Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 308

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Saarbrücken 2012

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 308
submitted: April 2, 2012

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AMS Subject Classification: 49 N 60, 35 J 85, 49 J 40.

Keywords: obstacle problems, non-standard growth, regularity of minimizers.

Abstract

We establish interior gradient bounds for functions $u \in W^1_{1,\text{loc}}(\Omega)$ which locally minimize the variational integral $J[u,\Omega] = \int_\Omega h(|\nabla u|)\, dx$ under the side condition $u \geq \Psi$ a.e. on Ω with obstacle Ψ being locally Lipschitz. Here h denotes a rather general N-function allowing (p,q)-ellipticity with arbitrary exponents $1 < p \leq q < \infty$. Our arguments are based on ideas developed in [BFM] combined with techniques originating in [F3].

1 Introduction

In our note we discuss the local Lipschitz (and even the interior $C^{1,\alpha}$-) regularity of functions u from the local Sobolev class $W^1_{1,\text{loc}}(\Omega)$ which locally minimize the functional

$$J[u,\Omega] := \int_\Omega H(\nabla u)\, dx \tag{1.1}$$

under the side condition $u \geq \Psi$ a.e. on Ω with a given Lipschitz function $\Psi : \Omega \to \mathbb{R}$. Here Ω is some open subset in $\mathbb{R}^n$, $n \geq 2$, and the energy density is a strictly convex function from $\mathbb{R}^n$ into the non-negative numbers. By definition u is a local J-minimizer subject to the constraint $u \geq \Psi$ if $J[u,\Omega'] < \infty$ for any subdomain Ω' with compact closure in Ω and if $J[u,\Omega'] \leq J[v,\Omega']$ holds for all $v \in W^1_{1,\text{loc}}(\Omega)$ such that $v \geq \Psi$ a.e. on Ω and $\text{spt}(u-v) \subset \Omega'$. The investigation of the regularity properties of such local minimizers for the obstacle problem has a long tradition starting with the discussion of energy densities H being of quadratic growth. We refer to the monographs [KS] and [FR] for a survey of the most important contributions and a detailed outline of the various regularity results including regularity up to the boundary and the regularity of the free boundary under suitable hypothesis on the data. A natural extension of quadratic growth is the p-growth condition (for some exponent $1 < p < \infty$), which requires the validity of (λ, Λ denoting positive constants)

$$\lambda \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\eta|^2 \leq D^2H(\xi)(\eta,\eta) \leq \Lambda \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\eta|^2 \tag{1.2}$$

for all ξ, $\eta \in \mathbb{R}^n$ or its degenerate variant. Here we mention the papers [CL], [F1], [F2], [LIN], [MIZ],[MUZ] and the references quoted therein. If we replace (1.2) by the (anisotropic) ellipticity condition

$$\lambda \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\eta|^2 \leq D^2H(\xi)(\eta,\eta) \leq \Lambda \left(1 + |\xi|^2\right)^{\frac{q-2}{2}} |\eta|^2 \tag{1.3}$$

with exponents $1 < p < q < \infty$, then unconstrained local minima (even in the vectorial setting) have been first investigated in [M1]-[M4] exhibiting conditions like

$$q < c(n)p,\ c(n) \to 1 \text{ as } n \to \infty\,, \tag{1.4}$$

as sufficient conditions for interior regularity, and from [BFM] it follows that

$$q < p\frac{n+2}{n} \tag{1.5}$$

(together with some minor technical assumptions imposed on H) implies the interior regularity of constrained minimizers. Moreover, as it is shown in [B] p.149 f., the hypothesis (1.5) can be replaced by the dimensionless condition

$$q < p + 2\,, \tag{1.6}$$

if the local minimizer belongs to the space $L^\infty_{\mathrm{loc}}(\Omega)$. For completeness we remark that energy densities H being of nearly linear growth as for example $H(\xi) := |\xi| \ln(1 + |\xi|)$ do not fall in the category (1.3) with exponents $1 < p < q < \infty$. However, based on the works [FS1], [FS2], [FO] and [MS] the question of full interior regularity for the obstacle problem was answered in [FM].

The purpose of this paper is to establish the following result:
suppose that $u \in W^1_{1,\mathrm{loc}}(\Omega)$ is local $J[\cdot,\Omega]$-minimizer subject to the constraint $u \geq \Psi$ a.e. on Ω with Lipschitz obstacle Ψ. Let H satisfy (1.3) with $1 < p < q < \infty$. Then we have $|\nabla u| \in L^\infty_{\mathrm{loc}}(\Omega)$ without any relation like (1.4), (1.5) or (1.6), provided H is of the special form

$$H(\xi) = h(|\xi|) \tag{1.7}$$

with $h : [0,\infty) \longrightarrow [0,\infty)$ of class C^2.

To make our statement precise, we fix the assumptions concerning h. The requirements are:

$$\left.\begin{array}{l} h \text{ is strictly increasing and convex together with} \\ h''(0) > 0 \text{ and } \displaystyle\lim_{t\to 0}\frac{h(t)}{t} = 0\,. \end{array}\right\} \tag{A1}$$

$$\text{For a constant } a > 0 \text{ it holds } h(2t) \leq a h(t),\ t \geq 0 \text{ (doubling property)}\,. \tag{A2}$$

$$\text{There exists a constant } \alpha > 0 \text{ such that } \alpha\tfrac{h'(t)}{t} \leq h''(t) \text{ is true for all } t > 0\,. \tag{A3}$$

$$\text{With } q \geq 2 \text{ and } A > 0 \text{ we have for any } t \geq 0\text{: } h''(t) \leq A(1+t^2)^{\frac{q-2}{2}}\,. \tag{A4}$$

Note that α can be arbitrary small and that there is no upper bound for the exponent q. Before proceeding further let us add some comments:

i) From (A1) we immediately get $h(0) = h'(0) = 0$.

ii) By taking the derivative and using (A3) we see that the function $t \mapsto \frac{h'(t)}{t^\alpha}$ is increasing on $(0,\infty)$, in particular it follows with a suitable constant $c > 0$

$$h(t) \geq c\left(t^{1+\alpha} - 1\right),\ t \geq 0\,. \tag{1.8}$$

iii) We have the following balancing condition:

$$cth'(t) \le h(t) \le th'(t),\ t \ge 0\,. \tag{1.9}$$

In fact, the second inequality follows from convexity of h together with $h(0) = 0$. For the first inequality we observe:

$$h(t) \overset{(A2)}{\ge} \frac{1}{a}h(2t) = \frac{1}{a}\int_0^{2t} h'(r)dr \ge \frac{1}{a}\int_t^{2t} h'(r)dr \ge \frac{1}{a}th'(t)\,.$$

iv) We claim: condition (1.3) holds with $p := 1+\alpha$ and with q from (A4). In fact, from the structural condition (1.7) we obtain

$$\min\left\{h''(|\xi|)\,, \frac{h'(|\xi|)}{|\xi|}\right\}|\eta|^2 \le D^2H(\xi)(\eta,\eta) \le \ \max\ \{\ldots\}\,|\eta|^2\,,$$

hence by (A3) and (A4)

$$\alpha\frac{h'(|\xi|)}{|\xi|}|\eta|^2 \le D^2H(\xi)(\eta,\eta) \le \frac{1}{\alpha}A\left(1+|\xi|^2\right)^{\frac{q-2}{2}}|\eta|^2\,.$$

