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Ernst-Ulrich Gekeler and Philipp Stopp

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Ernst-Ulrich Gekeler

Saarland University
Department of Mathematics
Campus E2 4
66123 Saarbrücken
Germany
gekeler@math.uni-sb.de

Philipp Stopp

Saarland University
Department of Mathematics
Campus E2 4
66123 Saarbrücken
Germany
stopp@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

ON THE ZEROES OF CERTAIN PERIODIC FUNCTIONS OVER VALUED FIELDS OF POSITIVE CHARACTERISTIC

ERNST-ULRICH GEKELER AND PHILIPP STOPP

Dedicated to the memory of David R. Hayes,
great mathematician and good friend

Abstract. Let $\mathbf{C}$ be an algebraically closed field of positive characteristic p and complete with respect to a non-archimedean absolute value $|\ .\ |$ and $\Lambda \subset \mathbf{C}$ a discrete $\mathbb{F}_p$-submodule. Suppose there exists an $\mathbb{F}_p$-basis $\{\lambda_0, \lambda_1, \ldots\}$ of Λ such that $0 < |\lambda_0| < |\lambda_1| < \cdots$ and $|\lambda_i| \longrightarrow \infty$. For $k \in \mathbb{N}$ define the meromorphic function

$$C_{k,\Lambda}(z) = \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^k}$$

on $\mathbf{C}$. We show that all the zeroes x of $C_{k,\Lambda}$ satisfy

$$(*) \qquad |x| = |\lambda_i|$$

for some i. Furthermore, the number (counted with multiplicities) of zeroes for which $(*)$ holds depends only on i and the p-adic expansion coefficients of k, but not on Λ.

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1. Introduction.

Let $\mathbf{C}$ be an algebraically closed field of finite characteristic p provided with a non-archimedean absolute value $|\ .\ | : \mathbf{C} \longrightarrow \mathbb{R}_{\geq 0}$, and complete with respect to $|\ .\ |$. For simplicity we suppose that the value group $|\mathbf{C}^*| \subset \mathbb{R}^*_{>0}$ is isomorphic with $\mathbb{Q}$, of shape $p^{\mathbb{Q}}$. (This assumption is for technical purposes only: We could as well admit more general value groups, at the cost of some notational complication.)

For some $\rho \in |\mathbf{C}^*|$ we write $B(x,\rho)$ resp. $B^+(x,\rho)$ resp. $\partial B(x,\rho)$ for the "open" resp. "closed" ball of radius ρ around $x \in \mathbf{C}$, resp. the circumference of $B(x,\rho)$, i.e., the set of $y \in \mathbf{C}$ that satisfy $|x-y| < \rho$ resp. $|x-y| \leq \rho$ resp. $|x-y| = \rho$.

An $\mathbb{F}_p$-subspace λ of $\mathbf{C}$ is called a *lattice* if it satisfies

(1.1) $\Lambda \cap B(0,\rho)$ is finite for each ρ. In particular, each finite $\mathbb{F}_p$-subspace Λ of $\mathbf{C}$ is a lattice.

1.2 Example. Let $A = \mathbb{F}_q[T]$ be the polynomial ring in an indeterminate T over a finite extension $\mathbb{F}_q$ of $\mathbb{F}_p$, with field of fractions $K = \mathbb{F}_q(T)$ and completion $K_\infty = \mathbb{F}_q((T^{-1}))$ at the infinite place of K. Then the completed algebraic closure C_∞ of K_∞ (with respect to the canonical absolute value $|\ .\ |$ of K_∞, normalized by $|T| = q$) is of the type $\mathbf{C}$ considered. In this case $\Lambda := A$ is a lattice.

(1.3) More general A-submodules of C_∞ appear as lattices as follows. Let $\{\omega_1, \ldots, \omega_r\}$ be K_∞-linearly independent elements of C_∞. Then $\Lambda := A\omega_1 + \cdots + A\omega_r$ is a lattice, which in fact gives rise to a Drinfeld A-module over C_∞ (see [1], [10], [8]).

(1.4) The polynomial ring A in (1.2) and (1.3) may be replaced by a more general Drinfeld ring (i.e., the affine ring of functions regular away from some closed point ∞ of $\mathfrak{X}$, where $\mathfrak{X}$ is a smooth connected projective curve over a finite field $\mathbb{F}_q$). We could for example take the affine ring $A = \mathbb{F}_q[X,Y]/(f(X,Y))$ of an elliptic curve over $\mathbb{F}_q$ defined by an equation

$$f(X,Y) = Y^2 + a_1XY + a_3Y - X^3 - a_2X^2 - a_4X - a_6 = 0.$$

Again, A is a lattice in $\mathbf{C} := C_\infty$, the completed algebraic closure of K_∞, where K_∞ itself is the completion of $K = \mathrm{Fract}(A)$ at its infinite place.

David Hayes, in [9], was able to use this set-up to give an explicit description of the abelian class field theory of K.

1.5 Definition. *A lattice $\Lambda \subset \mathbf{C}$ is* separable *if there is an $\mathbb{F}_p$-basis $\{\lambda_0, \lambda_1, \lambda_2, \ldots\}$ of Λ (a* separating basis*) such that $0 < |\lambda_0| < |\lambda_1| < \ldots$. If Λ is separable, we write $|\lambda_i| = p^{r_i}$ and call it the i-th* critical radius *and $\mathbf{S}_{i,\Lambda} := \partial B(0, p^{r_i})$ the i-th* critical sphere *of Λ. We further let $\Lambda_i := \langle \lambda_0, \ldots, \lambda_i \rangle_{\mathbb{F}_p}$ be the $\mathbb{F}_p$-span of $\lambda_0, \ldots, \lambda_i$. Then r_i, $\mathbf{S}_i$ and $\Lambda_i = \{\lambda \in \Lambda \mid |\lambda| \leq |\lambda_i|\}$ are invariants of Λ, i.e., independent of the choice of basis.*

1.6 Remark. The lattice $A = \mathbb{F}_q[T]$ in (1.2) is separable if and only if q equals the prime p. A Drinfeld ring A as in (1.4) is separable as a lattice if and only if $q = p$ and the distinguished point ∞ is $\mathbb{F}_p$-rational. (The latter is an easy consequence of the Riemann-Roch theorem.)

1.7 Definition. *Given a lattice Λ in $\mathbf{C}$ and $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, we define the following functions, where $z \in \mathbf{C}$, $z \notin \Lambda$ for (ii), (iii).*

(i) $e_\Lambda(z) := z \prod'_{\lambda \in \Lambda}(1 - z/\lambda)$ *(where we use the convention that $\prod'$ indicates the product over the non-zero elements of the domain);*

(ii) $t_\Lambda(z) := \frac{1}{e_\Lambda(z)}$;

(iii) $C_{k,\Lambda}(z) := \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^k}$.

It is fairly standard (e.g. [1], II, Theorem 2.1) and easy to verify that:

(1.8) • the product in (i) converges, locally uniformly, and defines an additive and Λ-periodic function $e_\Lambda : \mathbf{C} \longrightarrow \mathbf{C}$, surjective with kernel Λ;
• e_Λ is entire, represented through an everywhere convergent power series (labelled by the same symbol) $e_\Lambda(X) = \sum_{i\geq 0} \alpha_i X^{p^i}$ with $\alpha_0 = 1$;
• $t_\Lambda(z) = \sum_{\lambda\in\Lambda} \frac{1}{z-\lambda}$, i.e., $t_\Lambda = C_{1,\Lambda}$;
• the sum for $C_{k,\Lambda}(z)$ converges, uniformly on closed balls disjoint from Λ, thereby defining a meromorphic and Λ-periodic function $\mathbf{C}$ with the obvious poles at $\lambda \in \Lambda$ and no further poles.

