Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 313

Variations On Liouville's Theorem In The Setting Of Stationary Flows Of Generalized Newtonian Fluids In The Plane

Martin Fuchs

Saarbrücken 2012

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 313
submitted: 26^{th} of July, 2012

Variations On Liouville's Theorem In The Setting Of Stationary Flows Of Generalized Newtonian Fluids In The Plane

Martin Fuchs

Saarland University
Dep. of Mathematics
P.O. Box 15 11 50
D-66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

Classification: 76 D 05, 76 D 07, 76 M 30, 35 Q 30, 35 Q 35.

Keywords: generalized Newtonian fluids, stationary flows in $2D$, equations of Navier–Stokes type, Liouville theorems.

Abstract

We consider entire solutions $u : \mathbb{R}^2 \to \mathbb{R}^2$ of the equations describing the stationary flow of a generalized Newtonian fluid in $2D$ and prove the constancy of the velocity field u if either u is bounded (or even not exceeding a certain growth rate at infinity) or if u has finite energy.

1 Introduction

A large part of the work of Olga A. Ladyzhenskaya is devoted to the deep analysis of mathematical problems arising in fluid mechanics, especially to the study of the Navier-Stokes equations (NSE) including the stationary variant as well as the time dependent situation in dimensions 2 and 3. Some of her fundamental results are highlighted in the monograph [La1] and in the papers [La2], [La3], [La4], [La5], [LK]. In our note we limit ourselves to the stationary case in $2D$ and try to exhibit conditions which imply the constancy of entire solutions, i.e. of solutions defined on the whole plane. Roughly speaking our goal is to obtain Liouville-type results not only for (NSE) but also for the case of fluids with shear dependent viscosity. We first look at entire solutions of the homogeneous (NSE)

$$-\mu\Delta u + u^k \partial_k u + \nabla\pi = 0 \tag{1.1}$$

together with the incompressibility condition

$$\operatorname{div} u = 0\,. \tag{1.2}$$

Here $u : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the velocity field, $\pi : \mathbb{R}^2 \to \mathbb{R}$ is the pressure and μ stands for the constant viscosity. W.l.o.g. we can assume that $\mu = 1$, moreover, throughout this paper we adopt the convention of summation w.r.t. indices repeated twice. In 1978 the following result was obtained by Gilbarg and Weinberger [GW]:

THEOREM 1.1. *Suppose that (1.1) and (1.2) hold on the entire plane and that the (weak and thereby classical) solution $u : \mathbb{R}^2 \to \mathbb{R}^2$ is of finite energy in the sense that*

$$\int_{\mathbb{R}^2} |\nabla u|^2\, dx < \infty\,. \tag{1.3}$$

Then the constancy of u follows.

In memory of Olga A. Ladyzhenskaya 1922 - 2004

The author likes to thank the organizers of the workshop “Advances in mathematical analysis of partial differential equations” at Institut Mittag-Leffler, 9-13 July 2012.

One major ingredient of their proof is to introduce the vorticity $\omega := \partial_1 u^2 - \partial_2 u^1$ satisfying the linear elliptic equation

$$\Delta\omega - u \cdot \nabla\omega = 0\,, \tag{1.4}$$

and to give precise estimates for the growth of Dirichlet's energy of ω on annuli in terms of the radius with the help of the maximum-principle valid for solutions of equation (1.4). Approximately 30 years later Koch, Nadirashvili, Seregin, Sverăk [KNSS] studied entire solutions $u(t,x)$ of the much more delicate instationary variant of (1.1) and as a byproduct of their investigations they could replace (1.3) by the requirement that u is a bounded function, more precisely:

THEOREM 1.2. *If (1.1) and (1.2) hold on $\mathbb{R}^2$ and if*

$$\sup_{\mathbb{R}^2} |u| < \infty\,, \tag{1.5}$$

then $u = u_0$ for some vector $u_0 \in \mathbb{R}^2$.

We just wish to remark that the proof of Theorem 1.2 uses the linearity of the leading part of (NSE) in an essential way. An interesting question is, if we can weaken condition (1.5). In [BFZ] we could show:

THEOREM 1.3. *Suppose that $u : \mathbb{R}^2 \to \mathbb{R}^2$ is a solution of (1.1) and (1.2) on $\mathbb{R}^2$ such that*

$$\limsup_{|x|\to\infty} |u(x)|\,|x|^{-\alpha} < \infty \tag{1.6}$$

for some $\alpha < 1/3$. Then the constancy of u follows.

Since the proof of Theorem 1.3 is rather selfcontained, we will present it in Section 2. It would be very interesting to know, if actually $\alpha < 1$ is admissible in (1.6).