We have $\frac{h'(t)}{t} \to h''(0)$ as $t \to 0$, hence

$$\frac{h'(t)}{t} \ge \frac{1}{2}h''(0) \text{ on } (0,t_0]\,,$$

whereas for $t \ge t_0$ it holds

$$\frac{h'(t)}{t} = \frac{h'(t)}{t^\alpha}t^{-1+\alpha} \ge \frac{h'(t_0)}{t_0^\alpha}t^{-1+\alpha}\,,$$

so that

$$\frac{h'(t)}{t} \ge c\left(1+t^2\right)^{\frac{\alpha-1}{2}} = c\left(1+t^2\right)^{\frac{p-2}{2}}$$

for any $t > 0$. This proves our claim.

v) For later purpose we observe

$$c\left(h(t)t^s\right)^{1/2} \le \int_{t/2}^t \left(\frac{h'(r)}{r}r^s\right)^{1/2} dr\,, \tag{1.10}$$

$$\int_0^t \left(\frac{h'(r)}{r}r^s\right)^{1/2} dr \le c\left(h(t)t^s\right)^{1/2} \tag{1.11}$$

being valid for any choices of s, $t \ge 0$. Both inequalities essentially follow from ii): it holds

$$\int_0^t \left(\frac{h'(r)}{r}r^s\right)^{1/2} dr = \int_0^t \left(\frac{h'(r)}{r^\alpha}\right)^{1/2} r^{\frac{s}{2}-\frac{1}{2}+\frac{\alpha}{2}}dr$$
$$\le \left(\frac{h'(t)}{t^\alpha}\right)^{1/2}\int_0^r r^{\frac{s}{2}-\frac{1}{2}+\frac{\alpha}{2}}dr = c\left(\frac{h'(t)}{t^\alpha}\right)^{1/2} t^{\frac{s}{2}+\frac{1}{2}+\frac{\alpha}{2}} = c\left(h'(t)t\,t^s\right)^{1/2}\,,$$

thus we get (1.11) on account of (1.9). At the same time we have

$$\int_{t/2}^{t}\left(\frac{h'(r)}{r}r^{s}\right)^{1/2}dr \geq \left(\frac{h'(t/2)}{(t/2)^{\alpha}}\right)^{1/2}\int_{t/2}^{t} r^{\frac{s}{2}-\frac{1}{2}+\frac{\alpha}{2}}dr \geq c\,(h'(t/2)t\,t^{s})^{1/2}\ ,$$

and (1.10) is a consequence of (1.9) and (A2).

After these preparations we can formulate our result:

Theorem 1.1. *Assume that h satisfies (A1-4) and define H and the functional J according to (1.7) and (1.1), respectively. Then any local J-minimizer u subject to the constraint $u \geq \Psi$ a.e. on Ω is locally Lipschitz provided the obstacle Ψ has this property. If the gradient of Ψ satisfies a local Hölder condition, then the same is true for ∇u.*

Remark 1.1. *The above results easily extend to the double obstacle problem as studied for example in [BFM]. Moreover, the conditions on h can be relaxed in order to handle the degenerate case $h''(0) = 0$.*

Remark 1.2. *Since we deal with the scalar case, the structural condition (1.7) is not really needed. It can be replaced by suitable inequalities imposed on $DH(\xi)$ and $D^2H(\xi)$, $\xi \in \mathbb{R}^n$, relating these quantities as outlined for example in [LIE], where the degenerate case for the double obstacle problem under non-standard growth conditions is addressed. The reader should note that Theorem 1.1 is a consequence of the results obtained in [LIE], if instead of (A4) we require the validity of*

$$h''(t) \leq c\frac{h'(t)}{t},\ t \geq 0\,, \tag{1.12}$$

since (1.12) together with (A3) implies the condition of uniform ellipticity (c_1, c_2 positive constants)

$$c_1 \leq \frac{t\,h''(t)}{h'(t)} \leq c_2,\ t \geq 0\,, \tag{1.13}$$

which corresponds to (0.1) in [LIE]. However, in comparison to (1.12) and thereby (1.13), our assumptions (A3) and (A4) are less restrictive.

Remark 1.3. *Let us finally compare our assumptions with the hypotheses imposed on the density h in the paper [MP]. Clearly (A3) is more restrictive compared to the first part of inequality (2.9) from [MP], however we do not require an upper bound for $h''(t)$ in terms of $h'(t)/t$ as expressed in the second part of (2.9). In fact, in Section 2 we construct a density h for which $h'(t)/t$ is bounded, whereas (A4) holds with arbitrary (large) prescribed number q, so that the second inequality from (2.9) in [MP] is violated.*

Our paper is organized as follows: in Section 2 we give an example of a density h satisfying (A1-4). In Section 3 we introduce a suitable sequence of local regularizations as done in [BFM] and show following ideas from [F3] that any constrained local minimizer u satisfies

$$|\nabla u| \in L^r_{\mathrm{loc}}(\Omega),\ 1 \leq r < \infty\,, \tag{1.14}$$

if Ψ is locally Lipschitz. In Section 4 we show that (1.14) implies the local Lipschitz regularity of u, which proves the first part of Theorem 1.1. From this the local Hölder continuity of ∇u (for sufficiently regular obstacle Ψ) follows along the lines of [BFM]. Some additional results including the non-autonomous case are collected in Section 5.

2 An example

In this section we construct an energy density $h : [0, \infty) \to [0, \infty)$ satisfying (A1-4) with $\alpha = 1$ and arbitrary large exponent q. Thus the integrand $H(\xi) = h(|\xi|)$ is $(2, q)$-elliptic in the sense of (1.3) but for appropriate choices of q the conditions (1.4) - (1.6) fail to be true. The reader should note that $H(\xi)$ is of quadratic growth w.r.t. $\xi \in \mathbb{R}^n$. The construction works like this:

- we start with a "suitable"function $\Theta : [0, \infty) \longrightarrow [0, \infty)$ (playing the role of the derivative of $\frac{h'(t)}{t}$);
- then we let $g(t) := 1 + \int_0^t \Theta(s)ds$ ($g(t)$ represents $\frac{h'(t)}{t}$);
- finally we set $h(t) := \int_0^t sg(s)ds$.