Let Λ be a lattice and $(\Lambda_i)_{i\geq 0}$ a monotonically increasing sequence of finite sublattices with $\bigcup \Lambda_i = \Lambda$. Then $e_i := e_{\Lambda_i}$ are polynomials, and standard estimates show that

(1.9) $e_i \longrightarrow e_\Lambda$ both coefficient-wise and locally uniformly as functions on $\mathbf{C}$. Similar statements hold for t_Λ and $C_{k,\Lambda}$. Therefore, for many questions about e_Λ and $C_{k,\Lambda}$ we may restrict to considering finite lattices Λ.

Functions of type $C_{k,\Lambda}$ for convenient lattices Λ occur in many different contexts in number theory, arithmetic geometry, or the theory of finite fields (see sect. 7 for some examples). A weak form of our main result about the $C_{k,\Lambda}$ is as follows (the strong form is Theorem 6.1, whose formulation needs some preparation).

1.10 Theorem. *Let Λ be a separable lattice in $\mathbf{C}$. Then all the zeroes of $C_{k,\Lambda}$ lie on critical spheres $\mathbf{S}_{i,\Lambda}$ of Λ.*

In fact, Theorem 6.1 gives an accurate account of the number of zeroes of $C_{k,\Lambda}$ on $\mathbf{S}_{i,\Lambda}$, which depends only on i and k, but not on Λ.

The plan of the paper is as follows.

In section 2 we collect some facts about p-adic expansions. In section 3 we discuss power sums of elements of finite sublattices of a separable lattice Λ. We introduce the Goss polynomials attached to Λ in section 4, with a brief view to Eulerian polynomials, their characteristic-0 counterparts. Section 5 is devoted to the study of the "fundamental domain" $\mathcal{F}_\Lambda$ and the growth of e_Λ along $\mathcal{F}_\Lambda$. We present the proof (along, of course, with the exact statement) of Theorem 6.1 in section 6. It follows the strategy developed in [4], where the special case of the lattice $\Lambda = A = \mathbb{F}_p[T]$ has been treated. We conclude in section 7 with some examples and applications, and with an outlook to possible generalizations and further research problems.

Notation. We use $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and self-explaining symbols like $\mathbb{Q}_{\geq 0} = \{a \in \mathbb{Q} \mid a \geq 0\}$. The letters h, i, j, k, m, n usually denote elements of $\mathbb{N}_0$, $[\,]$ is the "greatest integer" Gauß bracket, and $\log = \log_p$ is the logarithm to base p.

2. p-adic digits.

We first recall the classical Lucas congruence. Let p be a fixed prime number and $m, n \in \mathbb{N}_0$, given in their p-adic expansions $m = \sum_{i\geq 0} m_i p^i$, $n = \sum_{i \geq 0} n_i p^i$ with $m_i, n_i \in \{0, 1, \ldots, p-1\}$. Then (Lucas):

$$\binom{n}{m} \not\equiv 0 \,(\mathrm{mod}\, p) \Leftrightarrow \forall i \geq 0 :\ m_i \leq n_i \tag{2.1}$$

We therefore define the order $<_p$ in $\mathbb{Z}_p = p$-adic integers by

$$x <_p y :\Leftrightarrow \forall i :\ x_i \leq y_i \tag{2.2}$$

for the p-adic digits x_i, y_i of $x, y \in \mathbb{Z}_p$, respectively. Accordingly, if $m_0, m_1, \ldots, m_h \in \mathbb{N}_0$ add up to n, then the multinomial coefficient

$$\binom{n}{\mathbf{m}} := \binom{n}{m_0, \ldots, m_h} := \frac{n!}{m_0! \cdots m_h!}$$

satisfies

(2.3) $\binom{n}{\mathbf{m}} \not\equiv 0 \,(\mathrm{mod}\, p)$ if and only if the p-adic digits of $m_0, \ldots, m_h$ sum up to those of n (i.e., there is no carry-over).

We let

$$\ell(n) := \sum_{i \geq 0} n_i \tag{2.4}$$

be the sum of p-adic digits of n, which satisfies

$$n \equiv \ell(n) \,(\mathrm{mod}\, p-1). \tag{2.5}$$

The following is motivated from the discussion of power sums in section 3. Let "$\prec$" be the relation on $\mathbb{N}_0$ defined by

(2.6) $m \prec n :\Leftrightarrow$ (i) $m < n$; (ii) $m \equiv n \,(\mathrm{mod}\, p-1)$; (iii) $m <_p n$.
Then "$\prec$" is transitive, and we define the *height* of $n \in (p-1)\mathbb{N}_0$ as

(2.7) $ht(n) :=$ largest h such that there exists a chain $0 = n^{(h)} \prec n^{(h-1)} \prec \cdots \prec n^{(0)} = n$.

2.8 Proposition. *Let $n \in \mathbb{N}_0$ be divisible by $(p-1)$. Then $ht(n) = \ell(n)/(p-1)$.*

Proof. By (2.5), $\ell(n)$ is divisible by $(p-1)$. Right from (2.6), each $m \prec n$ is obtained from n by decreasing all the p-adic digits n_i of n to those, m_i, of m, where $\sum_i (n_i - m_i)$ is a multiple of $(p-1)$. $\square$

Regard a p-adic number $x \in \mathbb{Z}_p$ in its p-adic expansion

$$x = \sum_{i \geq 0} x_i p^i = \sum_{\nu \geq 1} p^{i_\nu}$$

($x_i \in \{0, 1, \ldots, p-1\}$) as an infinite sum of powers of p (possibly adding zeroes if the sum is finite), arranged in increasing order $i_1 \leq i_2 \leq \cdots$, where the exponent $i_\nu = i$ occurs precisely x_i times. Put for $h \in \mathbb{N}_0$:

$$\tau_h(x) := \sum_{h(p-1) < \nu \leq (h+1)(p-1)} p^{i_\nu}, \tag{2.9}$$

provided that there are at least $(h+1)(p-1)$ summands p^{i_ν}. Then $\lim_{h \to \infty} \sum_{0 \leq j \leq h} \tau_i(x) = x$, and the partial sum $\sum_{0 \leq j \leq h} \tau_j(x)$ is the best approximation of x through a sum of $(h+1)(p-1)$ powers of p.