In her book [La1] Olga A. Ladyzhenskaya proposes also to study fluids with variable viscosity. One approach is to replace (1.1) through the equation

$$-\operatorname{div}\left[T^D\left(\varepsilon(u)\right)\right] + u^k\partial_k u + \nabla\pi = 0\,, \tag{1.7}$$

where $\varepsilon(u)$ is the symmetric gradient of the velocity field and where T^D is the deviatoric part of the stress tensor. For further mathematical and also physical explanations we refer to the monographs of Ladyzhenskaya [La1], Galdi [Ga1], [Ga2] and Málek, Necăs, Rokyta, Růžička [MNRR] as well as to the book [FS]. We assume that

$$T^D = \nabla H \tag{1.8}$$

for a potential H being of the form

$$H(\varepsilon) = h(|\varepsilon|) \tag{1.9}$$

with a given density $h : [0,\infty) \to [0,\infty)$ of class C^2. Since

$$T^D(\varepsilon) = \frac{h'(|\varepsilon|)}{|\varepsilon|}\,\varepsilon\,,$$

we see that the variable viscosity is given by

$$\mu = \mu\left(|\varepsilon|\right) = \frac{h'(|\varepsilon|)}{|\varepsilon|}\,.$$

Following standard terminology we say that the flow is shear thickening (thinning), if μ is an increasing (decreasing) function of $|\varepsilon|$.

Next we formulate our hypotheses imposed on the density h occurring in the structural condition (1.9). We suppose that h satisfies:

(A1) h is strictly increasing and convex; we have $h''(0) > 0$ and $\lim\limits_{t\to 0} \dfrac{h(t)}{t} = 0\,.$

(A2) There is a constant $a > 0$ such that $h(2t) \le a h(t)$ for all $t \ge 0$ (doubling property).

($A3_{\mathrm{I}}$) In the shear thickening case we have $\dfrac{h'(t)}{t} \le h''(t)$ for all $t > 0\,.$

($A3_{\mathrm{II}}$) In the shear thinning case we have $h''(t) \le \dfrac{h'(t)}{t}$ for all $t > 0\,.$

REMARK 1.1. *i) From (A1) it directly follows that $h(0) = h'(0)$ and $h'(t) > 0$ for any $t > 0$.*

ii) By considering $\frac{d}{dt}\frac{h'(t)}{t}$ it is immediate that ($A3_{\mathrm{I}}$) and ($A3_{\mathrm{II}}$) express the fact that the fluid is shear thickening and shear thinning, respectively.

iii) (A1) together with (A2) implies the balancing condition

$$c\,th'(t) \le h(t) \le th'(t) \quad \textit{for all } t \ge 0 \tag{1.10}$$

and for a suitable positive constant c. In fact, $0 = h(0) \ge h(t) - th'(t)$ holds by convexity, whereas by (A2) and the monotonicity of h'

$$h(t) \ge \frac{1}{a}h(2t) = \frac{1}{a}\int_0^{2t} h'(s)\,ds \ge \frac{1}{a}\int_t^{2t} h'(s)\,ds \ge \frac{1}{a}th'(t)\,.$$

iv) It is easy to see that from (A2) it follows

$$h(t) \le h(1)t^a \quad \textit{for all} \;\; t \ge 1\,,$$

thus

$$(1.11) \qquad h(t) \le c\,[t^a + 1] \quad \textit{for all } t \ge 0\,.$$

v) If we are in the shear thickening case (A3$_{\rm I}$), then $\frac{h'(t)}{t} \ge \lim_{s\to 0} \frac{h'(s)}{s} = h''(0)$ *gives*

$$(1.12) \qquad h(t) \ge \frac{1}{2} h''(0) t^2 \quad \textit{for all } t \ge 0\,,$$

and (A1) implies on account of $h''(0) > 0$ *that our energy is of at least quadratic growth.*

vi) In the shear thinning case we have

$$(1.13) \qquad h(t) \le \frac{1}{2} h''(0) t^2$$

and

$$(1.14) \qquad h'(t)^2 \le c h(t)$$

for any $t \ge 0$*. For (1.14) we observe* $h'(t) \le t h''(0)$*, which is an immediate consequence of* $h'(t)/t \le \lim_{s\to 0} h'(s)/s$*, thus*

$$h'(t)^2 \le t h''(0) h'(t) \overset{(1.10)}{\le} c h''(0) h(t)\,.$$

Note that according to (1.13) the condition (A3$_{\rm II}$) implies that the energy has subquadratic growth.
Actually, even the case of linear growth is covered , which means that we can easily give examples of densities h satisfying (A1)-(A3$_{\rm II}$) for which $\lim_{t\to\infty} h(t)/t \in (0,\infty)$*.*

vii) It is not hard to show that (A1) and (A3$_{\rm II}$) already imply (A2), we refer to the Appendix of [BF1].