To be precise consider a sequence $\{a_i\}$ of numbers such that $0 \ll a_i < a_{i+1}$ and $\lim_{i\to\infty} a_i = \infty$. We choose $\varepsilon_i > 0$ with the property

$$I_i \cap I_j = \emptyset,\ \text{if } i \neq j,\ I_i := (a_i - \varepsilon_i, a_i + \varepsilon_i)\,,$$

and

$$\sum_{i=1}^{\infty} \varepsilon_i a_i^{\omega} < \infty \tag{2.1}$$

for some number ω to be fixed later. We then define the continuous function $\Theta : [0, \infty) \to [0, \infty)$ through

$$\Theta(t) := \begin{cases} 0 \text{ on } [0,\infty) - \bigcup_{i=1}^{\infty} I_i\,, \\ \text{affine linear on } (a_i - \varepsilon_i, a_i) \text{ and on} \\ (a_i, a_i + \varepsilon_i) \text{ with value } a_i^{\omega} \text{ at } t = a_i\,, \\ i \in \mathbb{N} \end{cases}$$

and introduce $g(t)$, $h(t)$ as done above. Clearly

$$g' = \Theta,\ h'(t) = t\,g(t)\,, \tag{2.2}$$

and from the definition of g we obtain

$$g(t) \leq 1 + \int_0^{\infty} \Theta(s)ds = 1 + \sum_{i=1}^{\infty} \varepsilon_i a_i^{\omega} =: g_{\infty}\,,$$

thus by (2.1)

$$1 = g(0) \leq g(t) \leq g_{\infty} < \infty,\ t \in [0, \infty)\,. \tag{2.3}$$

Inserting (2.3) in the definition of h, we find

$$\frac{t^2}{2} \leq h(t) \leq g_\infty \frac{1}{2} t^2,\ t \geq 0\,, \tag{2.4}$$

and (2.4) shows that h is of quadratic growth. The validity of (A1) is immediate. For (A3) with $\alpha = 1$ we observe

$$h''(t) = \frac{d}{dt}\left(tg(t)\right) = g(t) + tg'(t) \geq g(t) \overset{(2.2)}{=} \frac{h'(t)}{t}\,.$$

Let us look at (A4): from $h''(t) = g(t) + tg'(t) = g(t) + t\Theta(t)$ and (2.3) it follows

$$1 + t\Theta(t) \leq h''(t) \leq g_\infty + t\Theta(t),\ \ t \geq 0\,. \tag{2.5}$$

Suppose that a number $q > 2$ is given. According to (2.5) we see that (A4) holds if we let $\omega := q - 3$ in the definition of Θ. Moreover, again by (2.5), it is immediate that in (A4) we can not replace q by some smaller exponent $\widetilde{q}$. For proving (A2) we first observe that (2.3) gives the inequality $g(2t) \leq g_\infty g(t)$, hence

$$h(2t) = \int_0^{2t} sg(s)ds = 4\int_0^t sg(2s)ds \leq 4g_\infty \int_0^t sg(s)ds = 4g_\infty h(t)\,,$$

therefore we obtain (A2) with $a := 4g_\infty$. □

3 Higher integrability of the gradient

In the following we assume that u is a local minimizer of the functional $J[\cdot,\Omega]$ from (1.1) under the side condition $u \geq \Psi$ a.e. on Ω with Ψ being locally Lipschitz. We further assume that the density h satisfies the assumptions (A1-4). As in Section 2 of [BFM] we work with a suitable local regularization. Let us briefly recall the basic notation: with ε and δ we denote two sequences of positive numbers such that $\varepsilon \to 0$ and $\delta \to 0$, and we will keep these symbols also for subsequences. As usual, c will denote a finite positive constant, whose value may change from line to line, and depending on various quantities but always being independent of ε and δ. We fix a radius $R > 0$ and $x_0 \in \Omega$ such that B_{2R} is compactly contained in Ω, $B_r := B_r(x_0)$. Let u_ε and Ψ_ε denote the mollifications of u and Ψ, respectively, with radius ε and define

$$\mathcal{C}_\varepsilon := \left\{ v \in u_\varepsilon + \overset{\circ}{W}{}^1_q(B_{2R}) : v \geq \Psi_\varepsilon \text{ a.e. on } \Omega \right\}$$

with exponent q from (A4). Finally we consider the unique solution $u_{\varepsilon,\delta} \in \mathcal{C}_\varepsilon$ of the problem

$$\begin{aligned} J_\delta[w, B_{2R}] &:= \int_{B_{2R}} H_\delta(\nabla w)\,dx \longrightarrow \min \text{ in } \mathcal{C}_\varepsilon\,, \qquad (3.1)\\ H_\delta(\xi) &:= h_\delta\left(|\xi|\right),\ h_\delta(t) := h(t) + \delta(1+t^2)^{q/2}\,. \end{aligned}$$

Lemma 3.1. *We can choose a suitable sequence $\delta = \delta(\varepsilon)$ such that for any $r \in (0, 2R)$ and all $1 \le s < \infty$ there is a finite constant (independent of ε)*

$$c = c\,(r, s, J[u, B_{2R}])$$

with the property

$$\left\|\nabla u_{\varepsilon,\delta(\varepsilon)}\right\|_{L^s(B_r)} \le c\,. \tag{3.2}$$

Remark 3.1. *Of course the constant c will depend on other (irrelevant) quantities as $\|\nabla\Psi\|_{L^\infty(B_{2R})}$.*

Proof of Lemma 3.1: From e.g. [CL] or [MUZ] it follows that

$$u_{\varepsilon,\delta} \in C^{1,\beta}(B_{2R}) \cap W^2_{q,\mathrm{loc}}(B_{2R}), \tag{3.3}$$

and this initial regularity justifies our calculations carried out below. For notational simplicity we will drop the subscripts ε and δ just denoting

$$u_{\varepsilon,\delta} = u,\; H_\delta = H,\; h_\delta = h,\; \Psi_\varepsilon = \Psi,\; \mathcal{C}_\varepsilon = \mathcal{C}\,,$$

but the reader should keep in mind that actually we work with the regularization. Following the lines of [F1] (see also [FM] and [BFM], Lemma 2.1) we use the minimality of u (recall (3.1)) to get the equation

$$\int_{B_{2R}} DH(\nabla u)\cdot\nabla\varphi\,dx = \int_{B_{2R}} \varphi g\,dx \tag{3.4}$$

for any $\varphi \in C^1_0(B_{2R})$, where

$$g := \mathbf{1}_S\left(-\operatorname{div}\left[DH(\nabla\Psi)\right]\right)\,,$$

$\mathbf{1}_S$ denoting the characteristic function of the coincidence set $S := \{x \in B_{2R} : u(x) = \Psi(x)\}$. We fix a number $M > 1 + \|\nabla\Psi\|^2_{L^\infty(B_{2R})}$ and define a function $\Phi : [0,\infty) \to [0,1]$ such that

$$\Phi(t) = 0 \text{ on } [0, M],\; \Phi(t) = 1 \text{ on } [2M, \infty),\; \Phi' \ge 0\,. \tag{3.5}$$

From the smoothness properties stated in (3.3) we see that we can replace φ by $\partial_\beta\varphi$ in (3.4) and obtain after integration by parts

$$\int_{B_{2R}} \partial_\beta\left(DH(\nabla u)\right)\cdot\nabla\varphi\,dx = -\int_{B_{2R}} g\partial_\beta\varphi\,dx \tag{3.6}$$

again for all $\varphi \in C^1_0(B_{2R})$. Letting $\Gamma := 1 + |\nabla u|^2$ we choose

$$\varphi := \eta^2\Gamma^{s/2}\Phi^2(\Gamma)\partial_\beta u,\; \eta \in C^1_0(B_{2R})\,,$$

in equation (3.6), where for the moment $s \geq 0$ denotes some arbitrary parameter. Since $u = \Psi$ and thereby $\nabla u = \nabla \Psi$ on the set S, it follows from the definition of Φ and the choice of M that $\Phi(\Gamma)$ vanishes on S, thus

$$\int_{B_{2R}} g \partial_\beta \left(\eta^2 \Gamma^{s/2} \Phi^2(\Gamma) \partial_\beta u \right) \, dx = 0$$

