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Abstract

We consider a solution u of the stationary Navier–Stokes equations defined on the complement of the unit disc vanishing on the unit circle. Among other results it is shown that u is identically zero provided that $\lim_{|x|\to\infty}|u(x)||x|^{1/3}=0$ uniformly. Here no assumption on the finiteness of the Dirichlet energy calculated over the exterior domain is imposed.

1 Introduction

In our paper we investigate the Navier–Stokes equations for incompressible Newtonian fluids concentrating on the stationary case in two spatial variables:

$$\begin{cases} \nu\Delta u = u\cdot\nabla u + \nabla p\,, \\ \operatorname{div} u = 0\,. \end{cases} \tag{1.1}$$

Here $u = u(x_1, x_2)$ denotes the velocity field, $p = p\,(x_1, x_2)$ is the pressure function and $\nu > 0$ the (constant) viscosity of the fluid. For a detailed explanation of the mathematical and physical background of equations (1.1) the reader is referred to the monographs of Ladyzhenskaya [La] and of Galdi [Ga1,2]. Let $B := \{x \in \mathbb{R}^2 : |x| < 1\}$ denote the open unit disc. Then we are going to consider solutions u of equations (1.1) on $\mathbb{R}^2 - \overline{B}$ satisfying the boundary condition

$$u|_{\partial B} = 0\,. \tag{1.2}$$

More precisely we introduce the class ($B_r := \{x \in \mathbb{R}^2 : |x| < r\}$)

$$S_{\text{loc}} := \bigcap_{r>1} W^1_2(B_r - \overline{B})$$

consisting of all (local) Sobolev functions $u : \mathbb{R}^2 - \overline{B} \to \mathbb{R}^2$ such that

$$\int_{B_r-\overline{B}} |\nabla u|^2\, dx \le c(u, r) < \infty$$

for any $r > 1$ and its proper subclass

$$S^{\text{div}}_{\text{loc}} := \{u \in S_{\text{loc}} : \operatorname{div} u \equiv 0\}\ . \tag{1.3}$$

Note that for functions u from $S^{(\text{div})}_{\text{loc}}$ boundary values on ∂B are defined in the trace sense. We also emphasize that the finiteness of the Dirichlet integral on the exterior domain $\mathbb{R}^2 - \overline{B}$ is not required.

Definition 1.1. *A function $u \in S^{\mathrm{div}}_{\mathrm{loc}}$ is a solution of the exterior problem associated to the unit disc, if (1.2) holds and if*

$$0 = \nu \int\limits_{\mathbb{R}^2 - \overline{B}} \nabla u : \nabla \varphi \, dx + \int\limits_{\mathbb{R}^2 - \overline{B}} u^k \partial_k u^i \varphi^i \, dx \tag{1.4}$$

is satisfied for all $\varphi \in C^\infty_0(\mathbb{R}^2 - \overline{B})$, $\operatorname{div} \varphi = 0$, where here and it what follows the convention of summation is adopted.

Much attention has been paid to so–called D–solutions of the exterior problem, i.e. to solutions u satisfying additionaly

$$\int\limits_{\mathbb{R}^2 - \overline{B}} |\nabla u|^2 \, dx < \infty \, . \tag{1.5}$$

In the absence of the convective term it was shown by Heywood [He] (cf. also [Ga1], Theorem 2.2, p.247) that u vanishes identically ("Stokes paradox for generalized solutions"), but according to Remark 3.1, p.187, in [Ga2] this conclusion seems to be open if we consider D–solutions of the original problem, i.e. of (1.2) and (1.4). However, referring to Theorem 3.4, p.186, in [Ga2], the asymptotic behaviour of D–solutions to the exterior problem can be characterized in a rather satisfying way. In this setting the reader should also consult the famous paper [GW] of Gilbarg and Weinberger as well as chapter X.3 of Galdi's book [Ga2], where one finds further references.