REMARK 1.2. *Standard examples are non-degenerate power law fluids for which* $h(t) = (1+t^2)^{p/2}$*,* $t \ge 0$*, with* $p \in (1,\infty)$ *being of shear thickening type, if* $p \ge 2$*, and of shear thinning type otherwise. Another model we can include is the Prandtl-Eyring fluid described by* $h(t) = t\ln(1+t)$*,* $t \ge 0$*.*

Summarizing the results obtained in [Fu], [FZ] and [Zh] we have:

THEOREM 1.4. *Suppose that* $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ *is a (weak) solution of (1.7) and (1.2) on* $\mathbb{R}^2$ *under the hypothesis (1.8) and (1.9). Then u is constant, if* *either*

a) h satisfies (A1, A2, A3$_{\mathrm{I}}$) and it holds

$$\int_{\mathbb{R}^2} h\left(|\nabla u|\right)\, dx < \infty \quad \textit{or} \quad \sup_{\mathbb{R}^2} |u| < \infty\,,$$

or

b) h satisfies (A1, A2, A3$_{\mathrm{II}}$) and in addition

$$h''(t) \geq c/(1+t),\ t \geq 0\,,$$

and u is assumed to be bounded.

Note that the lower bound imposed on h'' in part b) of Theorem 1.4 together with $h(0) = h'(0) = 0$ implies that $h(t) \geq ct\ln(1+t), t \geq 0$. Since the case of bounded solutions for shear thinning flows is treated in the cited references only for the Prandtl-Eyring model, a proof of part b) of Theorem 1.4 will be provided in Section 3. Let us clarify our terminology: a function $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ is called a weak solution of (1.7) and (1.2), if (1.2) holds in the pointwise sense and if

$$\int_{\mathbb{R}^2} DH\left(\varepsilon(u)\right) : \varepsilon(\varphi)\, dx + \int_{\mathbb{R}^2} u^k \partial_k u \cdot \varphi\, dx = 0 \tag{1.15}$$

holds all $\varphi \in C^1_0(\mathbb{R}^2, \mathbb{R}^2)$, $\operatorname{div}\varphi = 0$. A discussion of the hypothesis $u \in C^1$ can be found in e.g. [Fu], [FZ], [Zh].

We emphasize that Theorem 1.4 requires the existence as well as the positivity of $D^2H(0)$ so that we can cover the non-degenerate p-case and also the Prandtl-Eyring fluid. Concerning the degenerate p-case we proved in [BFZ]:

THEOREM 1.5. *Let $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ denote a weak solution of (1.7) and (1.2) (cf. equation (1.15)) with $H(\varepsilon) = |\varepsilon|^p$, $1 < p < \infty$.*

i) a) Suppose that $1 < p \leq 2$. If u is bounded, then the constancy of u follows.

b) Let $1 < p < 2$. Then

$$\limsup_{|x|\to\infty} |x|^{-\alpha}\, |u(x)| < \infty$$

for some $\alpha \in \left[0, \frac{2-p}{p+6}\right)$ implies the constancy of u.

ii) If $\frac{6}{5} < p \leq 3$ together with $\int_{\mathbb{R}^2} |\nabla u|^p\, dx < \infty$, then again $u \equiv$ const follows.

iii) Let $p > 2$ and let $u_\infty \in \mathbb{R}^2$ denote a vector such that

a) in case $2 < p < 6$:

$$\sup_{|x|\geq R} |u(x) - u_\infty|\, |x|^{\frac{p-2}{p+6}} \to 0 \ \textit{as}\ R \to \infty\,,$$

b) in case $p = 6$:

$$\limsup_{|x|\to\infty} |x|^{1/3} |u(x) - u_\infty| < \infty \,,$$

c) in case $p > 6$:

$$\sup_{|x|\geq R} |u(x) - u_\infty| \, |x|^{1/3} \longrightarrow 0 \ \text{as} \ R \to \infty \,.$$

Then we obtain $u = u_\infty$.