(we adopt the convention of summation from now on), and (3.6) yields

$$\int_{B_{2R}} \partial_\beta \left(DH(\nabla u) \right) \cdot \nabla \left(\eta^2 \Gamma^{\frac{s}{2}} \Phi^2(\Gamma) \partial_\beta u \right) \, dx = 0 \, ,$$

hence

$$\begin{aligned} &\int_{B_{2R}} D^2 H \left(\nabla u \right) \left(\partial_\beta \nabla u, \partial_\beta \nabla u \right) \eta^2 \Gamma^{s/2} \Phi^2) \, dx \\ &\quad = \; - \int_{B_{2R}} \partial_\beta \left(DH(\nabla u) \right) \cdot \nabla \left(\eta^2 \Gamma^{s/2} \Phi^2(\Gamma) \right) \partial_\beta u \, dx \, . \end{aligned} \tag{3.7}$$

Let us abbreviate $\widetilde{\Phi}(\Gamma) := \Gamma^{s/2} \Phi^2(\Gamma)$. Then we get

$$\begin{aligned} \text{r.h.s. of (3.7)} \; &= \; - \int_{B_{2R}} D^2 H(\nabla u) \left(\partial_\beta \nabla u, \nabla \widetilde{\Phi}(\Gamma) \right) \eta^2 \partial_\beta u dx \\ &\quad - \int_{B_{2R}} \partial_\beta \left(DH(\nabla u) \right) \cdot \nabla \eta^2 \widetilde{\Phi}(\Gamma) \partial_\beta u \, dx =: -T_1 - T_2 \, , \end{aligned}$$

and it is easy to check that

$$D^2 H(\nabla u) \left(\partial_\beta \nabla u, \nabla \widetilde{\Phi}(\Gamma) \right) \partial_\beta u = \widetilde{\Phi}'(\Gamma) a_{\beta\gamma} \partial_\beta \Gamma \partial_\gamma \Gamma \, ,$$

where we have abbreviated

$$a_{\beta\gamma} := \frac{1}{2} \delta_{\beta\gamma} \frac{h'(|\nabla u|)}{|\nabla u|} + \frac{1}{2} \left[h''(|\nabla u|) - \frac{h'(|\nabla u|)}{|\nabla u|} \right] \frac{\partial_\beta u \partial_\gamma u}{|\nabla u|^2} \, .$$

Since $(a_{\beta\gamma})$ is an elliptic matrix and since $\widetilde{\Phi}' \geq 0$ (recall (3.5)), we see that $-T_1 \leq 0$, and (3.7) yields

$$\int_{B_{2R}} D^2 H(\nabla u) \left(\partial_\beta \nabla u, \partial_\beta \nabla u \right) \eta^2 \Gamma^{s/2} \Phi^2(\Gamma) \, dx \leq -T_2 \, . \tag{3.8}$$

In a next step we observe that after integration by parts

$$\begin{aligned} -T_2 \; &= \; \int_{B_{2R}} DH(\nabla u) \cdot \partial_\beta \left(\nabla \eta^2 \partial_\beta u \Phi^2(\Gamma) \Gamma^{s/2} \right) \, dx \\ &\leq \; c \Bigg\{ \int_{B_{2R}} h'(|\nabla u|) \eta |\nabla \eta| \Phi^2(\Gamma) \Gamma^{s/2} |\nabla^2 u| \, dx \\ &\quad + \int_{B_{2R}} h'(|\nabla u|) \eta |\nabla \eta| \left| \nabla \left(\Phi^2(\Gamma) \Gamma^{s/2} \right) \right| |\nabla u| \, dx \\ &\quad + \int_{B_{2R}} h'(|\nabla u|) |\nabla^2 \eta| \Phi^2(\Gamma) \Gamma^{s/2} |\nabla u| \, dx \Bigg\} =: c \left\{ S_1 + S_2 + S_3 \right\} \, . \end{aligned}$$

From (1.9) it follows

$$S_3 \le c \int_{B_{2R}} h(|\nabla u|)\Phi^2(\Gamma)\Gamma^{s/2}|\nabla^2 \eta|\, dx\,. \tag{3.9}$$

For S_1 we observe (using Young's inequality)

$$\begin{aligned} S_1 &= \int_{B_{2R}} \eta \left(\frac{h'(|\nabla u|)}{|\nabla u|}\right)^{1/2} \Phi(\Gamma)\Gamma^{s/4}|\nabla^2 u||\nabla\eta|\,(h'(|\nabla u|)|\nabla u|)^{1/2}\,\Phi(\Gamma)\Gamma^{s/4}\,dx \\ &\overset{(1.9)}{\le} \tau\int_{B_{2R}}\eta^2\frac{h'(|\nabla u|)}{|\nabla u|}\Phi^2(\Gamma)\Gamma^{s/2}|\nabla^2 u|^2\,dx + \tau^{-1}\int_{B_{2R}}|\nabla\eta|^2 h(|\nabla u|)\Phi^2(\Gamma)\Gamma^{s/2}\,dx\,. \end{aligned}$$

Now, if we recall iv) from Section 1, it is immediate that for τ sufficiently small the "τ-term" can be absorbed in the l.h.s. of (3.8). Taking into account (3.9), it is shown

$$\begin{aligned} &\int_{B_{2R}} \frac{h'(|\nabla u|)}{|\nabla u|}\left|\nabla^2 u\right|^2\eta^2\Phi^2(\Gamma)\Gamma^{s/2}\,dx \\ &\le\; c\left\{\int_{B_{2R}}\left(|\nabla\eta|^2+|\nabla^2\eta|\right)h\left(|\nabla u|\right)\Phi^2(\Gamma)\Gamma^{s/2}\,dx + |S_2|\right\}\,. \end{aligned} \tag{3.10}$$

Let us discuss S_2: we have

$$\begin{aligned} |S_2| &\le c\left\{\int_{B_{2R}} h'\left(|\nabla u|\right)\eta|\nabla\eta|\Phi'(\Gamma)\Phi(\Gamma)|\nabla u|^2|\nabla^2 u|\Gamma^{s/2}\,dx \right.\\ &\quad\left. + \int_{B_{2R}} sh'\left(|\nabla u|\right)\eta|\nabla\eta|\Phi^2(\Gamma)\Gamma^{s/2}|\nabla^2 u|\,dx\right\} =: c\left\{U_1 + sU_2\right\}\,, \end{aligned}$$

and with Young's inequality applied to U_2 (compare the estimate concerning S_1) we see in combination with (3.10)

$$\begin{aligned} &\int_{B_{2R}} \frac{h'(|\nabla u|)}{|\nabla u|}\left|\nabla^2 u\right|^2\eta^2\Phi^2(\Gamma)\Gamma^{s/2}\,dx \\ &\le\; c\left\{\int_{B_{2R}}\left(|\nabla\eta|^2+|\nabla^2\eta|\right)h\left(|\nabla u|\right)\Phi^2(\Gamma)\Gamma^{s/2}\,dx + U_1\right\} \end{aligned} \tag{3.11}$$