Let us now drop assumption (1.5) and consider a solution u from the space $S^{\mathrm{div}}_{\mathrm{loc}}$ (cf. (1.3)) in the sense of Definition 1.1. In his recent paper [Ko] Koch raises the question, if the asymptotic condition "$u \to 0$ at infinity" implies the vanishing of u. Actually a positive answer would shed light on the existence problem for the twodimensional Navier–Stokes equations on an exterior domain satisfying the homogeneous boundary condition (1.2) together with

$$\lim_{|x| \to \infty} u(x) = u_\infty \, ,$$

where u_∞ is a prescribed vector and where $|u_\infty|$ can be identified with the Reynolds number of the problem. At this stage we refer to the deep paper [Ga3] of Galdi, in which he presents the historical background and discusses the existence of solutions to the exterior problem for arbitrary large Reynolds number in a class of fields having certain symmetries. In our note we will not contribute to this basic problem, we just show using the "Liouville techniques" from the papers [FuZha] and [FuZho] that a solution of (1.2) and (1.4) decaying sufficiently fast at infinity actually must vanish, more precisely we have:

THEOREM 1.1. *Let $u \in S^{\mathrm{div}}_{\mathrm{loc}}$ satisfy (1.2) and (1.4). Suppose further that for some $\alpha \geq 0$ it holds (uniformly)*

$$\lim_{|x| \to \infty} |x|^\alpha \, |u(x)| = 0 \, . \tag{1.6}$$

If we can choose $\alpha = 1/3$ in (1.6), then u is identically zero.

REMARK 1.1. *As mentioned before it is a challenging task to prove Theorem 1.1 just under the hypothesis $\alpha = 0$ in (1.6). From the proof of Theorem 1.1 it will become clear that our arguments also apply to the exterior problem in $\mathbb{R}^n$, $n \geq 3$, with suitable numbers $\alpha = \alpha(n)$. We leave the details to the reader. Perhaps it might also be of interest to give variants of Theorem 1.1 for p-fluids or fluids of Prandtl-Eyring type.*

REMARK 1.2. *Assume that $u \in S^{\mathrm{div}}_{\mathrm{loc}}$ is a solution of the exterior Stokes problem with the boundary condition (1.2). Then in the absence of the convective term the proof of Theorem 1.1 presented below easily gives the vanishing of u under the assumption that $u(x) \to 0$ as $|x| \to \infty$ uniformly. However, as it is outlined in [Ga1], Theorem 3.5 on p.257, this conclusion even holds under the weaker hypothesis $\lim_{|x|\to\infty} u(x)/\ln|x| = 0$ (uniformly). Due to this fact we believe that also in the setting of Theorem 1.1 different methods might lead to better results.*

Let us now consider the case $\alpha = 0$:

THEOREM 1.2. *Suppose that $u \in S^{\mathrm{div}}_{\mathrm{loc}}$ satisfies (1.2) and (1.4). Assume further that (1.6) holds with $\alpha = 0$.*
a) If $\omega := \partial_2 u^1 - \partial_1 u^2$ denotes the vorticity of u, then $\lim_{|x|\to\infty} \omega(x) = 0$ uniformly. In particular, if we require $\nabla u = 0$ on $|x| = 1$ then u must be zero.
b) Let $\nabla^2 u$ denote the matrix of the second partial derivatives of u. Then we have

$$\lim_{R\to\infty} \int_{T_R} \left|\nabla^2 u\right|^2 \, dx = 0 \,,$$

where $T_R := \{x \in \mathbb{R}^2 : R < |x| < 2R\}$.

2 Proof of Theorem 1.1

Let us assume throughout this section that $u \in S^{\mathrm{div}}_{\mathrm{loc}}$ is a solution of (1.4) with homogeneous boundary condition (1.2) satisfying in addition