REMARK 1.3. *Letting $p = 2$ in i) and ii) we recover the statements of Theorem 1.1 and 1.2. For $p < 2$ even a certain growth is allowed depending on the number $(2-p)/(p+6)$. Clearly one should try to optimize this apriori growth condition, and it also should be investigated what happens in the finite energy case, if $p > 3$ or $p < 6/5$. For $p > 2$ actually a certain decay of the solution is required being stated in iii), a) - c).*

REMARK 1.4. *The C^1-regularity of $W^1_{p,\mathrm{loc}}$-solutions of (1.7) and (1.2) for the degenerate p-case has been investigated by Naumann [Na] and Wolf [Wo] for $\frac{3}{2} < p < 2$. For $p > 2$ the C^1-regularity of weak local solutions seems to be open.*

2 Proof of Theorem 1.3

Let u denote an entire solution of (1.1) and (1.2) satisfying (1.6). Introducing the vorticity $\omega := \partial_2 u^1 - \partial_1 u^2$ we have for $q, \ell \in \mathbb{N}$ sufficiently large with $\eta \in C_0^\infty(\mathbb{R}^2)$

$$\begin{aligned}
(2.1) \qquad \int_{\mathbb{R}^2} \omega^{2q}\eta^{2\ell}\, dx &= \int_{\mathbb{R}^2} \left(\partial_2 u^1 - \partial_1 u^2\right) \omega^{2q-1}\eta^{2\ell}\, dx \\
&= \int_{\mathbb{R}^2} \operatorname{div}\left(-u^2, u^1\right) \omega^{2q-1}\eta^{2\ell}\, dx = -\int_{\mathbb{R}^2} \left(-u^2, u^1\right) \cdot \nabla \left[\omega^{2q-1}\eta^{2\ell}\right] dx \\
&= (2q-1) \int_{\mathbb{R}^2} \nabla\omega \cdot \left(u^2, -u^1\right) \omega^{2q-2}\eta^{2\ell}\, dx \\
&\quad +2\ell \int_{\mathbb{R}^2} \left(u^2, -u^1\right) \cdot \nabla\eta\, \omega^{2q-1}\eta^{2\ell-1}\, dx \,,
\end{aligned}$$

and from $\operatorname{div} u = 0$ we infer

$$(2.2) \ \int_{\mathbb{R}^2} u \cdot \nabla\omega\, \omega^{2q-3}\eta^{2\ell}\, dx = \frac{1}{2q-2} \int_{\mathbb{R}^2} u \cdot \nabla\omega^{2q-2}\eta^{2\ell}\, dx = -\frac{1}{2q-2} \int_{\mathbb{R}^2} u \cdot \nabla\eta^{2\ell}\omega^{2q-2} \,.$$

Recall (cf. (1.4)) that $\Delta\omega - u \cdot \nabla\omega = 0$ on $\mathbb{R}^2$, hence

$$\int_{\mathbb{R}^2} \nabla\omega \cdot \nabla\varphi\, dx + \int_{\mathbb{R}^2} u \cdot \nabla\omega\varphi\, dx = 0$$

for $\varphi \in C_0^1(\mathbb{R}^2)$. We specify $\varphi = \eta^{2\ell}\omega^{2q-3}$ and get

$$\begin{aligned}
(2.3) \qquad & \int_{\mathbb{R}^2} \eta^{2\ell} (2q-3) \left|\nabla\omega\right|^2 \omega^{2q-4}\, dx \\
&= -\int_{\mathbb{R}^2} \nabla\omega \cdot \nabla\eta^{2\ell}\omega^{2q-3}\, dx - \int_{\mathbb{R}^2} u \cdot \nabla\omega\omega^{2q-3}\eta^{2\ell}\, dx \,.
\end{aligned}$$

By Young's inequality the first term on the r.h.s. of (2.3) is estimated through

$$\delta \int_{\mathbb{R}^2} |\nabla\omega|^2\, \omega^{2q-4}\eta^{2\ell}\, dx + c\,(\delta,\ell) \int_{\mathbb{R}^2} |\nabla\eta|^2\, \eta^{2\ell-2}\omega^{2q-2}\, dx\,,$$

to the second term on the r.h.s. of (2.3) we apply (2.2). This yields after appropriate choice of δ

(2.4)

$$\int_{\mathbb{R}^2} |\nabla\omega|^2\, \omega^{2q-4}\eta^{2\ell}\, dx \le c\,(\ell,q) \left[\int_{\mathbb{R}^2} \omega^{2q-2}\eta^{2\ell-2}\, |\nabla\eta|^2\, dx + \int_{\mathbb{R}^2} |u|\, \left|\nabla\eta^{2\ell}\right| \omega^{2q-2}\, dx \right].$$