with constant c now depending also on s. Recalling (3.5) we have

$$\Phi'(\Gamma) = \mathbf{1}_{[M\le\Gamma\le 2M]}\Phi'(\Gamma)\,,$$

thus $\Phi'(\Gamma)|\nabla u|^2 \le c(M)\Phi'(\Gamma)$ and therefore

$$\begin{aligned} U_1 &\le c(M)\int_{B_{2R}} h'(|\nabla u|)\eta|\nabla\eta|\Phi'(\Gamma)\Phi(\Gamma)|\nabla^2 u|\Gamma^{s/2}\,dx \\ &\le \tau\int_{B_{2R}}\frac{h'(|\nabla u|)}{|\nabla u|}\eta^2|\nabla^2 u|^2\Phi^2(\Gamma)\Gamma^{s/2}\,dx \\ &\quad + c(\tau, M)\int_{B_{2R}} h(|\nabla u|)|\nabla\eta|^2\Phi'(\Gamma)^2\Gamma^{s/2}\,dx\,, \end{aligned}$$

and after appropriate choice of $\tau > 0$ we end up with (cf. (3.11))

$$\int_{B_{2R}} \frac{h'(|\nabla u|)}{|\nabla u|}|\nabla^2 u|^2 \Phi^2(\Gamma)\Gamma^{s/2}\,dx \le c \int_{B_{2R}} \left(|\nabla \eta|^2 + |\nabla^2 \eta|\right) h(|\nabla u|)\Gamma^{s/2}\,dx \tag{3.12}$$

for a constant c depending on s and additionally on the number M. We emphasize that up to now η and s can be chosen arbitrarily in estimate (3.12). Next we fix

$$\delta := \delta(\varepsilon) := \left(1 + \varepsilon^{-1} + \|\nabla u_\varepsilon\|^{2q}_{L^q(B_{2R})}\right)^{-1}$$

(compare [BFM], Step 5 in Section 5) and denote by the symbol v_ε the solution $u_{\varepsilon,\delta(\varepsilon)}$ of problem (3.1). From (2.22) in [BFM] we infer

$$\int_{B_{2R}} h_{\delta(\varepsilon)}\left(|\nabla v_\varepsilon|\right)\,dx \le \int_{B_{2R}} h\left(|\nabla u|\right)\,dx + 0(\varepsilon)\,, \tag{3.13}$$

and (3.13) states in particular the uniform boundedness of the energies of the approximations. Dropping the subscripts ε and $\delta(\varepsilon)$ again, which means that we use the symbols v and h in place of v_ε and $h_{\delta(\varepsilon)}$, respectively, we see by (3.12) that

$$\int_{\Omega_1} \frac{h'(|\nabla v|)}{|\nabla v|}|\nabla^2 v|^2 \Phi^2(\Gamma)\Gamma^{s/2}\,dx \le c \int_{\Omega_2} h\left(|\nabla v|\right)\Gamma^{s/2}\,dx \tag{3.14}$$

for arbitrary subdomains $\Omega_1 \Subset \Omega_2 \Subset B_{2R}$ and any exponent $s \ge 0$, the constant c depending on s and the domains Ω_i but being independent of ε. Let $s_0 = 0$ and define

$$\Psi_0 := \int_0^{|\nabla v|} \Phi\left(1+t^2\right)\left(\frac{h'(t)}{t}\right)^{1/2} dt\,.$$

From (3.14) and the remark stated after (3.13) it follows

$$|\nabla \Psi_0| \in L^2_{\mathrm{loc}}(B_{2R}) \tag{3.15}$$

uniformly w.r.t. ε. At the same time (1.11) (with $s = s_0$) implies

$$\Psi_0 \in L^2_{\mathrm{loc}}(B_{2R}) \tag{3.16}$$

again uniformly w.r.t. ε, and from (3.15) and (3.16) we infer in combination with Sobolev's embedding theorem

$$\Psi_0^t \in L^1_{\mathrm{loc}}(B_{2R}),\ t \begin{cases} < \infty, & \text{if } n = 2 \\ \le \frac{2n}{n-2}, & \text{if } n \ge 3 \end{cases}. \tag{3.17}$$

On the set $\left[|\nabla v| \ge 2\sqrt{2M-1}\right]$ we clearly have

$$\begin{aligned} \Psi_0 \ &\ge \ \int_{|\nabla v|/2}^{|\nabla v|} \Phi(1+t^2)\left(\frac{h'(t)}{t}\right)^{1/2} dt \\ &\overset{(3.5)}{=} \int_{|\nabla v|/2}^{|\nabla v|} \left(\frac{h'(t)}{t}\right)^{1/2} dt \overset{(1.10)}{\ge} c\,h\left(|\nabla v|\right)^{1/2}\,, \end{aligned}$$

thus (3.17) shows the validity of

$$h\,(|\nabla v|)^t \in L^1_{\rm loc}(B_{2R}),\ t \begin{cases} < \infty, & \text{if } n = 2 \\ \le \frac{n}{n-2}, & \text{if } n \ge 3 \end{cases} . \tag{3.18}$$

In case $n = 2$ we deduce from (3.18) in combination with (1.8) that $\nabla v = \nabla v_\varepsilon$ is in any space $L^t_{\rm loc}(B_{2R}, \mathbb{R}^n)$, $t < \infty$, uniformly w.r.t. to ε and our claim (3.2) follows. Let us therefore assume that $n \ge 3$. In this case we write $h^{\frac{n}{n-2}} = h\, h^{\frac{2}{n-2}}$ and quote (1.8) (recall $p = 1 + \alpha$) in order to deduce from (3.18)

$$h\,(|\nabla v|)\, |\nabla v|^{\frac{2p}{n-2}} \in L^1_{\rm loc}(B_{2R}) \text{ (uniformly)}\,. \tag{3.19}$$

But (3.19) shows that the r.h.s. of (3.14) stays bounded now for the choice $s := s_1 := \frac{2p}{n-2}$, hence the function

$$\Psi_1 := \int_0^{|\nabla v|} \Phi\left(1 + t^2\right) \left\{\frac{h'(t)}{t} t^{s_1}\right\}^{1/2} dt$$

satisfies on account of (3.14) and (1.11)

$$|\nabla \Psi_1| \in L^2_{\rm loc}(B_{2R}),\ \Psi_1 \in L^2_{\rm loc}(B_{2R})$$

(uniformly). Using the resulting uniform $L^1_{\rm loc}(B_{2R})$-bound for $\Psi_1^{\frac{2n}{n-2}}$ and quoting (1.10), the same reasoning as applied after (3.17) leads to the result

$$h\,(|\nabla v|)^{\frac{n}{n-2}}\, |\nabla v|^{s_1 \frac{n}{n-2}} \in L^1_{\rm loc}(B_{2R})\,,$$

and in combination with (1.8) we get

$$h\,(|\nabla v|)\, |\nabla v|^{s_2} \in L^1_{\rm loc}(B_{2R}),\ s_2 := \frac{2p}{n-2} + s_1 \frac{n}{n-2}\,.$$

With $s_0 = 0$ we let

$$s_{k+1} := \frac{2p}{n-2} + s_k \frac{n}{n-2},\ k \in \mathbb{N}_0\,.$$

Repeating the steps from above it follows for each k

$$h\,(|\nabla v|)\, |\nabla v|^{s_k} \in L^1_{\rm loc}(B_{2R})$$

uniformly w.r.t. ε, and since $s_k \to \infty$, it is shown

$$\sup_\varepsilon \int_{B_r} |\nabla v_\varepsilon|^s\, dx \le c(r, s) < \infty \tag{3.20}$$

for any $r < 2R$ and any $s > 1$. Obviously this proves Lemma 3.1 and by passing to the limit $\varepsilon \to 0$ we additionally see that $|\nabla u|$ is in $L^s_{\rm loc}(\Omega)$ for any $s < \infty$. □