(2.1) $$\Theta(r) := \sup_{|x|\geq r} |u(x)| \to 0 \text{ as } r \to \infty \,,$$

which means that we impose (1.6) at least with $\alpha = 0$. Our first goal is to estimate the quantity $\int_{T_R} |\nabla u|^2 \, dx$, $T_R := \{x \in \mathbb{R}^2 : R < |x| < 2R\}$, for large values of R. To this purpose we consider a square $Q_{2r}(z) := \{x \in \mathbb{R}^2 : |x_i - z_i| < 2r, \, i = 1, 2\}$ such that $Q_{2r}(z) \subset \{x \in \mathbb{R}^2 : |x| > 1\}$. In equation (1.4) we let $\varphi := \eta^2 u - w$, where $\eta \in C^1_0(Q_{2r}(z))$, $\eta = 1$ on $Q_r(z)$, $0 \leq \eta \leq 1$ and $|\nabla\eta| \leq c/r$. The field w is defined according to Lemma 4.1 on the domain $Q_{2r}(z)$ and with the choice

$$f = \mathrm{div}(\eta^2 u) = \nabla\eta^2 \cdot u \,.$$

Following exactly the calculations from [FuZho], we arrive at inequality (2.7) of this paper, i.e. we have for any $\delta > 0$

$$(2.2)\quad \int_{Q_r(z)} |\nabla u|^2\,dx \;\le\; \delta \int_{Q_{2r}(z)} |\nabla u|^2\,dx + c\left[\delta^{-1} r^{-2} \int_{Q_{2r}(z)} |u|^2\,dx + r^{-1} \int_{Q_{2r}(z)} \left(|u|^2 + |u|^3 + |u|^4\right) dx\right]$$

with a positive constant c being independent of u, δ, r and z. Let $Q_0 := Q_R(x_0)$ denote an arbitrary square contained in the exterior of B. Then (2.2) holds for all squares $Q_{2r}(z) \subset Q_0$ and Lemma 4.2 yields

$$(2.3)\quad \int_{Q_r(z)} |\nabla u|^2\,dx \le c\left[r^{-2} \int_{Q_{2r}(z)} |u|^2\,dx + r^{-1} \int_{Q_{2r}(z)} \left(|u|^2 + |u|^3 + |u|^4\right) dx\right]$$

again for all squares $Q_{2r}(z) \subset Q_0$.

Next we fix an annulus T_R of radius $R \gg 1$ such that (compare (2.1))

$$(2.4)\quad |u(x)| \le 1,\ |x| \ge R/2\,.$$

The closure of T_R can be covered by a finite number N (independent of R) of squares $Q_{R/8}(x_i)$ with centers $x_i \in \overline{T}_R$, and (2.3) together with (2.4) yields (replacing $Q_r(z)$ by $Q_{R/8}(x_i)$ in (2.3) and defining $Q_0 := Q_{R/2}(x_i)$)

$$(2.5)\quad \int_{Q_{\frac{R}{8}}(x_i)} |\nabla u|^2\,dx \le c\,R^{-1} \int_{Q_{\frac{R}{4}}(x_i)} |u|^2\,dx\,.$$

Next we observe that inequality (2.2) can be replaced by

$$(2.6)\quad \int_{Q_r(z)} |\nabla u|^2\,dx \;\le\; \delta \int_{Q_{2r}(z)} |\nabla u|^2\,dx + c\left\{\delta^{-1} r^{-2} \int_{Q_{2r}(z)} |u|^2\,dx + r^{-1} \int_{Q_{2r}(z)} |u|^3\,dx + r^{-1}\left[\int_{Q_{2r}(z)} |u|^4\,dx \int_{Q_{2r}(z)} |u|^2\,dx\right]^{1/2}\right\}$$

valid for any square $Q_{2r}(z) \Subset \{x \in \mathbb{R}^2 : |x| > 1\}$ and arbitrary $\delta > 0$, we refer to [FuZho], (2.3) - (2.6). With the notation introduced after (2.4) it follows from (2.6)

$$\text{(2.7)} \qquad \int\limits_{Q_{\frac{R}{8}}(x_i)} |\nabla u|^2\, dx \;\le\; \delta \int\limits_{Q_{\frac{R}{4}}(x_i)} |\nabla u|^2\, dx + c\left\{ \delta^{-1} R^{-2} \int\limits_{Q_{\frac{R}{4}}(x_i)} |u|^2\, dx + R^{-1} \int\limits_{Q_{\frac{R}{4}}(x_i)} |u|^3\, dx + R^{-1}\left[\int\limits_{Q_{\frac{R}{4}}(x_i)} |u|^4\, dx \int\limits_{Q_{\frac{R}{4}}(x_i)} |u|^2\, dx \right]^{1/2} \right\} \;\le\; c\,\delta\, R^{-1} \int\limits_{Q_{\frac{R}{2}}(x_i)} |u|^2\, dx + c\,\{\dots\}\ ,$$