Now we return to (2.1) and apply Young's inequality one more time with the result

$$\begin{aligned}
\int_{\mathbb{R}^2} \omega^{2q}\eta^{2\ell}\, dx &\le (2q-1) \int_{\mathbb{R}^2} |\nabla\omega|\, |u| \omega^{2q-2}\eta^{2\ell}\, dx \\
&\quad +2\ell \int_{\mathbb{R}^2} |u|\, |\nabla\eta|\, \omega^{2q-1}\eta^{2\ell-1}\, dx \\
&\le \delta \int_{\mathbb{R}^2} \omega^{2q}\eta^{2\ell}\, dx + c(\delta,q) \int_{\mathbb{R}^2} |\nabla\omega|^2\, |u|^2 \omega^{2q-4}\eta^{2\ell}\, dx \\
&\quad +2\ell \int_{\mathbb{R}^2} |u|\, |\nabla\eta|\, \omega^{2q-1}\eta^{2\ell-1}\, dx\,,
\end{aligned}$$

hence for δ sufficiently small

$$\int_{\mathbb{R}^2} \eta^{2\ell}\omega^{2q}\, dx \le c(\ell,q) \left[\int_{\mathbb{R}^2} |\nabla\omega|^2\, |u|^2 \omega^{2q-4}\eta^{2\ell}\, dx + \int_{\mathbb{R}^2} |u|\, |\nabla\eta|\, \omega^{2q-1}\eta^{2\ell-1}\, dx \right]. \tag{2.5}$$

Next we specify η: let $R \ge 1$ and choose $\eta = 1$ on $B_R(0)$, $0 \le \eta \le 1$, $\operatorname{spt}\eta \subset B_{2R}(0)$, $|\nabla\eta| \le c/R$. From (1.6) we obtain (w.l.o.g. we assume $\alpha > 0$)

$$|u(x)| \le cR^\alpha,\ \ x \in B_R(0)\,. \tag{2.6}$$

We use (2.6) on the r.h.s. of (2.5) and get

$$\int_{B_{2R}(0)} \eta^{2\ell}\omega^{2q}\, dx \le c(\ell,q) \left[R^{2\alpha} \int_{B_{2R}(0)} |\nabla\omega|^2\, \omega^{2q-4}\eta^{2\ell}\, dx + R^\alpha \int_{B_{2R}(0)} |\nabla\eta|\, \omega^{2q-1}\eta^{2\ell-1}\, dx \right],$$

and if we apply (2.4) on the r.h.s. quoting (2.6) one more time it follows

$$\begin{aligned}
\int_{B_{2R}(0)} \eta^{2\ell}\omega^{2q}\, dx &\le c(\ell,q) \Bigg[R^{2\alpha} \int_{B_{2R}(0)} \omega^{2q-2}\eta^{2\ell-2}\, |\nabla\eta|^2\, dx \\
&\quad +R^{3\alpha} \int_{B_{2R}(0)} \left|\nabla\eta^{2\ell}\right| \omega^{2q-2}\, dx \\
&\quad +R^\alpha \int_{B_{2R}(0)} \omega^{2q-1}\, |\nabla\eta|\, \eta^{2\ell-1}\, dx \Bigg] \\
&=: c\,(\ell,q)\, [T_1 + T_2 + T_3]\,.
\end{aligned} \tag{2.7}$$

Young's inequality yields

$$\begin{aligned}
T_1 &\le \int_{B_{2R}(0)} \omega^{2q-2}\eta^{2\ell-2}R^{2\alpha-2}\,dx\\
&\le \delta\int_{B_{2R}(0)} \omega^{2q}\eta^{(2\ell-2)2q/(2q-2)}\,dx + c(\delta)R^{2+q(2\alpha-2)}\,,\\
T_2 &\le \int_{B_{2R}(0)} \omega^{2q-2}\eta^{2\ell-1}R^{3\alpha-1}\,dx\\
&\le \delta\int_{B_{2R}(0)} \omega^{2q}\eta^{(2\ell-1)2q/(2q-2)}\,dx + c(\delta)R^{2+q(3\alpha-1)}\,,\\
T_3 &\le \int_{B_{2R}(0)} \omega^{2q-1}\eta^{2\ell-1}R^{\alpha-1}\,dx\\
&\le \delta\int_{B_{2R}(0)} \omega^{2q}\eta^{(2\ell-1)2q/(2q-1)}\,dx + c(\delta)R^{2+2q(\alpha-1)}\,,
\end{aligned}$$

and for $\ell \gg 1$ we have

$$2\ell \le (2\ell-2)2q/(2q-2)$$

and

$$2\ell \le (2\ell-1)2q/(2q-1)\,,$$

hence, for δ small enough, we obtain from (2.7)

$$\int_{B_{2R}(0)} \eta^{2\ell}\omega^{2q}\,dx \le c(\ell,q)\left[R^{2+q(2\alpha-2)} + R^{2+(3\alpha-1)q} + R^{2+2q(\alpha-1)}\right]\,. \tag{2.8}$$