4 Local boundedness of the gradient

Here we are going to show

Lemma 4.1. *Under the assumptions and with the notation of Theorem 1.1 we consider a local J-minimizer u subject to the constraint $u \geq \Psi$ a.e. on Ω with Ψ being locally Lipschitz. Then it holds*

$$|\nabla u| \in L^\infty_{\mathrm{loc}}(\Omega)\,. \tag{4.1}$$

Proof: We work with our local regularization $v_\varepsilon = u_{\varepsilon,\delta(\varepsilon)}$ introduced in front of (3.13). Of course (4.1) will follow as soon as we can show

$$|\nabla v_\varepsilon| \in L^\infty_{\mathrm{loc}}(B_{2R}) \tag{4.2}$$

uniformly w.r.t. ε. With a slight abuse of notation we agree to write u in place of v_ε having the advantage that we can use e.g. equation (3.4) without further change of symbols. The local boundedness of ∇u follows via De Giorgi-type arguments as applied for example in [B], Theorem 5.22, [BFM], Step 4 in Section 2, or [ABF]. We fix a ball $B_\rho(y)$ compactly contained in $B_{2R} = B_{2R}(x_0)$, choose any number $k \geq 1 + \|\nabla\Psi\|^2_{L^\infty(B_{2R})}$ and define the sets

$$A(k,\rho) := \{x \in B_\rho(y) : \Gamma > k\}$$

with $\Gamma := 1 + |\nabla u|^2$. Finally we let $\eta \in C^\infty_0(B_\rho(y))$ and recall that equation (3.4) implies the identity (3.6). In (3.6) we choose $\varphi := \eta^2 \partial_\beta u \max(\Gamma - k, 0)$ and observe $\nabla u = \nabla \Psi$ on the set S, thus

$$\int_{B_\rho(y)} g \partial_\beta \left(\eta^2 \partial_\beta u \max(\Gamma - k, 0)\right)\, dx = 0$$

and therefore

$$0 = \int_{B_\rho(y)} D^2H(\nabla u)\left(\partial_\beta \nabla u, \nabla\left[\eta^2 \partial_\beta u \max(\Gamma - k, 0)\right]\right)\, dx\,.$$

This immediately implies

$$\int_{A(k,\rho)} \eta^2 a_{\beta\gamma} \partial_\beta \Gamma \partial_\gamma \Gamma\, dx \leq -2 \int_{A(k,\rho)} a_{\beta\gamma} \partial_\beta \Gamma \partial_\gamma \eta \eta (\Gamma - k)\, dx$$

with coefficients $a_{\beta\gamma}$ being defined in front of (3.8). On the r.h.s. we can apply the Cauchy-Schwarz inequality to the symmetric bilinear form induced by the matrix $(a_{\beta\gamma})$ with the result

$$\int_{A(k,\rho)} \eta^2 a_{\beta\gamma} \partial_\beta \Gamma \partial_\gamma \Gamma\, dx \leq c \int_{A(k,\rho)} a_{\beta\gamma} \partial_\beta \eta \partial_\gamma \eta (\Gamma - k)^2\, dx\,. \tag{4.3}$$

Let $r < \hat{r}$ such that $B_{\hat{r}}(y) \Subset B_{2R}$, and consider η such that $\eta = 1$ on $B_r(y)$, $0 \leq \eta \leq 1$, $\mathrm{spt}(\eta) \subset B_{\hat{r}}(y)$, $|\nabla \eta| \leq c/(\hat{r} - r)$. Observing

$$\int_{A(k,r)} (\Gamma - k)^{\frac{n}{n-1}}\, dx \leq \int_{B_{\hat{r}}(y)} \left(\eta[\Gamma - k]^+\right)^{\frac{n}{n-1}}\, dx\,,$$

$[\dots]^+$ denoting the positive part of $[\dots]$, and using Sobolev's theorem we find

$$\int_{A(k,r)} (\Gamma - k)^{\frac{n}{n-1}}\, dx \le c\left[I_1^{\frac{n}{n-1}} + I_2^{\frac{n}{n-1}}\right]\,, \tag{4.4}$$

where we have abbreviated

$$\begin{aligned} I_1^{\frac{n}{n-1}} &:= \left[\int_{A(k,\hat r)} |\nabla\eta|(\Gamma - k)\, dx\right]^{\frac{n}{n-1}} \\ &\le c\,(\hat r - r)^{-\frac{n}{n-1}}\left[\int_{A(k,\hat r)} (\Gamma - k)\, dx\right]^{\frac{n}{n-1}}\,, \\ I_2^{\frac{n}{n-1}} &:= \left[\int_{A(k,\hat r)} \eta|\nabla\Gamma|\, dx\right]^{\frac{n}{n-1}}\,. \end{aligned}$$

Since $k \ge 1$ it holds on the set $A(k,\hat r)$

$$\frac{h'(|\nabla u|)}{|\nabla u|} \overset{(1.9)}{\ge} c\Gamma^{-1}h(|\nabla u|), \qquad h''(|\nabla u|) \underset{(A3)}{\overset{(1.9)}{\ge}} c\,\Gamma^{-1}h(|\nabla u|)\,,$$

$$\frac{h'(|\nabla u|)}{|\nabla u|} \overset{(1.9)}{\le} c\Gamma^{-1}h(|\nabla u|) \quad \text{and} \quad h''(|\nabla u|) \overset{(A4)}{\le} c\Gamma^{\frac{q-2}{2}}\,,$$

and clearly the r.h.s. of the last inequality also serves as an upper bound for $\frac{h'(|\nabla u|)}{|\nabla u|}$ on the set $A(k,\hat r)$. Recalling the ellipticity estimate

$$\frac{1}{2}\min\left\{\frac{h'(|\nabla u|)}{|\nabla u|}, h''(|\nabla u|)\right\}|\tau|^2 \le a_{\beta\gamma}\tau_\beta\tau_\gamma \le \frac{1}{2}\max\{\dots\}\,|\tau|^2,\ \tau \in \mathbb{R}^n\,,$$

we find after an application of Hölder's inequality:

$$\begin{aligned} I_2^{\frac{n}{n-1}} &= \left[\int_{A(k,\hat r)} \eta\,|\nabla\Gamma|\, h(|\nabla u|)^{1/2}\,\Gamma^{-1/2}\Gamma^{1/2}h(|\nabla u|)^{-1/2}\ dx\right]^{\frac{n}{n-1}} \\ &\le \left[\int_{A(k,\hat r)} \eta^2\,|\nabla\Gamma|^2\, h(|\nabla u|)\,\Gamma^{-1}\, dx\right]^{\frac12\frac{n}{n-1}} \left[\int_{A(k,\hat r)} \Gamma h(|\nabla u|)^{-1}\ dx\right]^{\frac12\frac{n}{n-1}} \\ &\le c\left[\int_{A(k,\hat r)} \eta^2 a_{\beta\gamma}\partial_\beta\Gamma\partial_\gamma\Gamma\, dx\right]^{\frac12\frac{n}{n-1}} \left[\int_{A(k,\hat r)} \Gamma h(|\nabla u|)^{-1}\ dx\right]^{\frac12\frac{n}{n-1}} \\ &\overset{(4.3)}{\le} c(\hat r - r)^{-\frac{n}{n-1}}\left[\int_{A(k,\hat r)} (\Gamma - k)^2\Gamma^{\frac{q-2}{2}}\, dx\right]^{\frac12\frac{n}{n-1}} \left[\int_{A(k,\hat r)} \Gamma h(|\nabla u|)^{-1}\ dx\right]^{\frac12\frac{n}{n-1}}\,. \end{aligned}$$