where we have used (2.5) with R replaced by $2R$ to estimate the quantity $\delta \int_{Q_{\frac{R}{4}}(x_i)} |\nabla u|^2\, dx$ through $c\,\delta\, R^{-1} \int_{Q_{\frac{R}{2}}(x_i)} |u|^2\, dx$. In (2.7) we choose $\delta := R^{-1/3}$. Then using the previous covering argument we end up with

$$\text{(2.8)} \qquad \int\limits_{T_R} |\nabla u|^2\, dx \le c\, \left\{ R^{2/3} \Theta\left(R/2\right)^2 + R\Theta\left(R/2\right)^3 \right\}\ ,$$

and obviously (2.8) implies $\lim_{R\to\infty} \int_{T_R} |\nabla u|^2 dx = 0$, provided (1.6) is valid for $\alpha = 1/3$.

Our next goal is to show that in this case ∇u vanishes on the exterior domain $\mathbb{R}^2 - \overline{B}$ so that u must be constant. But then the boundary condition (1.2) implies our claim $u = 0$. Let us fix some radius $R \gg 1$. In order to discuss the quantity $\int_{|x|>1} |\nabla u|^2\, dx$ we look at the behaviour of $\int_{1<|x|<R} |\nabla u|^2\, dx$. To this purpose we let in equation (1.4)

$$\varphi := \begin{cases} u, \ 1 \le |x| \le R \\ \eta^2 u - w, \ R \le |x| \le 2R \end{cases}$$

with $\eta = 1$ on $1 \le |x| \le R$, $0 \le \eta \le 1$, $\eta = 0$ on $|x| \ge 2R$ and $|\nabla\eta| \le c/R$. Moreover, the field $w \in \overset{\circ}{W}{}^1_2(T_R)$ is chosen according to Lemma 4.1 (with domain $T_R := \{x \in \mathbb{R}^2 : R < |x| < 2R\}$) such that

$$\operatorname{div} w = f \text{ on } T_R,\ f := \operatorname{div}(\eta^2 u)\,.$$

Note that φ vanishes on ∂B, moreover

$$\int_{T_R} f\,dx = 0 \tag{2.9}$$

is satisfied: in fact, by the choice of η we have

$$\begin{aligned}\int_{T_R} f\,dx &= \int_{\partial T_R} \eta^2 u\cdot \mathcal{N}_{T_R} d\mathcal{H}^1 = -\int_{\partial B_R} u\cdot \mathcal{N}_{\partial B_R} d\mathcal{H}^1\\ &\overset{(1.2)}{=} -\int_{\partial(B_R-\overline{B})} u\cdot\mathcal{N}_{\partial(B_R-\overline{B})} d\mathcal{H}^1 = -\int_{B_R-\overline{B}} \operatorname{div} u\,dx = 0\,,\end{aligned}$$

and (2.9) follows. Here $\mathcal{N}\dots$ denotes the exterior normal of the domains under consideration and $\mathcal{H}^1$ is the onedimensional Hausdorff measure. If we assume w.l.o.g. that $\nu = 1$ in (1.4), then we obtain from this equation:

$$0 = \int_{B_R-\overline{B}} |\nabla u|^2\,dx + \int_{T_R} \nabla u : \nabla(\eta^2 u)\,dx - \int_{T_R} \nabla u : \nabla w\,dx + \int_{B_{2R}-\overline{B}} u^k\partial_k u^i \varphi^i\,dx$$

or equivalently

$$\begin{aligned}&\int_{1<|x|<2R} \eta^2|\nabla u|^2\,dx = \\ &-\int_{R<|x|<2R} \nabla u : \left(\nabla\eta^2\otimes u\right)\,dx + \int_{R<|x|<2R} \nabla u : \nabla w\,dx - \int_{1<|x|<2R} u^k\partial_k u^i\varphi^i\,dx\,.\end{aligned}$$