Recall that $\alpha < 1/3$. Therefore we can fix a sufficiently large exponent q with the property that

$$2+(3\alpha-1)q < 0\,,$$

and (2.8) shows

$$\int_{B_R(0)} \omega^{2q}\,dx \le c(\ell,q)R^{2+(3\alpha-1)q} \longrightarrow 0$$

and $R\to\infty$, hence $\omega \equiv 0$ on $\mathbb{R}^2$. This together with $\operatorname{div} u = 0$ shows that u is harmonic and the constancy of u then follows from (1.6) and results concerning entire harmonic functions. □

3 Bounded solutions in the shear thinning case

Throughout this section we assume that h satisfies (A1,2) and ($\mathrm{A3_{II}}$) together with $h''(t) \ge c/(1+t)$ and that $u \in C^1(\mathbb{R}^2,\mathbb{R}^2)$ denotes a bounded solution of (1.7) and (1.2). Then it is an easy exercise to show that u is in the space $W^2_{2\,\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$, which follows from the non-degeneracy of D^2H and the local boundedness of ∇u. For proving the constancy of u we first claim:

Lemma 3.1. *There is a constant $c = c(\|u\|_{L^\infty(\mathbb{R}^2)})$ such that*

$$\int_{Q_R(x_0)} h\left(|\varepsilon(u)|\right)\, dx \le c(R+1) \tag{3.1}$$

for any square $Q_R(x_0) := \{x \in \mathbb{R}^2 : |x^i - x_0^i| < R,\ i = 1, 2\}$.

Proof of Lemma 3.1: Recalling (1.15) we have

$$0 = \int_{Q_{2R}(x_0)} DH\left(\varepsilon(u)\right) : \varepsilon\left(\eta^2 u - w\right)\, dx + \int_{Q_{2R}(x_0)} u^k \partial_k u^i \left(\eta^2 u^i - w^i\right)\, dx\,. \tag{3.2}$$

Here $\eta \in C_0^1\left(Q_{2R}(x_0)\right)$ denotes a cut-off function such that $\eta = 1$ on $Q_R(x_0)$, $0 \le \eta \le 1$, $|\nabla \eta| \le c/R$ and $w \in \overset{\circ}{W}{}_2^1\left(Q_{2R}(x_0), \mathbb{R}^2\right)$ is chosen to satisfy (cf. [Ga1])

$$\operatorname{div} w = \operatorname{div}(\eta^2 u) = \nabla \eta^2 \cdot u \quad \text{on } Q_{2R}(x_0)$$

together with

$$\|\nabla w\|_{L^2(Q_{2R}(x_0))} \le c\left\|\nabla \eta^2 \cdot u\right\|_{L^2(Q_{2R}(x_0))}\,.$$

From (3.2) we obtain

$$\begin{aligned}
&\int_{Q_{2R}(x_0)} DH\left(\varepsilon(u)\right) : \varepsilon(u)\eta^2\, dx \\
&\quad + \int_{Q_{2R}(x_0)} 2\frac{\partial H}{\partial \varepsilon_{i\alpha}}\left(\varepsilon(u)\right) \partial_\alpha \eta u^i \eta\, dx \\
&\quad - \int_{Q_{2R}(x_0)} DH\left(\varepsilon(u)\right) : \varepsilon(w)\, dx + \int_{Q_{2R}(x_0)} u^k \partial_k u^i u^i \eta^2\, dx \\
&\quad - \int_{Q_{2R}(x_0)} u^k \partial_k u^i w^i\, dx = 0\,.
\end{aligned}$$

We have

$$\int_{Q_{2R}(x_0)} DH\left(\varepsilon(u)\right) : \varepsilon(u)\eta^2\, dx \ge \int_{Q_{2R}(x_0)} \eta^2 H\left(\varepsilon(u)\right)\, dx\,,$$

$$\begin{aligned}
&\left|2\int_{Q_{2R}(x_0)} \frac{\partial H}{\partial \varepsilon_{i\alpha}}\left(\varepsilon(u)\right) \partial_\alpha \eta u^i \eta\, dx\right| \\
&\quad \le\ c\int_{Q_{2R}(x_0)} h'\left(|\varepsilon(u)|\right) |\nabla \eta||u|\eta\, dx \\
&\quad \le\ \delta\int_{Q_{2R}(x_0)} h'\left(|\varepsilon(u)|\right) |\varepsilon(u)|\eta^2\, dx + c(\delta)\int_{Q_{2R}(x_0)} \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|}|u|^2|\nabla \eta|^2\, dx\,,
\end{aligned}$$

and if δ is chosen small enough and if we take into account the inequality $\frac{h'(t)}{t} \leq$ const (compare the remarks after (1.14)) in combination with (1.10) it follows

$$
\begin{aligned}
(3.3)\qquad \int_{Q_{2R}(x_0)} \eta^2 H\left(\varepsilon(u)\right)\, dx \leq c \Bigg[& \int_{Q_{2R}(x_0)} |u|^2 |\nabla\eta|^2 \, dx \\
& + \left| \int_{Q_{2R}(x_0)} DH\left(\varepsilon(u)\right) : \varepsilon(w) \, dx \right| \\
& + \left| \int_{Q_{2R}(x_0)} u^k \partial_k u^i u^i \eta^2 \, dx \right| + \left| \int_{Q_{2R}(x_0)} u^k \partial_k u^i w^i \, dx \right| \Bigg] \\
=: \;\; & c\left[T_1 + T_2 + T_3 + T_4\right] .
\end{aligned}
$$