Another application of Hölder's inequality yields

$$I_1^{\frac{n}{n-1}} \le c(\hat r - r)^{-\frac{n}{n-1}}\left[\int_{A(k,\hat r)} (\Gamma - k)^2\,\Gamma^{\frac{q-2}{2}}\, dx\right]^{\frac12\frac{n}{n-1}} \cdot \left[\int_{A(k,\hat r)} \Gamma^{\frac{2-q}{2}}\, dx\right]^{\frac12\frac{n}{n-1}}\,.$$

Returning to (4.4) it is shown that

$$\int_{A(k,r)} (\Gamma - k)^{\frac{n}{n-1}}\, dx \quad \le \quad c(\hat{r} - r)^{-\frac{n}{n-1}} \left[\int_{A(k,\hat{r})} (\Gamma - k)^2\, \Gamma^{\frac{q-2}{2}}\, dx \right]^{\frac{1}{2}\frac{n}{n-1}} \cdot \left[\int_{A(k,\hat{r})} \Gamma h\left(|\nabla u|\right)^{-1}\, dx \right]^{\frac{1}{2}\frac{n}{n-1}} . \tag{4.5}$$

Here we have used one more time that from (A1) and (A4) it follows $h(t) \le ct^q$ for $t \ge 1$, hence on the set $A(k, \hat{r})$

$$\Gamma^{\frac{2-q}{2}} \le c\Gamma h\left(|\nabla u|\right)^{-1}$$

and (recall (1.8)) in addition

$$\Gamma h\left(|\nabla u|\right)^{-1} \le c\Gamma^{\frac{\mu}{2}},\ \mu := 1 - \alpha\,.$$

This shows that (4.5) exactly corresponds to inequality (24) in Lemma 5.23 of [B] and we can follow the calculations from p.158 in [B] using Lemma 3.1 to get our claim (4.2), which proves Lemma 4.1. □

5 Additional results

It would be interesting to know if Theorem 1.1 remains valid under the hypotheses (A1) and (A2), but with (A3) and (A4) being replaced by

There exists a constant $K \ge 1$ such that $h''(t) \le K\frac{h'(t)}{t}$ is true for all $t > 0$. (A3*)

For some exponent $p > 1$ and a constant $\lambda > 0$ we have
$\lambda(1 + t^2)^{\frac{p-2}{2}} \le h''(t)$ for any $t \ge 0$. (A4*)

The reader should note that (A2) actually is a consequence of (A1) and (A3*), since by (A3*) $\frac{h'(t)}{t^K}$ is a decreasing function, which yields the doubling property for h'. From this we obtain (A2) by using the formula

$$h(2t) = \int_0^{2t} h'(s)\, ds = 2\int_0^t h'(2s)\, ds\,.$$

Moreover, from (A2) we deduce the existence of an exponent q (w.l.o.g. $q \ge 2$) such that

$$h(t) \le c(t^q + 1)\,,\ h'(t) \le c(t^{q-1} + 1) \tag{5.1}$$

holds for all $t \ge 0$. Inequality (5.1) in combination with (A3*) yields

$$h''(t) \le c(t^{q-2} + 1)\,,\ t \ge 0\,, \tag{5.2}$$

i.e. we have (p, q)-ellipticity in the sense of (1.3) on account of (5.2), (A3*), (A4*) and iv) of Section 1. In addition, the balancing condition (1.9) is still satisfied. However, we cannot follow the lines of Section 3 for proving the statement of Lemma 3.1 in this different setting. Quoting [BFM] we obtain Lemma 3.1 only under the extra smallness condition (1.5) or its weaker variant (5.5) stated below (needed in case $n \geq 3$), and we would like to know if such an additional limitation is really necessary.

There is a way to avoid (1.5) but leading to another restriction: we return to the proof of Lemma 3.1, in particular equation (3.7), but this time we do not perform an integration by parts in the quantity T_2. After some calculations we see that we arrive at (3.12), where now on the l.h.s. the term $\frac{h'(|\nabla u|)}{|\nabla u|}$ has to be replaced by $h''(|\nabla u|)$. For this reason the function Ψ_0 introduced after (3.14) takes the principle form

$$\Psi_0 := \int_0^{|\nabla u|} h''(t)^{\frac{1}{2}}\, dt\,,$$

and in order to start our iteration procedure in case $n \geq 3$ it is sufficient to assume

$$h''(t) \geq c\Big(\frac{h'(t)}{t}\Big)^{\frac{n-2}{n}} t^{-2\beta} \tag{5.3}$$

at least for $t \geq t_0$ with exponent $\beta < 2/n$. In fact, (5.3) together with (1.9) yields

$$h''(t)^{\frac{1}{2}} \geq ch(t)^{\frac{n-2}{2n}} t^{-\frac{n-2}{n}-\beta} \quad t \geq t_0\,,$$

hence by (A2) (w.l.o.g. $t_0 = 0$)

$$\begin{aligned}\Psi_0 &\geq c\int_0^{|\nabla u|} h(t)^{\frac{n-2}{2n}} t^{\frac{2}{n}-1-\beta}\, dt \geq c\int_{|\nabla u|/2}^{|\nabla u|} h(t)^{\frac{n-2}{2n}} t^{\frac{2}{n}-1-\beta}\, dt \\ &\geq ch(|\nabla u|/2)^{\frac{n-2}{2n}} \int_{|\nabla u|/2}^{|\nabla u|} t^{\frac{2}{n}-1-\beta}\, dt \geq h(|\nabla u|)^{\frac{n-2}{2n}} |\nabla u|^{\frac{2}{n}-\beta}\,.\end{aligned}$$

Since Ψ_0 is in the space $L_{loc}^{\frac{2n}{n-2}}$, we arrive at

$$h(|\nabla u|)|\nabla u|^{\gamma} \in L^1_{loc} \tag{5.4}$$

for a suitable exponent γ. As in Section 3 we can iterate (5.4) to get Lemma 3.1. The arguments from Section 4 are easily adjusted, thus it is shown:

Theorem 5.1. *Let h satisfy (A1), (A2), (A3*) and (A4*). Suppose in addition that we have (5.3) with $\beta < 2/n$ at least in case $n \geq 3$. Then any local J-minimizer u subject to the constraint $u \geq \Psi$ a.e. on Ω is locally Lipschitz, if so is the obstacle Ψ. Moreover, ∇u satisfies a local Hölder condition, if $\nabla\Psi$ has this property.*

Remark 5.1. *The reader should note that (A3*) together with (5.3) implies a condition similar to (2.9) of [MP].*

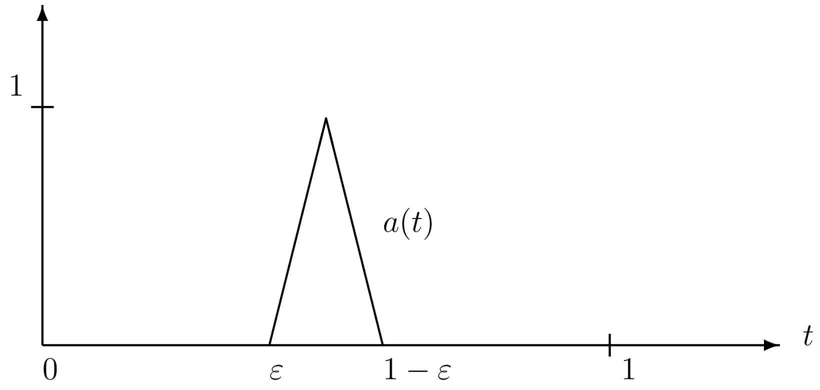

Figure 1: The function $a(t)$.