Applying Young's inequality to the first two items on the right–hand side and recalling the definition of the field w we find by the properties of η:

$$\begin{aligned}&\int_{1<|x|<R} |\nabla u|^2\,dx \leq c\left\{\int_{T_R} |\nabla u|^2\,dx + R^{-2}\int_{T_R} |u|^2\,dx + |T|\right\}\,,\\ &T := \int_{1<|x|<2R} u^k\partial_k u^i\varphi^i\,dx\,.\end{aligned} \tag{2.10}$$

From (1.6) and (2.8) we deduce (in the case $\alpha = 1/3$)

$$\lim_{R\to\infty}\left\{\int_{T_R} |\nabla u|^2\,dx + R^{-2}\int_{T_R} |u|^2\,dx\right\} = 0\,, \tag{2.11}$$

and (2.10) together with (2.11) will give the desired statement, i.e.

$$\lim_{R\to\infty} \int\limits_{1<|x|<R} |\nabla u|^2 \, dx = 0 \,,$$

as soon as we can show that

$$\lim_{R\to\infty} \int\limits_{1<|x|<2R} u^k \partial_k u^i \varphi^i \, dx = 0 \,. \tag{2.12}$$

To this purpose we use the properties of φ as well as the equation $\operatorname{div} u = 0$ to get

$$\begin{aligned} T &= - \int\limits_{1<|x|<2R} u^k u^i \partial_k \varphi^i \, dx \\ &= - \int\limits_{1<|x|<2R} u^k u^i \partial_k \left(\eta^2 u^i\right) \, dx + \int\limits_{R<|x|<2R} u^k u^i \partial_k w^i \, dx =: -T_1 + T_2 \,. \end{aligned} \tag{2.13}$$

It holds

$$\begin{aligned} |T_2| &\leq \int\limits_{T_R} |u|^2 |\nabla w| \, dx \leq \left(\int\limits_{T_R} |u|^4 \, dx \right)^{1/2} \left(\int\limits_{T_R} |\nabla w|^2 \, dx \right)^{1/2} \\ &\leq c \left(\int\limits_{T_R} |u|^4 \, dx \right)^{1/2} \frac{1}{R} \left(\int\limits_{T_R} |u|^2 \, dx \right)^{1/2} \leq c \; R \sup_{T_R} |u|^2 \sup_{T_R} |u| \,, \end{aligned}$$

thus by (1.6) (with $\alpha = 1/3$)

$$\lim_{R\to\infty} T_2 = 0 \,. \tag{2.14}$$

For T_1 we observe the identity

$$\begin{aligned} T_1 &= \int_{1<|x|<2R} u^k |u|^2 \partial_k \eta^2 dx + \int_{1<|x|<2R} u^k u^i \eta^2 \partial_k u^i dx \\ &= \int_{R<|x|<2R} u^k |u|^2 \partial_k \eta^2 dx + \int_{1<|x|<2R} u^k u^i \eta^2 \partial_k u^i dx =: T_3 + T_4 \,, \end{aligned} \tag{2.15}$$

which follows from the support properties of $\nabla \eta$.

At the same time integration by parts together with the equation $\operatorname{div} u = 0$ yields

$$\begin{aligned} T_1 &= - \int_{1<|x|<2R} \partial_k (u^k u^i) \eta^2 u^i \, dx \\ &= - \int_{1<|x|<2R} u^k \partial_k u^i \eta^2 u^i \, dx = -T_4 \,. \end{aligned} \tag{2.16}$$

Thus (2.15) together with (2.16) implies $2T_1 = T_3$ and we arrive at

$$\lim_{R\to\infty} T_1 = 0\,. \tag{2.17}$$

With (2.17) and (2.14) the claim (2.12) follows from the decomposition (2.13). This finishes the proof of Theorem 1.1. □