Now fix $k \in \mathbb{N}$ for the rest of this section. We are mainly interested in binomial coefficients $\binom{k-1-n}{n}$ and their non-vanishing $(\mathrm{mod}\, p)$. If we thus put

$$k - 1 = \sum_{i \geq 0} k_i p^i \quad \text{(note the shift by 1!)} \tag{2.10}$$

with $k_i \in \{0, 1, \ldots, p-1\}$ and $\ell_i := p - 1 - k_i$, then

- $\ell_i = p - 1$ for $i \gg 0$;
- the infinite sum $(k-1)^* := \sum \ell_i p^i$ describes the p-adic expansion of $-k$, and

$$\binom{k-1-n}{n} \not\equiv 0 \, (\mathrm{mod}\, p) \Leftrightarrow n <_p (k-1)^*. \tag{2.11}$$

(We prefer not to write $n <_p -k$ since this might lead to confusion, as "$<_p$" is not compatible with the minus sign.) We finally put

$$\sigma_h(k) := \sum_{0 \leq j \leq h} \tau_j((k-1)^*). \tag{2.12}$$

2.13 Example. Let $p = 3$, $k = 43 = 1 + 2 \cdot 3 + 3^2 + 3^3$. Then $\tau_0(k) = 1 + 3$, $\tau_1(k) = 3 + 3^2$, $\tau_2(k)$ is not defined, $(k-1)^* = 2 + 0 \cdot 3 + 3^2 + 3^3 + 2 \cdot 3^4 + 2 \cdot 3^5 + \cdots$.
$\sigma_0(k) = 2$, $\sigma_1(k) = 2 + 3^2 + 3^3$, $\sigma_2(k) = 2 + 3^2 + 3^3 + 2 \cdot 3^4, \ldots$.

2.14 Proposition. *Let $\tilde{n} = \tilde{n}(k, h)$ be the least natural number n divisible by $p - 1$ such that (i) $\binom{k-1-n}{n} \not\equiv 0 \, (\mathrm{mod}\, p)$ and (ii) $ht(n) \geq h + 1$ with $h \geq 0$ hold. Then $\tilde{n} = \sigma_h(k)$.*

Proof. Property (i) is equivalent with $n <_p (k-1)^*$ by (2.11). Further, $\sigma_h(k)$ has height $h + 1$ by (2.8), and is the least such number that satisfies (i), as follows from the definition (2.12). □

3. Power sums.

Fix a separable lattice $\Lambda \neq \{0\}$ in $\mathbf{C}$ with separating basis $\{\lambda_0, \lambda_1, \ldots\}$ as in (1.5), $|\lambda_i| = p^{r_i}$, with $r_0 < r_1 < \cdots$. We have $\Lambda_i = \langle \lambda_0, \ldots, \lambda_i \rangle_{\mathbb{F}_p}$ for $i \geq 0$, as long as λ_i exists, i.e., as long as $i+1 \leq \dim \Lambda$ if the latter is finite. We define for $i > 0$, $n \geq 0$:

$$S_{i,\Lambda}(n) := \sum_{\lambda \in \Lambda_{i-1}} \lambda^n. \tag{3.1}$$

In order to calculate $|S_{i,\Lambda}(n)|$, we first observe

$$\sum_{c \in \mathbb{F}_p} c^n = \begin{cases} -1 & \text{if } 0 < n \equiv 0 \,(\operatorname{mod} p-1) \\ \;\;0 & \text{if } n = 0 \text{ or } n \not\equiv 0 \,(\operatorname{mod} p-1). \end{cases} \tag{3.2}$$

Therefore,

$$S_{1,\Lambda}(n) = -\lambda_0^n \text{ if } 0 < n \equiv 0 \,(\operatorname{mod} p-1). \tag{3.3}$$

3.4 Proposition. *$S_{i,\Lambda}(n)$ vanishes if*

(i) $n \not\equiv 0 \,(\operatorname{mod} p-1)$ *or*

(ii) $i > ht(n) = \ell(n)/(p-1)$.

(iii) *Otherwise (that is, if $n \equiv 0 \,(\bmod\; p-1)$ and $ht(n) \geq i$), $S_{i,\Lambda}(n) \neq 0$ and* $\log |S_{i,\Lambda}(n)| = \tau_0(n)r_0 + \cdots + \tau_{i-2}(n)r_{i-2} + (n - \sum_{0 \leq j \leq i-2} \tau_j(n))r_{i-1}$.

Proof. (i) is standard: Let $c \in \mathbb{F}_p^*$ have multiplicative order $p-1$. Summing over $c\Lambda = \Lambda$, we find $(c^n - 1)S_{i,\Lambda}(n) = 0$, which forces $S_{i,\Lambda}(n) = 0$.
For (ii) and (iii) we calculate

$$S_{i,\Lambda}(n) = \sum_{\mathbf{c}} (c_0\lambda_0 + \cdots + c_{i-1}\lambda_{i-1})^n,$$

where the sum is over all i-tuples $\mathbf{c} = (c_0, \ldots, c_{i-1}) \in \mathbb{F}_p^i$. With the notation $\mathbf{m} = (m_0, \ldots, m_{i-1})$, which runs through the i-tuples of non-negative integers that sum up to n, $\binom{n}{\mathbf{m}} = \frac{n!}{m_0! \cdots m_{i-1}!}$, $\lambda^{\mathbf{m}} := \lambda_0^{m_0} \cdots \lambda_{i-1}^{m_{i-1}}$ and similarly for $\mathbf{c}^{\mathbf{m}}$, we find

$$S_{i,\Lambda}(n) = \sum_{\mathbf{c}} \sum_{\mathbf{m}} \binom{n}{\mathbf{m}} \mathbf{c}^{\mathbf{m}} \lambda^{\mathbf{m}} = \sum_{\mathbf{m}} \binom{n}{\mathbf{m}} \lambda^{\mathbf{m}} \sum_{\mathbf{c}} \mathbf{c}^{\mathbf{m}}.$$

Now (3.2) shows that the inner sum $\sum_{\mathbf{c}} \mathbf{c}^{\mathbf{m}}$ vanishes unless all $m_0, \ldots, m_{i-1}$ are positive and divisible by $p-1$, in which case it equals $(-1)^i$. Hence

$$S_{i,\Lambda} = (-1)^i \sum_{\mathbf{m}} \binom{n}{\mathbf{m}} \lambda^{\mathbf{m}}, \tag{3.5}$$

where $\mathbf{m}$ runs through the set of $(m_0, \ldots, m_{i-1}) \in (p-1)\mathbb{N}^i$ subject to $m_0 + \cdots + m_{i-1} = n$. If now $i > \ell(n)/(p-1) = ht(n)$ holds, the sum is empty, due to the definition of $ht(n)$, which gives (ii).
Statement (iii) holds for $i = 1$, due to (3.3). Thus we may assume that $i \geq 2$. Next, we have $\log |\lambda^{\mathbf{m}}| = m_0 r_0 + m_1 r_1 + \cdots + m_{i-1} r_{i-1}$. Under

the constraint that $\binom{n}{\mathbf{m}} \not\equiv 0 \,(\mathrm{mod}\, p)$, and taking (2.3) into account, $\log|\lambda^{\mathbf{m}}|$ is maximal if

$$\begin{array}{llll} m_0 \in (p-1)\mathbb{N} & \text{is minimal} & <_p n \\ m_1 \in (p-1)\mathbb{N} & \text{“} & <_p n - m_0 \\ m_{i-2} \in (p-1)\mathbb{N} & \text{“} & <_p n - \sum_{0\le j<i-2} m_j \\ m_{i-1} = n - \sum_{0\le j\le i-2} m_j, & & \end{array}$$

i.e., if $m_0 = \tau_0(n), \ldots, m_{i-2} = \tau_{i-2}(n)$, $m_{i-1} = n - \sum_{0\le j\le i-2} \tau_j(n)$. In this case, the corresponding term in (3.5) is strictly larger than the other terms. This shows (iii). □