The quantities T_i are estimated as follows: it clearly holds

$$
(3.4)\qquad T_1 \leq c R^{-2} \int_{Q_{2R}(x_0)} |u|^2 \, dx .
$$

We have for any $\delta > 0$ by Young's inequality and the properties of w

$$
\begin{aligned}
T_2 \;\; & \leq \;\; \delta \int_{Q_{2R}(x_0)} h'\left(|\varepsilon(u)|\right)^2 \, dx + c(\delta) \int_{Q_{2R}(x_0)} |\nabla w|^2 \, dx \\
& \leq \;\; \delta \int_{Q_{2R}(x_0)} h'\left(|\varepsilon(u)|\right)^2 \, dx + c(\delta) \int_{Q_{2R}(x_0)} |\operatorname{div}(\eta^2 u)|^2 \, dx \\
& \leq \;\; \delta \int_{Q_{2R}(x_0)} h'\left(|\varepsilon(u)|\right)^2 \, dx + c(\delta) R^{-2} \int_{Q_{2R}(x_0)} |u|^2 \, dx ,
\end{aligned}
$$

hence (replacing δ by $\delta/4$ and quoting (1.14))

$$
(3.5)\qquad T_2 \leq \delta \int_{Q_{2R}(x_0)} H\left(\varepsilon(u)\right) \, dx + c(\delta) R^{-2} \int_{Q_{2R}(x_0)} |u|^2 \, dx .
$$

Next we observe

$$
\begin{aligned}
(3.6)\qquad T_3 \;\; & = \;\; \frac{1}{2} \left| \int_{Q_{2R}(x_0)} u^k \partial_k |u|^2 \eta^2 \, dx \right| \\
& = \;\; \frac{1}{2} \left| \int_{Q_{2R}(x_0)} u^k |u|^2 \partial_k \eta^2 \, dx \right| \leq c R^{-1} \int_{Q_{2R}(x_0)} |u|^3 \, dx ,
\end{aligned}
$$

and from

$$
\int_{Q_{2R}(x_0)} u^k \partial_k u^i w^i \, dx = - \int_{Q_{2R}(x_0)} u^k u^i \partial_k w^i \, dx
$$

it follows using Hölder's inequality

$$
\begin{aligned}
(3.7)\qquad T_4 &\le \left(\int_{Q_{2R}(x_0)} |u|^4\,dx\right)^{1/2}\left(\int_{Q_{2R}(x_0)} |\nabla w|^2\,dx\right)^{1/2}\\
&\le c\left(\int_{Q_{2R}(x_0)} |u|^4\,dx\right)^{1/2}\left(R^{-2}\int_{Q_{2R}(x_0)} |u|^2\,dx\right)^{1/2}\\
&= cR^{-1}\left[\int_{Q_{2R}(x_0)} |u|^4\,dx\int_{Q_{2R}(x_0)} |u|^2\,dx\right]^{1/2}\\
&\le cR^{-1}\left[\int_{Q_{2R}(x_0)} |u|^4\,dx+\int_{Q_{2R}(x_0)} |u|^2\,dx\right].
\end{aligned}
$$

Inserting (3.4) - (3.7) into (3.3) we get

$$
\begin{aligned}
\int_{Q_R(x_0)} H(\varepsilon(u))\,dx &\le \delta\int_{Q_{2R}(x_0)} H(\varepsilon(u))\,dx\\
&\quad +c(\delta)\left[R^{-1}\int_{Q_{2R}(x_0)}\left(|u|^2+|u|^3+|u|^4\right)dx+R^{-2}\int_{Q_{2R}(x_0)}|u|^2\,dx\right]
\end{aligned}
$$

for any $\delta>0$ and all $Q_R(x_0)$. The δ-Lemma of Giaquinta and Modica (cf. Lemma 0.5 in [GM]) and its slight extension stated in Lemma 3.1 of [FZ] then yields

$$
\int_{Q_R(x_0)} H(\varepsilon(u))\,dx \le c\left[R^{-1}\int_{Q_{2R}(x_0)}\left(|u|^2+|u|^3+|u|^4\right)dx+R^{-2}\int_{Q_{2R}(x_0)}|u|^2\,dx\right],
$$

and since u is bounded we have established (3.1). □

Next we like to prove the validity of

$$
(3.8)\qquad \int_{\mathbb{R}^2} D^2H(\varepsilon(u))\left(\partial_k\varepsilon(u),\partial_k\varepsilon(u)\right)dx<\infty\,.
$$