3 Proof of Theorem 1.2

Suppose that we have (1.2), (1.4) and (1.6) with $\alpha = 0$. From the calculations carried out in [FuZho], Section 3, we obtain the inequality

$$\int\limits_{Q_r(x_0)} |\nabla\omega|^2\,dx \le c\left[r^{-2}\int\limits_{Q_{2r}(x_0)} \omega^2\,dx + r^{-1}\int\limits_{Q_{2r}(x_0)} |u|\omega^2\,dx\right] \tag{3.1}$$

valid for any square $Q_{2r}(x_0) \Subset \{x \in \mathbb{R}^2 : |x| > 1\}$. Let us fix an annulus T_R with large radius R. Then we apply a covering argument in combination with (3.1) and get

$$\begin{aligned} \int_{T_R} |\nabla\omega|^2\,dx &\le c\left[R^{-2}\int_{\widetilde{T}_R} \omega^2\,dx + \Theta(R/2)R^{-1}\int_{\widetilde{T}_R} \omega^2\,dx\right], \\ \widetilde{T}_R &:= \left\{x \in \mathbb{R}^2 : \frac{R}{2} < |x| < \frac{5}{2}R\right\}. \end{aligned} \tag{3.2}$$

Since $|\omega| \le |\nabla u|$ we may use a variant of (2.8) (note that we have (2.4) on account of our choice of α) on the right-hand side of (3.2) leading to the estimate

$$\begin{aligned} \int\limits_{T_R} |\nabla\omega|^2\,dx &\le c\Big[R^{-2}\left(R^{2/3}\Theta(R/4)^2 + R\Theta(R/4)^3\right) \\ &\qquad + \Theta(R/4)R^{-1}\left(R^{2/3}\Theta(R/4)^2 + R\Theta(R/4)^3\right)\Big]\,, \end{aligned}$$

hence

$$\int\limits_{T_R} |\nabla\omega|^2\,dx \le c\Theta(R/4)^2\,, \tag{3.3}$$

and (2.8) clearly yields

$$\int\limits_{T_R} \omega^2\,dx \le cR\Theta(R/4)^2\,. \tag{3.4}$$

According to inequality (2.17) in [GW] there exists a suitable radius $r_n \in (R_n, R_{n+1})$, $R_n := 2^n$ for sufficiently large n, such that

$$
\int_0^{2\pi} \left[r_n^2 \omega(r_n, \Theta)^2 + 2r_n \left|\omega(r_n, \Theta)\right| \left|\omega_\Theta(r_n, \Theta)\right| \right] d\Theta \le c \int_{T_{R_n}} \left[\omega^2 + |\nabla \omega|^2 \right] \, dx \, , \tag{3.5}
$$

where $\omega(r, \Theta)$ is the representation of the vorticity in polar coordinates. The estimate stated after (2.17) in the paper [GW] yields at the same time

$$
\sup_\Theta \omega(r_n, \Theta)^2 \le c \left[\int_0^{2\pi} \omega(r_n, \Theta')^2 d\Theta' + \int_0^{2\pi} \left|\omega(r_n, \Theta')\omega_\Theta(r_n, \Theta')\right| d\Theta' \right] , \tag{3.6}
$$

and if we combine (3.5) and (3.6) we find

$$
\sup_\Theta \omega(r_n, \Theta)^2 \le c r_n^{-1} \int_{T_{R_n}} \left[\omega^2 + |\nabla \omega|^2 \right] \, dx \, . \tag{3.7}
$$

Since (recall (3.3) and (3.4)) the right-hand side of (3.7) is of order $\Theta(R_n/4)^2$, we get

$$
\sup_{|x|=r_n} |\omega| \le c\Theta(R_n/4) \, . \tag{3.8}
$$

Let us fix a large number n in (3.8). For $m \ge n+1$ we obtain (again quoting (3.8))

$$
\sup_{|x|=r_m} |\omega| \le c\Theta(r_m/4) \le c\Theta(r_n/4) \, ,
$$

hence $|\omega| \le c\Theta(r_n/4)$ on the boundary of $r_n < |x| < r_m$. From the maximum principle it then follows that