4. Goss polynomials.

We recall without proofs some definitions and facts from [7] ch. 6, [3] sect. 3, [4] sect. 2. Given a lattice Λ, there exists a unique series of polynomials $G_{k,\Lambda}(X)$ such that

$$C_{k,\Lambda}(z) = G_{k,\Lambda}(t_\Lambda(z)) \tag{4.1}$$

holds as an identity of meromorphic functions on $\mathbf{C}$. These *Goss polynomials* $G_k = G_{k,\Lambda}$ satisfy

$$\text{(4.2)} \quad \begin{array}{ll} \text{(i)} & G_k \text{ is monic of degree } k \text{ with } G_k(0) = 0; \\ \text{(ii)} & G_{pk} = (G_k)^p; \\ \text{(iii)} & G_k(X) = X^k \text{ for } k \le p; \\ \text{(iv)} & G_k(X) = X(G_{k-1} + \alpha_1 G_{k-p} + \alpha_2 G_{k-p^2} + \cdots), \end{array}$$

where $G_k = 0$ for $k \le 0$ and the $\alpha_i = \alpha_i(\Lambda)$ are the coefficients of $e_\Lambda(z) = \sum_{i\ge 0} \alpha_i z^{p^i}$ (see (1.8)). In particular, G_k has its coefficients in $\overline{\mathbb{F}_p(\Lambda)}$, the closure of the subfield of $\mathbf{C}$ generated by the elements of Λ. The behavior of the above quantities under scaling the lattice $\Lambda \rightsquigarrow c\Lambda$ with $0 \ne c \in \mathbf{C}$ is described by

$$\text{(4.3)} \quad \begin{array}{llcl} \text{(i)} & e_{c\Lambda}(cz) & = & c e_\Lambda(z); \\ \text{(ii)} & \alpha_k(c\Lambda) & = & c^{1-p^k}\alpha_k(\Lambda); \\ \text{(iii)} & C_{k,c\Lambda}(cz) & = & c^{-k} C_{k,\Lambda}(z); \\ \text{(iv)} & G_{k,c\Lambda}(c^{-1}X) & = & c^{-k} G_{k,\Lambda}(X). \end{array}$$

(4.4) Although the following will not be used in the present paper, it might be helpful to be aware of the “classical” analogues of Goss polynomials. In the often stressed dictionary number fields ↔ global functions fields, we have corresponding entries

$$\begin{array}{lcccl} \mathbb{Z} & \leftrightarrow & A & = & \mathbb{F}_q[T] \\ \mathbb{Q} & \leftrightarrow & K & = & \mathbb{F}_q(T) \\ \mathbb{R} & \leftrightarrow & K_\infty & = & \mathbb{F}_q((T^{-1})) \\ \mathbb{C} & \leftrightarrow & C_\infty & = & \text{completed algebraic} \\ & & & & \text{closure of } K_\infty. \end{array}$$

Accordingly, the uniformizer at ∞ (which occurs in the theory of modular forms) $q(z) := e^{2\pi i z}$ is $\mathbb{Z}$-periodic, while the uniformizer (for Drinfeld modular forms) $t_A(z) := \frac{1}{e_A(z)}$ is A-periodic. The complex-analytic function (say, on the complex upper half-plane)

$$C_{k,\mathbb{Z}}(z) := \sum_{a \in \mathbb{Z}} \frac{1}{(z-a)^k}$$

(suppose $k \geq 2$ to ensure locally uniform convergence) may be written as a rational function

$$C_{k,\mathbb{Z}}(z) = \epsilon_k \frac{E_k(q(z))}{(1-q(z))^k}$$

in $q(z)$, where ϵ_k is a constant and $E_k(X) \in \mathbb{Z}[X]$ is some sort of Eulerian polynomial (see [13]). More specifically, $\epsilon_k = \frac{(-2\pi i)^{k-1}}{(k-1)!}$, and E_k may be recursively determined through $E_2(X) = X$,

$$E_{k+1}(X) = E_k'(X)X(1-X) + kXE_k(X),$$

from which one easily derives that E_k is monic of degree $k-1$,

$$E_3(X) = X + X^2,\ E_4(X) = X + 4X^2 + X^3, \ldots$$

In particular, $C_{k,\mathbb{Z}}(z) = 0 \Leftrightarrow q(z)$ is a zero of $E_k(X)$, in perfect analogy with the property of Goss polynomials

$$C_{k,\Lambda}(z) = 0 \Leftrightarrow t_\Lambda(z) \text{ is a zero of } G_{k,\Lambda}(X).$$

5. The fundamental domain $\mathcal{F}_\Lambda$.

Let $\Lambda \neq \{0\}$ be a separable lattice with separating basis $\{\lambda_0, \lambda_1, \ldots\}$. We first assume that Λ is infinite and describe later the modifications necessary to cover the finite case. As a first step toward a proof of Theorem 1.10, we note:

5.1 Proposition. *$C_{k,\Lambda}$ has no zeroes z with $|z| < |\lambda_0|$.*

Proof. Suppose $0 < |z| < |\lambda_0|$. Then $C_{k,\Lambda}(z) = z^{-k}+$ smaller terms. □

We may thus focus on z with $|z| \geq |\lambda_0|$ is our search for zeroes of $C_{k,\Lambda}$. Without restriction (see (4.3)), we may further assume that

(5.2) $$\lambda_0 = 1, \text{ so } r_0 = 0.$$

For $z \in \mathbf{C}$ with $|z| \geq 1$ we put

(5.3) $$|z|_{\min} := \inf_{\lambda \in \Lambda} |z - \lambda| = \min_{\lambda \in \Lambda} |z - \lambda|.$$

Then $|z| = |z|_{\min}$ if $|z|$ doesn't belong to the value set $\{|\lambda_i| \mid i \geq 0\}$. We further define

$$\begin{array}{rcl} \Omega_\Lambda & := & \{z \in \mathbf{C} \mid |z|_{\min} \geq 1\} \text{ and} \\ \mathcal{F}_\Lambda & := & \{z \in \mathbf{C} \mid |z| = |z|_{\min} \geq 1\}, \end{array} \tag{5.4}$$

which are rigid analytic subspaces of affine space $\mathbb{A}^1/\mathbf{C} \subset \mathbb{P}^1/\mathbf{C}$ as discussed in [6] and [2]. In fact, both are unions of ascending sequences of rational subdomains of $\mathbb{P}^1/\mathbf{C}$. The group Λ acts on Ω_Λ through shifts, and $\mathcal{F}_\Lambda$ is a fundamental domain in the sense that each element of Ω_Λ is Λ-equivalent with at least one and at most finitely many elements of $\mathcal{F}_\Lambda$. Given $z \in \Omega_\Lambda$ Λ-equivalent with $z' \in \mathcal{F}_\Lambda$, we define

(5.5)
$$\text{type}(z) := \begin{cases} i & \text{if } |z'| = |\lambda_i| = p^{r_i} \text{ with } i \in \mathbb{N}_0 \\ i+a & \text{if } |z'| = p^{r_i(1-a)+ar_{i+1}} \text{ with } a \in \mathbb{Q},\ 0 < a < 1. \end{cases}$$

Then type: $\Omega_\Lambda \longrightarrow \mathbb{Q}_{\geq 0} :\ z \longmapsto \text{type}(z)$ is a well-defined, Λ-invariant and surjective mapping. As C_k is Λ-periodic and, by (5.1), all the zeroes of $C_{k,\Lambda}$ lie in Ω_Λ, Theorem 1.10 may be rephrased as saying that all the zeroes of $C_{k,\Lambda}$ are of integer type.