Remark 5.2. *Suppose that we have (A1), (A2), (A3*) and (A4*) together with*

$$q < p\frac{n}{n-2}\,. \tag{5.5}$$

Then it is easy to check that (5.3) holds with $\beta < 2/n$, thus we have the conclusions of Theorem 5.1. At the same time condition (5.5) corresponds to (1.11) in [BFM], and since our hypotheses imply (1.9) from this paper, we see that Theorem 5.1 slightly extends part (a) of Theorem 1.1 from [BFM].

Remark 5.3. *Clearly it is possible to replace inequality (5.3) in Theorem 5.1 through the weaker requirement that there exist a constant $c > 0$ and an exponent $\gamma > 0$ such that*

$$\int_0^t h''(s)^{\frac{1}{2}}\,ds \geq c h(t)^{\frac{n-2}{2n}} t^{\gamma} \tag{5.3*}$$

holds for all $t \geq t_0$. We note that (5.3) is more adequate in the case of oscillatory behaviour of h''.*

We next give an example of a density h with (A1), (A2), (A3*) and (A4*), which does not satisfy (5.3), if the parameter q is chosen large enough, but for which (5.3*) holds. For numbers $1 < p < q < \infty$, $q \geq 2$, we let

$$\Theta(t) := (1+t^2)^{\frac{p-2}{2}} + a(t)t^{q-2}\,, \quad t \geq 0\,,$$

and define

$$h(t) := \int_0^t \left[\int_0^r \Theta(\rho)\,d\rho\right] dr \quad t \geq 0\,,$$

where the function a: $[0,\infty) \to [0,\infty)$ is the periodic extension of the function $[0,1] \to [0,1]$ indicated in Figure 1 below.

Here $\varepsilon \in [0, 1/2)$ denotes some free parameter. The properties (A1) and (A4*) are immediate. For (A3*) it is enough to show

$$a(t)t^{q-2} \leq c\frac{1}{t}\int_0^t a(r)r^{q-2}\,dr \tag{5.6}$$

for a constant $c > 0$ and all sufficiently large values of t. Given t we choose $n \in \mathbb{N}$ such that $t \in [n, n+1]$. We get

$$\begin{aligned}
\int_0^t a(r) r^{q-2}\, dr &\geq \int_0^n a(r) r^{q-2}\, dr = \sum_{k=0}^{n-1} \int_k^{k+1} a(r) r^{q-2}\, dr \geq \sum_{k=0}^{n-1} k^{q-2} \int_k^{k+1} a(r)\, dr \\
&= c \sum_{k=0}^{n-1} k^{q-2} = c \sum_{k=1}^{n-1} \int_k^{k+1} r^{q-2} \Big[\frac{k}{r}\Big]^{q-2} dr \geq c \sum_{k=1}^{n-1} \int_k^{k+1} r^{q-2} \Big[\frac{k}{k+1}\Big]^{q-2} dr \\
&\geq c \sum_{k=1}^{n-1} \int_k^{k+1} r^{q-2}\, dr = c \int_1^n r^{q-2}\, dr = c r^{q-1}\big|_1^n = c(n^{q-1} - 1) \geq c n^{q-1} \\
&= c t^{q-1} \Big[\frac{n}{t}\Big]^{q-1} \geq c t^{q-1} \Big[\frac{n}{n+1}\Big]^{q-1} \geq c t^{q-1} ,
\end{aligned}$$

and for $t \geq 1$ it follows

$$c t^{q-2} \leq \frac{1}{t} \int_0^t a(r) r^{q-2}\, dr\,. \tag{5.7}$$

Since $0 \leq a(t) \leq 1$, inequality (5.7) clearly implies (5.6), and (A3*) is established. As remarked earlier, condition (A2) follows from (A1) and (A3*), thus the density h satisfies the requirements of Theorem 5.1 with the exception of (5.3): from (5.7) it follows that $\frac{h'(t)}{t}$ is bounded from below by ct^{q-2}, whereas $h''(t) = (1+t^2)^{(p-2)/2}$ on the set $[a = 0]$. Thus, for q large enough, (5.3) is violated, and for the same reason inequality (2.9) of [MP] fails to hold. However, it is easy to see that inequality (5.3*) is fulfilled with $\gamma = q/n$, so that we have the conclusions of Theorem 5.1 for local minimizers of this particular energy.

For completeness we look at a non-autonomous variant of Theorem 1.1, which means that we consider densities $H(x, \xi) = h(x, |\xi|)$, $h = h(x,t)$, $x \in \overline{\Omega}$, $t \geq 0$, satisfying the required conditions uniformly in $x \in \overline{\Omega}$ replacing h' by $\frac{\partial}{\partial t} h$, etc.

Theorem 5.2. *Assume that (A1)-(A4) hold for the density $h(x,t)$. Suppose that (A3) is satisfied with $\alpha = 1$, moreover we have*

$$\Big| \nabla_x \frac{\partial}{\partial t} h(x,t) \Big| \leq c \frac{\partial}{\partial t} h(x,t)\,, \quad x \in \overline{\Omega}, \quad t \geq 0\,. \tag{A5}$$

a) Then the conclusions of Theorem 1.1 extend to local minimizers u of $\int_\Omega h(x, |\nabla u|)\, dx$ subject to the constraint $u \geq \Psi$.

b) If the vectorial case is considered, i.e. if $u \colon \Omega \to \mathbb{R}^M$, $M \geq 1$, is an unconstrained local minimizer of the energy $\int_\Omega h(x, |\nabla u|)\, dx$, then ∇u is locally bounded.

Formally the proof of Theorem 5.2 is easily obtained by following the arguments leading to the statements of Theorem 1.1, where the additional terms resulting from the x-dependence are handled with the help of (A5). Thus the proof would be complete, if we assume a sufficient degree of initial regularity. However, as it is outlined in [ELM], the local approximation procedure from Section 3 cannot be applied. To overcome this difficulty we follow ideas of [M4] and of [CGM] by introducing a quadratic regularization from below. The reader should note that at this stage the validity of (A3) with $\alpha = 1$ enters in an essential way. For details we refer to Section 3 of [BF]. We wish to remark that (A5) does not cover the case of "variable exponents" as discussed for example in [CM].

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