{2}R}}|\nabla u|^2\,dx\le c\left[R^{-2}\int_{T_{2R}}|u|^2\,dx+\Phi(R)\right],$$

thus

$$\Psi(R)\le c\left(R^{-1}\int_{Q_{\frac{3}{2}R}}|\nabla u|^2\,dx\right)^{1/2}\left(R^{-3}\int_{T_{2R}}|u|^2\,dx+R^{-1}\Phi(R)\right)^{1/2}$$

and the second factor on the right-hand side goes to zero as $R\to\infty$ as observed earlier. Returning to our previous bound

$$\begin{aligned}\int_{Q_{\frac{3}{2}R}}|\nabla u|^2\,dx \;&\le\; c\left[R^{-2}\int_{Q_{2R}}|u|^2\,dx\right.\\ &\left.+\left(\int_{Q_{2R}}\omega\,dx\right)^{1/2}\left(\int_{Q_{2R}}(1+|\varepsilon(u)|)\,dx\right)^{1/2}+R^{-1}\int_{Q_{2R}}|\varepsilon(u)|\,dx\right]\end{aligned}$$

we see in combination with (3.25) and (3.8) that $R^{-1}\int_{Q_{\frac{3}{2}R}}|\nabla u|^2\,dx$ stays bounded, which means that also $\Psi(R)\longrightarrow 0$ as $R\to\infty$. Therefore the passage to the limit in (3.30) finally yields our claim $\omega_\infty=0$. □

Appendix: Proof of Lemma 3.1

We completely adopt the notation introduced in the proof of Lemma 0.5 of [GM]. As done there we work in the space $\mathbb{R}^n$ with $n\ge 2$. For $x_0\in\mathbb{R}^n$ and $\sigma>0$ we consider the cube $Q:=Q_{\frac{3}{2}\sigma}(x_0)$. We will make use of the following inequalities:

(A1) $$d(x)<\sigma 2^{-k+1}\text{ on }Q_j^k,$$

$$\text{(A2)} \qquad d(x) \geq \sigma 2^{-k-1} \text{ on } \overline{Q}_j^k\,,$$

$$\text{(A3)} \qquad d(x) < \sigma 2^{-k+2} \text{ on } \overline{Q}_j^k\,,$$

where the last inequality is not explicitly stated in [GM] but follows from the inclusion

$$\overline{Q}_j^k \subset C_{k-1} \cup C_k \cup C_{k+1}\,.$$

Let $\alpha := \max\{\alpha_1, \dots, \alpha_\ell\}$. It holds

$$\begin{aligned}
\int_{Q_j^k} d^\alpha f\,dx &\leq \sigma^\alpha 2^{(-k+1)\alpha} \int_{Q_j^k} f\,dx \\
&\leq \sigma^\alpha 2^{(-k+1)\alpha} \delta \int_{\overline{Q}_j^k} f\,dx + c(\delta)\sigma^\alpha 2^{(-k+1)\alpha} \sum_{m=1}^{\ell} (\sigma 2^{-k})^{-\alpha_m} \int_{\overline{Q}_j^k} f_m\,dx
\end{aligned}$$

on account of our assumption imposed on the functions f, $f_1, \dots, f_\ell$. Moreover, we made use of (A1), whereas from (A2) we get

$$\sigma^\alpha 2^{(-k+1)\alpha} \int_{\overline{Q}_j^k} f\,dx \leq 4^\alpha \int_{\overline{Q}_j^k} f d^\alpha\,dx\,,$$

and (A2) in combination with (A3) implies

$$\begin{aligned}
&\sigma^\alpha 2^{(-k+1)\alpha} \sum_{m=1}^{\ell} (\sigma 2^{-k})^{-\alpha_m} \int_{\overline{Q}_j^k} f_m\,dx \\
&\leq 4^\alpha \sum_{m=1}^{\ell} (\sigma 2^{-k})^{-\alpha_m} \int_{\overline{Q}_j^k} f_m d^\alpha\,dx \\
&\leq 4^\alpha \sum_{m=1}^{\ell} (\sigma 2^{-k})^{-\alpha_m} \left(\sigma 2^{-k+2}\right)^{\alpha_m} \int_{\overline{Q}_j^k} f_m d^{\alpha-\alpha_m}\,dx \\
&= 4^\alpha \sum_{m=1}^{\ell} 4^{\alpha_m} \int_{\overline{Q}_j^k} f_m d^{\alpha-\alpha_m}\,dx\,.
\end{aligned}$$