5.6 Proposition. *The function t_Λ restricted to $\mathcal{F}_\Lambda$ defines a biholomorphic isomorphism of the quotient $\mathcal{F}_\Lambda/\Lambda$ (i.e., of the image of $\mathcal{F}_\Lambda$ in $\mathbf{C}/\Lambda$, which equals Ω_Λ/Λ) with the pointed closed ball $B^+(0,1) - \{0\}$.*

Proof. As each non-constant entire function is surjective and e_Λ is Λ-periodic and additive, we have an isomorphism

$$e_\Lambda :\ \mathbf{C}/\Lambda \xrightarrow{\cong} \mathbf{C},$$

which gives

$$t_\Lambda :\ \mathbf{C}/\Lambda - \{0\} \xrightarrow{\cong} \mathbf{C}^*.$$

If $z \in \mathbf{C}$ is represented $(\bmod\ \Lambda)$ by some z' with $|z'| < 1$, then $|e_\Lambda(z)| < 1$, so $|t_\Lambda(z)| > 1$. On the other hand, $z \in \Omega_\Lambda$ immediately implies $|e_\Lambda(z)| \geq 1$, i.e., $|t_\lambda(z)| \leq 1$. Hence

$$\mathcal{F}_\Lambda/\Lambda \xrightarrow{\cong} B^+(0,1) - \{0\}.$$

□

5.7 Proposition. *For $z \in \Omega_\Lambda$ the absolute value $|t_\Lambda(z)|$ depends only on type (z). The induced map*

$$\begin{array}{rrl} L: & \mathbb{Q}_{\geq 0} & \longrightarrow \mathbb{Q}_{\leq 0} \\ & s & \longmapsto \log|t_\Lambda(z)|, \text{ where type}(z) = s, \end{array}$$

is bijective and strictly monotonically decreasing. Its values on integer types are given by

$$L(i) = (p-1) \sum_{0 \leq j < i} r_j p^j - p^i r_i \quad (i \in \mathbb{N}_0).$$

(Recall that "log" = "$\log_p$".)

Proof. Without restriction, $z \in \mathcal{F}_\Lambda$. Then

$$|t_\Lambda(z)|^{-1} = |e_\Lambda(z)| = |z| \prod_{\lambda \in \Lambda}{}' |1 - z/\lambda| = |z| \prod_{\substack{\lambda \in \Lambda, \\ |\lambda| < |z|}}{}' |1 - z/\lambda| = |z| \prod_{|\lambda| < |z|}{}' |z/\lambda|,$$

as factors corresponding to λ with $|\lambda| \geq |z|$ have absolute value 1. (This is obvious for $|\lambda| > |z|$, and follows for $|\lambda| = |z|$ from $|z| = |z|_{\min}$.) Hence we have to evaluate the finite product on the right hand side, from which the first two assertions are obvious. The value for $L(i)$ comes from an omitted elementary computation, taking into account that in the finite lattice Λ_i there are precisely $(p-1)p^j$ elements λ with $|\lambda| = p^{r_j}$ if $j \leq i$. □

5.8 Corollary. *The following statements for $k \in \mathbb{N}$ are equivalent:*

(i) *Theorem 1.10 holds for $C_{k,\Lambda}$, i.e., all its zeroes lie on critical spheres $\mathbf{S}_i$ of Λ;*
(ii) *all the zeroes of $C_{k,\Lambda}$ are of integer type $i \in \mathbb{N}_0$;*
(iii) *all the zeroes $x \neq 0$ of $G_{k,\Lambda}$ satisfy $\log|x| = L(i)$ with some $i \in \mathbb{N}_0$;*
(iv) *all the slopes of the Newton polygon of $G_{k,\Lambda}$ are of shape $L(i)$ for some $i \in \mathbb{N}_0$.*

Proof. The equivalence (i) ⇔ (ii) results from the fact that $C_{k,\Lambda}$ is Λ-periodic, (iii) ⇔ (iv) is the characterizing property of the Newton polygon (we use the definition and conventions of [11], II sect. 6), and (i) ⇔ (iii) comes from the definition of $G_{k,\Lambda}$ along with (5.7). □

(5.9) Finally we consider the case where Λ is finite, of dimension $m+1$ with separating basis $\{\lambda_0 = 1, \lambda_1, \ldots, \lambda_m\}$. In this case, the definitions of $|z|_{\min}$, Ω_Λ, $\mathcal{F}_\Lambda$ remain unchanged, ditto for type (z) as long as $|z|_{\min} \leq |\Lambda_m|$. If $|z|_{\min} = p^{r_m + a}$ with $a \in \mathbb{Q}_{>0}$, we put type $(z) := m + a$. Then Proposition 5.6 and 5.7 and Corollary 5.8 still hold, where we have to restrict to integers $i \leq m$ for the value of $L(i)$ in Proposition 5.7, and add the condition $i \leq m$ in the four assertions of Corollary 5.8.

6. Statement and proof of the main result.

We keep the setting and notations of the last section: Λ is an infinite separable lattice with separating basis $\{\lambda_0 = 1, \lambda_1, \ldots\}$, $|\lambda_i| = p^{r_i}$. The case of a finite Λ will be discussed afterwards. For simplicity the subscript Λ in $e = e_\Lambda, \ldots, \mathcal{F} = \mathcal{F}_\Lambda$ will be omitted. As in section 2, we let

$$k - 1 = \sum_{0 \leq i \leq N} k_i p^i$$

be the p-adic expansion, where $k_N \neq 0$, $N = [\log(k-1)]$, $k \geq 2$. We may now state the announced result, which encompasses and refines

Theorem 1.10. "Number of zeroes" will always mean "number of zeroes counted with multiplicities".

6.1 Theorem.

(i) *The equivalent assertions of Corollary 5.8 hold. That is, all the zeroes* $x \neq 0$ *of* G_k *satisfy* $\log|x| = L(i)$ *for some* $i \in \mathbb{N}_0$ *with* $L(i)$ *as in (5.7). Accordingly, all the zeroes of* C_k *on* $\mathcal{F}$ *lie in some* $\mathcal{F}_i := \{z \in \mathcal{F} \mid |z| = p^{r_i}\}$.

(ii) *Let* $\tilde{\gamma}_i(k)$ *be the number of zeroes of* C_k *in* $\mathcal{F}_i$ *and* $\gamma_i(k)$ *the number of zeroes* x *of* G_k *with* $\log|x| = L(i)$*. Then*

$$\tilde{\gamma}_i(k) = p^{i+1}\gamma_i(k) = (p-1)k + p\sigma_{i-1}(k) - \sigma_i(k).$$

($\sigma_i(k)$ *is defined in (2.12),* $\sigma_{-1}(k) = 0$*.)*

(iii) *Let* $\overline{r}(k)$ *be the least integer* h *such that* $\sigma_{h-1}(k) + k \equiv 0 \pmod{p^N}$*, where* $N := [\log(k-1)]$*. Then* $\gamma_i(k) = 0$ *for* $i \geq \overline{r}(k)$ *and* $\gamma_i(k) \neq 0$ *for* $0 \leq i < \overline{r}(k)$*.*

(iv) *Let* $\ell(k-1) = \sum_{0 \leq i \leq N} k_i$ *be the sum of* p*-adic digits of* $k-1$*, with representative* $R(k)$ *modulo* $p-1$ *in* $\{0, 1, \ldots, p-2\}$*. Then the multiplicity* $\gamma(k)$ *of 0 as a zero of* $G_k(X)$ *is given by*

$$\gamma(k) = (R(k)+1)p^{[\ell(k-1)/(p-1)]}.$$

(Here and in (iii), [ ] means Gauß brackets.)