$$
|\omega| \le c\Theta(r_n/4) \text{ on } r_n < |x| < r_m
$$

for any $m \ge n+1$, hence

$$
|\omega| \le c\Theta(r_n/4) \text{ on } |x| > r_n \, ,
$$

which proves $\lim_{|x|\to\infty} \omega(x) = 0$ uniformly. The second part of Theorem 1.2a) is immediate, since $\nabla u = 0$ on $|x| = 1$ yields the same for ω, hence $\omega = 0$ on $|x| > 1$ by quoting the maximum principle again. But then $u^1 - iu^2$ is a holomorphic function on $|x| > 1$ vanishing at infinity (by (1.6) with $\alpha = 0$) and on the unit circle, hence $u = 0$.

Part b) of Theorem 1.2 follows from inequality (3.18) in [FuZha]: note that in this inequality "ω" represents the quantity $|\nabla^2 u|^2$, and if we use the estimate for squares $Q_R(x_0)$ such that $Q_{2R}(x_0)$ is contained in a fixed square $Q_0 \subset [|x| > 1]$, then we deduce in combination with Lemma 4.2

$$\text{(3.9)} \qquad \int\limits_{Q_R(x_0)} \left|\nabla^2 u\right|^2 dx \;\leq\; c\left\{ R^{-2} \int\limits_{Q_{2R}(x_0)} |\nabla u|^2 \, dx + cR^{-1} \int\limits_{Q_{2R}(x_0)} |\nabla u|^2 \, dx \right\}, \; Q_{2R}(x_0) \subset Q_0 \, .$$

From (3.9) and a covering argument we then obtain

$$\int\limits_{T_R} \left|\nabla^2 u\right|^2 dx \leq cR^{-1} \int\limits_{\frac{R}{2}<|x|<\frac{5}{2}R} |\nabla u|^2 \, dx \, ,$$

and a variant of (2.8) with suitable radii gives our claim. □

4 Appendix

The history of the following result concerning the solvability of the equation $\operatorname{div} w = f$ can be traced for example in Chapter III, Section 3, of [Ga1].

Lemma 4.1. *Let Ω denote either a disc $B_R(x_0)$, a square $Q_R(x_0)$ or an annulus $B_{2R}(x_0) - \overline{B_R(x_0)}$. Consider a function $f \in L^2(\Omega)$ such that $\int_\Omega f \, dx = 0$. Then there exists a field $w \in \overset{\circ}{W}{}^1_2(\Omega)$ and a constant C independent of Ω such that $\operatorname{div} w = f$ on Ω together with the estimate*

$$\int\limits_\Omega |\nabla w|^2 \, dx \leq C \int\limits_\Omega f^2 \, dx \, .$$

The next auxiliary result is known as the ε–lemma of Giaquinta and Modica (cf. [GM], Lemma 0.5), we state a variant proved in [FuZha].

Lemma 4.2. *Let $f, f_1, \dots, f_\ell$ denote non–negative functions from the space $L^1(Q_0)$, where Q_0 is an open square in $\mathbb{R}^2$. Suppose that we are given exponents $\alpha_1, \dots, \alpha_\ell > 0$. Then we can find a number $\varepsilon_0 > 0$ depending on $\alpha_1, \dots, \alpha_\ell$ as follows: if for $\varepsilon \in (0, \varepsilon_0)$ it is possible to calculate a constant $c(\varepsilon) > 0$ such that the inequality*

$$\int\limits_{Q_R(z)} f \, dx \leq \varepsilon \int\limits_{Q_{2R}(z)} f \, dx + c(\varepsilon) \sum_{j=1}^{\ell} R^{-\alpha_j} \int\limits_{Q_{2R}(z)} f_j \, dx$$

holds for all squares $Q_{2R}(z) \Subset Q_0$, *then there is a constant* $c > 0$ *with the property*

$$\int_{Q_R} f\,dx \leq c \sum_{j=1}^{\ell} R^{-\alpha_j} \int_{Q_{2R}(z)} f_j\,dx$$

again for all $Q_{2R}(z) \Subset Q_0$.

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