Putting together these estimates, we deduce:

$$\text{(A4)} \qquad \begin{aligned}
\int_{Q_j^k} d^\alpha f\,dx \leq\ & \delta 4^\alpha \int_{\overline{Q}_j^k} f\,dx \\
& + c(\delta) 16^\alpha \sum_{m=1}^{\ell} \int_{\overline{Q}_j^k} f_m d^{\alpha-\alpha_m}\,dx\,.
\end{aligned}$$

This inequality is in correspondence to the estimate stated in the last line of p.178 of the paper [GM], and with the help of (A4) we can now finish the proof of Lemma 3.1 exactly along the lines of p.179 in [GM]. □

References

[Ad] Adams, R. A., Sobolev spaces. Academic Press, New York-San Francisco-London 1975.

[ABF] Apushkinskaya, D., Bildhauer, M., Fuchs, M., Interior gradient bounds for local minimizers of variational integrals under nonstandard growth conditions. J. Math. Sciences 164(3), 345–363 (2010).

[EM] Esposito, L., Mingione, G., Partial regularity for minimizers of convex integrals with L log L–growth. Nonlinear Diff. Equ. Appl. 7, 107–125 (2000).

[FrSe] Frehse, J., Seregin, G., Regularity for solutions of variational problems in the deformation theory of plasticity with logarithmic hardening. Transl. Amer. Math. Soc.II, 193, 127–152 (1999).

[Fu1] Fuchs, M., Local Lipschitz regularity of vector valued local minimizers of variational integrals with densities depending on the modulus of the gradient. Math. Nachrichten. DOI: 10.1002/mana.200810189.

[Fu2] Fuchs, M., Liouville theorems for stationary flows of shear thickening fluids in the plane. submitted

[FuSe1] Fuchs, M., Seregin, G., A regularity theory for variational integrals with L log L-growth. Calc. Var. 6, 171–187 (1998).

[FuSe2] Fuchs, M., Seregin, G., Variational methods for problems from plasticity theory and for generalized Newtonian fluids. Lecture Notes in Mathematics 1749, Springer Verlag, Berlin-Heidelberg-New York 2000.

[FuSe3] Fuchs, M., Seregin, G., Variational methods for fluids of Prandtl-Eyring type and plastic materials with logarithmic hardening. Math. Meth. Appl. Sciences 22, 317–351 (1999).

[Ga] Galdi, G., An introduction to the mathematical theory of the Navier-Stokes equations Vol.I, Springer Tracts in Natural Philosophy Vol.38, Springer, Berlin-Heidelberg-New York 1994.

[GM] Giaquinta, M., Modica, G., Nonlinear systems of the type of the stationary Navier-Stokes system. J. Reine Angew. Math. 330, 173–214 (1982).

[GT] Gilbarg, D., Trudinger, N. S., Elliptic partial differential equations of second order. Grundlehren der Math. Wissenschaften 224, Springer Verlag, Berlin-Heidelberg-New York 1998.

[HL] Han, Q., Lin, F. H., Elliptic partial differential equations. Memoirs of the AMS 2000.

[La] Ladyzhenskaya, O. A., The mathematical theory of viscous incompressible flow. Gordon and Breach, 1969.

[Ma1] Marcellini, P., Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. Arch. Rat. Mech. Anal. 10, 267–284 (1989).

[Ma2] Marcellini, P., Regularity for elliptic equations with general growth conditions. J. Diff. Equ.105, 296–333 (1993).

[Ma3] Marcellini, P., Everywhere regularity for a class of elliptic systems without growth conditions. Ann. Scuola Norm. Sup. Pisa 23, 1–25 (1996).

[MP] Marcellini, P., Papi, G., Nonlinear elliptic systems with general growth. J. Diff. Equ.221, 412–443 (2006).

[MS] Mingione, G., Siepe, F., Full $C^{1,\alpha}$-regularity for minimizers of integral functionals with L log L-growth. Z. Anal. Anw.18, 1083–1100 (1999).

[To] Tolksdorf, P., Everywhere-regularity for some quasi-linear systems with a lack of ellipticity. Ann. Mat. Pura Appl.134, 241–266 (1983).