6.2 Remarks.

(i) As $\lim_{h\to\infty} \sigma_h(k) = -k$, the number $\overline{r}(k) \geq 0$ in (iii) is well-defined.

(ii) Note that the formulas in (ii), (iii), (iv) of the theorem refer only to the p-adic expansion of $k-1$, but not to Λ.

The proof, which will occupy this section, is based on properties of the power sums $S_{i,\Lambda}(n)$ of section 3 and non-archimedean contour integration (see [6] pp. 93–95). It generalizes the proof scheme of Theorem 6.12 of [4], which gives the assertion of our present theorem in the special case where the lattice Λ equals $A = \mathbb{F}_p[T]$ as in Example 1.2.

Let us (very) briefly recall the non-archimedean prerequisites (for more details, see [6], [2] and also [4]).

(6.3) Let $B = B(z_0, p^r)$ be an open ball with corresponding closed ball $B^+ = B^+(z_0, p^r)$ and boundary $\partial B = B^+ - B$, $r \in \mathbb{Q}$. Then ∂B is an analytic subspace of B^+. Let v be a coordinate function of ∂B of absolute value 1 (e.g., $v = \frac{z-z_0}{w}$, where $w \in \mathbf{C}$ is fixed with $|w| = p^r$). Each invertible (= holomorphic without zeroes) function f on ∂B may be written as a convergent Laurent series

$$f(z) = v^m \sum_{n\in\mathbb{Z}} a_n v^n \qquad (6.4)$$

with $|a_0| > \max_{n\neq 0} |a_n|$ and some $m =: \mathrm{ord}_{\partial B}(f) \in \mathbb{Z}$.

Conversely, any such series defines an invertible function on ∂B. Suppose that f is meromorphic on B^+, without zeroes or poles on ∂B. Then the residue type formula

$$\sum_{x\in B} \mathrm{ord}_x(f) = \mathrm{ord}_{\partial B}(f) \tag{6.5}$$

holds, where $\mathrm{ord}_x(f) \in \mathbb{Z}$ is the vanishing order of f in x (negative if f has a pole in x).

The idea, taken from [4], is to determine Laurent expansions of C_k on ∂B for balls $B = B(0, p^r)$ with *non-critical* radii p^r (i.e., $r \neq r_i$ for all i), and to isolate a dominant term as in (6.4). This will show that C_k cannot have zeroes on ∂B. The numerical formulas of Theorem 6.1 (ii), (iii), (iv) will then be derived by means of (6.5).

Let us first note that the finite group $\Lambda_i = \langle 1, \lambda_1, \ldots, \lambda_i\rangle_{\mathbb{F}_p}$ of order p^{i+1} acts through shifts on the sets $\{z \in \mathcal{F} \mid |z| \leq p^r\} = \{z \in \mathcal{F} \mid 0 \leq \mathrm{type}(z) \leq s\}$, provided that $r \geq r_i$. Here $s \in \mathbb{Q}_{\geq 0}$ is the type corresponding to r. Suppose that $r_i \leq r < r_{i+1}$, i.e., $i \leq s < i+1$. The restriction of t to the set

$$\mathcal{F}(s) := \{z \in \mathcal{F} \mid i \leq \text{type } (z) \leq s\} = \{z \in \mathcal{F} \mid p^{r_i} \leq |z| \leq p^r\}$$

is therefore p^{i+1}-to-one and maps $\mathcal{F}(s)$ to an annulus $\{z \in B^+(0,1) \mid p^{L(s)} \leq |z| \leq p^{L(i)}\}$, as follows from (5.6) and (5.7).

Next, we put provisionally (we will find *a posteriori* that the present $\tilde{\gamma}_i(k)$, $\gamma_i(k)$ agree with those in the statement of Theorem 6.1)

$$\begin{aligned} \tilde{\gamma}_i(k) &= \#\{\text{zeroes } z \text{ of } C_k \text{ in } \mathcal{F} \text{ s.t. } i \leq \text{type } (z) < i+1\} \\ \gamma_i(k) &= \#\{\text{zeroes } x \text{ of } G_k \text{ s.t. } p^{L(i+1)} < |x| \leq p^{L(i)}\}, \end{aligned}$$

where "#" means "number of zeroes counted with multiplicities". By the above

$$\tilde{\gamma}_i(k) = p^{i+1}\gamma_i(k). \tag{6.6}$$

6.7 Lemma. *For each $r \in \mathbb{Q}_{\geq 0}$, the number of poles of C_k in $B^+ = B^+(0, p^r)$ is strictly larger than the number of zeroes.*

Proof. Let $s \in \mathbb{Q}_{\geq 0}$ be the type corresponding to r, with integral part $i := [s] \in \mathbb{N}_0$. The poles of C_k in B^+ are the elements of Λ_i, each of multiplicity k, which gives $\#\{\text{poles}\} = kp^{i+1}$. Each zero $z \in B^+$ is Λ_i-equivalent with an element z' of $\mathcal{F}$. If $j \leq \text{type } (z') < j+1$ with $j \leq i$, there are p^{j+1} choices for z'. Therefore,

$$\begin{aligned} \#\{\text{ zeroes of } C_k \text{ on } B^+\} &= \textstyle\sum_{0\leq j<i} \#\{\text{zeroes } z \text{ with } j \leq \text{type } (z) < j+1\} \\ &\quad + \#\{\text{zeroes } z \text{ with } i \leq \text{type } (z) \leq s\} \\ &\leq \textstyle\sum_{0\leq j<i} \tilde{\gamma}_j(k)\frac{p^{i+1}}{p^{j+1}} + \tilde{\gamma}_i(k) \end{aligned}$$

(a possible strict inequality comes at most from the last term, namely if there exist zeroes z with $s <$ type $(z) < i+1$)

$$= \sum_{0 \leq j \leq i} \gamma_j(k) p^{i+1} < k \cdot p^{i+1}, \text{ since } \sum_{0 \leq j \leq i} \gamma_j(k) < k$$

(as the Goss polynomial $G_k(X)$ is always divisible by X). $\square$

6.8 Corollary. *Let $C_k(z) = \sum_{n \in \mathbb{Z}} a_n v^n$ be the Laurent expansion of C_k on ∂B, where B is as in (6.7) and $v = z/w$ with some fixed $w \in \mathbf{C}$ of absolute value p^r. Suppose that $|a_n| = \max\{|a_m| \mid m \in \mathbb{Z}\}$. Then $n < 0$.*

Proof. This is immediate from the lemma and (6.5), provided that a_n dominates, i.e., $|a_n| > |a_m|$ $\forall m \neq n$. If the maximum is attained for several values of n, the assertion results from an easy deformation argument ([4] Lemma 6.5). $\square$

(6.9) Fix now a non-critical radius p^r, where $r_i < r < r_{i+1}$, and the corresponding ball $B(0, p^r)$ with boundary ∂B and coordinate $v := z/w$ as above.

6.10 Proposition. *The Laurent expansion of $C_k(z)$ on ∂B is given by*

$$C_k(z) = \sum_{n \in \mathbb{Z}} a_n v^n,$$

where

$$\begin{array}{lcll} a_n & = & (-1)^k \binom{k-1+n}{n} w^n \sum_{\lambda \in \Lambda - \Lambda_i} \lambda^{-k-n} & (n \geq 0) \\ a_n & = & 0 & (-k \leq n < 0) \\ a_{-k-n} & = & \binom{k-1+n}{n} w^{-k-n} S_{i+1}(n) & (n > 0) \end{array}$$

with the power sums $S_i(n) = S_{i,\Lambda}(n)$ of section 3, and with binomial coefficients $\binom{k-1+n}{n}$ evaluated in $\mathbb{F}_p \hookrightarrow \mathbf{C}$.

Proof. The evaluation has been carried out in [4] (4.3)–(4.6) in the case $\Lambda = \mathbb{F}_p[T]$ with separating basis $\{\lambda_i\}$, where $\lambda_i = T^i$. It carries over - mutatis mutandis - to the general case considered here. $\square$

6.11 Proposition. *Let $\tilde{n}$ be the least positive integer n such that $a_{-k-n} \neq 0$. Then the term $a_{-k-\tilde{n}}$ dominates in the Laurent expansion of $C_k(z)$ on ∂B.*

Proof. In looking for maximal coefficients in $C_k(z) = \sum a_n v^n$, we may restrict to negative subscripts n by (6.8). By (6.10), (3.4) (iii) and (2.14), we have

$$\tilde{n} = \sigma_i(k) = \sum_{0 \leq j \leq i} \tau_j((k-1)^*).$$

The assertion is therefore equivalent with:

$$\begin{array}{ll} & |a_{-k-\tilde{n}}| > |a_{-k-n}| \quad (0 < n \neq \tilde{n}) \\ \Leftrightarrow & -\tilde{n}r + \log|S_{i+1}(\tilde{n})| > -nr + \log|S_{i+1}(n)| \\ & \forall n \in \mathbb{N} \text{ s.t. } n > \tilde{n},\ n \equiv 0 \,(\mathrm{mod}\, p-1),\ ht(n) \geq i+1,\ n <_p (k-1)^* \\ \Leftrightarrow & \log|S_{i+1}(n)| - \log|S_{i+1}(\tilde{n}| < r(n-\tilde{n}) \quad \forall n \text{ as above} \\ \Leftrightarrow & (\tau_0(n) - \tau_0(\tilde{n}))r_0 + \cdots + (\tau_{i-1}(n) - \tau_{i-1}(\tilde{n}))r_{i-1} + \\ & (n - \sum_{0\leq j\leq i-1} \tau_j(n) - \tilde{n} + \sum_{0\leq j \leq i-1} \tau_j(\tilde{n}))r_i < r(n-\tilde{n}) \quad \forall n \text{ as above} \\ \Leftrightarrow & (\tau_0(n) - \tau_0(\tilde{n}))(r_0 - r_i) + \cdots + (\tau_{i-1}(n) - \tau_{i-1}(\tilde{n}))(r_{i-1} - r_i) \\ & +(n-\tilde{n})r_i < r(n-\tilde{n}) \quad \forall n \text{ as above} \end{array}$$

Now by construction of $\tilde{n}$, the value $\tau_j(\tilde{n}) = \tau_j((k-1)^*)$ is minimal among all $\tau_j(n)$ with $n <_p (k-1)^*$, $0 \leq j \leq i$. Therefore, and by $0 = r_0 < r_1 < \cdots < r_i$, the last inequality is implied by $(n-\tilde{n})r_i < (n-\tilde{n})r$, which in turn follows from $r > r_i$. □

Proof of Theorem 6.1. (i) results directly from (6.11), since C_k is invertible on each non-critical sphere. In particular, we now know that the $\tilde{\gamma}_i(k)$, $\gamma_i(k)$ of (6.6) agree with those in the statement of the theorem.

(ii) The relation $\tilde{\gamma}_i(k) = p^{i+1}\gamma_i(k)$ comes from (i) and (6.6). Let again r be non-critical such that $r_i < r < r_{i+1}$. The proof of Lemma 6.7 shows that the number of zeroes of C_k in $B^+ = B^+(0, p^r)$ is $(\gamma_0(k) + \cdots + \gamma_i(k))p^{i+1}$. Hence

$$(k - \gamma_0(k) - \cdots - \gamma_i(k))p^{i+1} = -\mathrm{ord}_{\partial B}(C_k) = k + \tilde{n},$$

where $\tilde{n} = \tilde{n}(k,i) = \sigma_i(k)$ is the quantity that appears in Proposition 6.11. This allows to recursively solve for the $\gamma_i(k)$, which through an omitted standard calculation yields the stated formula. Note that the result does no longer depend on Λ, but only on the p-adic expansion of $k-1$.

(iii) and (iv). As $\gamma(k) = k - \sum_{i\geq 0} \gamma_i(k)$, the assertions of (iii) and (iv) involve only the expansion of $k-1$. The corresponding calculations are given in [4], proof of Theorem 6.12. (Note that the quantity $\lambda_i(k)$ *loc. cit.* agrees with the present $\sigma_{i-1}(k)$.) □

We finally explain the modifications necessary to deal with a finite lattice Λ. Thus assume that Λ is separable and finite of dimension $m+1$, with separating basis $\{\lambda_0 = 1, \lambda_1, \ldots, \lambda_m\}$. Then Theorem 6.1 remains valid "up to m". Below we state the complete result for this case, whose proof is - with some obvious changes - identical with that of Theorem 6.1.

6.12 Theorem.

(i) *All the zeroes $0 \neq x$ of G_k satisfy $\log|x| = L(i)$ for some $i \in \mathbb{N}_0$, $i \leq m$, with $L(i)$ as in (5.7). Accordingly, all the zeroes of C_k on $\mathcal{F}$ lie in some $\mathcal{F}_i = \{z \in \mathcal{F} \mid |z| = p^{r_i}\}$ with $0 \leq i \leq m$.*

(ii) *The numbers* $\tilde{\gamma}_i(k)$ *of zeroes of* C_k *in* $\mathcal{F}_i$ *and* $\gamma_i(k)$ *of zeroes of* G_k *with* $\log|x| = L(i)$ *are for* $0 \leq i \leq m$ *given by*

$$\tilde{\gamma}_i(k) = p^{i+1}\gamma_i(k) = (p-1)k + p\sigma_{i-1}(k) - \sigma_i(k).$$

(iii) *We have* $\tilde{\gamma}_i(k) = 0 = \gamma_i(k)$ *for* $i \geq \min(m+1, \overline{r}(k))$ *and* $\gamma_i(k) \neq 0$ *for* $i < \min(m+1, \overline{r}(k))$ *with the number* $\overline{r}(k)$ *of Theorem 6.1 (iii).*

(iv) *Suppose* $\overline{r}(k) \leq m+1$. *Then the multiplicity* $\gamma(k)$ *of 0 as a zero of* $G_k(X)$ *is given by the formula of Theorem 6.1 (iv). In any case,* $\gamma(k) = k - \sum_{0 \leq i \leq \min(m, \overline{r}(k)-1)} \gamma_i(k)$ *with the* $\gamma_i(k)$ *given in (ii).*

7. Examples and concluding remarks.

(7.1) We first point out that Theorem 6.1 applies to $\Lambda = A = \mathbb{F}_q[T]$ as in Example 1.2, under the assumption that the prime power q is in fact a prime. This is the special case treated in [4]. The relevant numbers $L(i)$ are then $L(i) = -q(\frac{q^i-1}{q-1})$. Here the Goss polynomials $G_{k,A}$ play an important part in the arithmetic of values of characteristic-p zeta and L-functions as studied in [8] Ch. 8, or in the theory of Eisenstein series for congruence subgroups of the modular group $\mathrm{GL}(2, A)$.

For example, the pattern $(\gamma_0(k), \gamma_1(k), \ldots)$ of zeroes of $G_{k,A}$ also governs the behavior of zeroes of the Eisenstein series $E_{\mathbf{u}}^{(k)}$ (see Theorem 3.1 of [5]). All of this generalizes to Drinfeld rings A different from $\mathbb{F}_p[T]$ (see (1.4)), provided that

- $\mathbb{F}_p$ is algebraically closed in A, and
- the distinguished place ∞ of $\mathfrak{X}$ has degree 1.

Under these conditions A is a separable lattice in the corresponding field C_∞, and (6.1) applies. Even more generally, we may allow A to be the affine ring of a non-smooth curve $\mathfrak{X}$ over $\mathbb{F}_p$, e.g., $A = \mathbb{F}_p[T^2, T^3]$.

We continue Example 2.13, which exhibits the main features of (6.1).

7.2 Example (= Example 2.13 continued; see [4] Example 6.14). Consider $G_{k,A}(X)$, where $A = \mathbb{F}_3[T]$ and $k = 43 = 1 + 2\cdot 3 + 3^2 + 3^3$, $k-1 = 2\cdot 3 + 3^2 + 3^3$, $(k-1)^* = 2 + 0\cdot 3 + 3^2 + 3^3 + 2\cdot 3^4 + 2\cdot 3^5 + \cdots$. Then $\ell(k-1) = 4$ and $R(k) = 0$. The formulas of (6.1) yield $\gamma_0(k) = 28$, $\gamma_1(k) = 6$, $\gamma_2(k) = \gamma_3(k) = \cdots = 0$, $\gamma(k) = 9$. Equivalently, the break points of the Newton polygon of A are (9,18), (15,5) and (43,0).

The next example, despite its simple appearance, gives rise to a non-trivial result hardly accessible by more elementary means.

7.3 Example. Let $\Lambda = \mathbb{F}_p$, so $e(z) = z\prod_{\lambda\in\mathbb{F}_p}(1 - z/\lambda) = z - z^p$, $C_k(z) = \sum_{\lambda\in\mathbb{F}_p} \frac{1}{(z-\lambda)^k} = G_k(\frac{1}{z-z^p})$. Then $G_k(X) = X^{\gamma(k)}H_k(X)$ with

$H_k(X) \in \mathbb{F}_p[X]$ and $H_k(0) \neq 0$. The exponent $\gamma(k)$ is $\gamma(k) = k - \sigma_0(k) = k - \tau_0((k-1)^*)$. It is an open problem to describe the splitting of $H_k(X)$ in $\mathbb{F}_p[X]$ and its total splitting field in dependence of k.

7.4 Problem. Another topic for further research is about the simplicity of zeroes of $G_k(X)$ (or of $C_k(z)$, which amounts to the same). From $G_{pk,\Lambda}(X) = (G_{k,\Lambda}(X))^p$ we have trivial multiplicities if k is divisible by p. On the other hand, we know of no single example of lattice Λ and subscript $k \not\equiv 0 \pmod{p}$ such that $G_{k,\Lambda}(X)$ presents multiple zeroes $x \neq 0$. Can this be proved in general (or perhaps for certain classes of lattices Λ, including $\Lambda = \mathbb{F}_p[T]$)? Note that the simplicity of zeroes of the Eulerian polynomials $E_k(X)$ of (4.4) is known. Therefore, if one believes in the number field/function field analogy, it predicts a positive answer to that question.

7.5 Concluding remarks. Studying the problem that underlies Theorem 6.1, there are different levels of generality to which our results may (possibly) be adapted.

First, as noted in the introduction, we made the assumption that $|\mathbf{C}^*| = p^{\mathbb{Q}}$, i.e., the value group is isomorphic with $\mathbb{Q}$. This assumption was for notational convenience only; nothing would change if we replaced $\mathbb{Q}$ by an arbitrary divisible subgroup $\Gamma \supset \mathbb{Q}$ of $\mathbb{R}$ as the value group of our field $\mathbf{C}$.

Secondly, instead of considering $\mathbb{F}_p$-lattices in $\mathbf{C}$, we could consider separable $\mathbb{F}_q$-lattices, that is, discrete separable $\mathbb{F}_q$-subspaces of $\mathbf{C}$, where q is a power of p. The most simple and natural example is $\Lambda = \mathbb{F}_q[T]$. Here we run into serious trouble, since our arguments require an explicit formula like (3.4) for the size of the power sums $S_{i,\Lambda}(n)$, which itself depends on the (non-) vanishing of various q-expansion coefficients modulo p. The latter is - for $q \neq p$ - much more complex and less transparent compared to the $q = p$ case, compare [12]. Work on this circle of problems is in progress; for some partial results, see [4].

Thirdly, we might suppress the separability assertion on Λ. Here the power sums $S_{i,\Lambda}(n)$ may behave even more irregularly, due to possible cancellations. We don't see any reason to believe in a formula analogous with (3.4) (iii) for $|S_{i,\Lambda}(n)|$, or finally, a result like Theorem 6.1.

References

[1] P. Deligne and D. Husemöller: Survey of Drinfeld modules. Contemp. Math. 67 (1987), 25–91.

[2] J. Fresnel and M. van der Put: Rigid analytic geometry and its applications. Progress in Mathematics 218, Birkhäuser 2004.

[3] E.-U. Gekeler: On the coefficients of Drinfeld modular forms. Invent. Math. 93 (1988), 667–700.

[4] E.-U. Gekeler: On the zeroes of Goss polynomials. Trans. AMS, to appear.

[5] E.-U. Gekeler: Eisenstein series for principal congruence subgroups over rational function fields. J. Numb. Theory 132 (2012), 127–143.
[6] L. Gerritzen and M. van der Put: Schottky groups and Mumford curves. Lect. Notes Math. 817, Springer 1980.
[7] D. Goss: The algebraist's upper half-plane. Bull. AMS NS2 (1980), 391–415.
[8] D. Goss: Basic structures of function field arithmetic. Ergeb. Math. Grenzgeb. 35, Springer 1996.
[9] D. Hayes: Explicit class field theory in global function fields. In: G.C. Rota (ed.), Studies in Algebra and Number Theory. Academic Press 1979.
[10] D. Hayes: A brief introduction to Drinfeld modules. In: D.Goss, D.R. Hayes, M.I. Rosen (eds.) The arithmetic of function fields. De Gruyter 1992.
[11] J. Neukirch: Algebraic number theory. Grundlehren der Math. Wiss. 322, Springer 1999.
[12] J. Sheats: The Riemann Hypothesis for the Goss zeta function for $\mathbb{F}_q[T]$. J. Numb. Theory 71 (1998), 121–157
[13] http://oeis.org/wiki/Eulerian_polynomials.

Ernst-Ulrich Gekeler and Philipp Stopp
Fachrichtung 6.1 Mathematik
Universität des Saarlandes
Campus E2 4
D-66123 Saarbrücken

gekeler@math.uni-sb.de
stopp@math.uni-sb.de