Note that from (3.8) we immediately get (recalling our hypotheses imposed on h)

$$
(3.9)\qquad \int_{\mathbb{R}^2}\frac{1}{1+|\varepsilon(u)|}|\nabla\varepsilon(u)|^2\,dx<\infty\,.
$$

For the discussion of (3.8) we return to equation (1.15). Replacing φ by $\partial_\alpha\varphi$ for $\varphi\in C_0^\infty(Q_{\frac{3}{2}R}(x_0),\mathbb{R}^2)$ with $\operatorname{div}\varphi=0$ we obtain by partial integration

$$
\begin{aligned}
(3.10)\qquad 0 &= \int_{Q_{\frac{3}{2}R}(x_0)} D^2H(\varepsilon(u))\left(\partial_\alpha\varepsilon(u),\varepsilon(\varphi)\right)dx\\
&\quad -\int_{Q_{\frac{3}{2}R}(x_0)} u^k\partial_k u^i\partial_\alpha\varphi^i\,dx,\ \ \alpha=1,2\,.
\end{aligned}
$$

Let $\eta \in C_0^\infty(Q_{\frac{3}{2}R}(x_0))$ such that $\eta = 1$ on $Q_R(x_0)$, $0 \le \eta \le 1$ and $|\nabla \eta| \le c/R$. Let $f_\alpha := \operatorname{div}(\partial_\alpha u \eta^2) = \partial_\alpha u \cdot \nabla \eta^2$ and select w_α according to [Ga1] from the space $\overset{\circ}{W}{}_2^1\left(Q_{\frac{3}{2}R}(x_0), \mathbb{R}^2\right)$ such that

$$
\begin{aligned}
&\operatorname{div} w_\alpha = f_\alpha \text{ on } Q_{\frac{3}{2}R}(x_0)\,, \\
(3.11) \qquad &\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla w_\alpha|^2 \, dx \le c \int_{Q_{\frac{3}{2}R}(x_0)} |\partial_\alpha u \cdot \nabla \eta|^2 \, dx\,.
\end{aligned}
$$

Finally we choose $\varphi := \eta^2 \partial_\alpha u - w_\alpha$ in (3.10). Equation (3.10) then yields

$$
\begin{aligned}
(3.12) \qquad & \int_{Q_{\frac{3}{2}R}(x_0)} D^2 H(\varepsilon(u))(\partial_\alpha \varepsilon(u), \partial_\alpha \varepsilon(u))\, \eta^2 \, dx \\
&= -\int_{Q_{\frac{3}{2}R}(x_0)} D^2 H(\varepsilon(u)) \left(\partial_\alpha \varepsilon(u), \nabla \eta^2 \odot \partial_\alpha u\right) dx \\
&\quad + \int_{Q_{\frac{3}{2}R}(x_0)} D^2 H(\varepsilon(u))(\partial_\alpha \varepsilon(u), \varepsilon(w_\alpha))\, dx \\
&\quad + \int_{Q_{\frac{3}{2}R}(x_0)} u^k \partial_k u^i \partial_\alpha \left(\eta^2 \partial_\alpha u^i\right) dx - \int_{Q_{\frac{3}{2}R}(x_0)} u^k \partial_k u^i \partial_\alpha w_\alpha^i \, dx \\
&=: -S_1 + S_2 + S_3 - S_4\,,
\end{aligned}
$$

where "$\odot$" is the symmetric product of vectors. Using the Cauchy–Schwarz inequality for the bilinear form $D^2H(\varepsilon(u))$ in combination with Young's inequality we obtain for any $\delta > 0$ (observe that $D^2H(\varepsilon)(\sigma, \sigma) \le c|\sigma|^2$ on account of ($\mathrm{A3_{II}}$) and the remarks after (1.14))

$$
\begin{aligned}
|S_2| &\le \delta \int_{Q_{\frac{3}{2}R}(x_0)} D^2 H(\varepsilon(u))(\partial_\alpha \varepsilon(u), \partial_\alpha \varepsilon(u))\, dx \\
&\quad + \frac{1}{\delta} \int_{Q_{\frac{3}{2}R}(x_0)} D^2 H(\varepsilon(u))(\varepsilon(w_\alpha), \varepsilon(w_\alpha))\, dx \\
&\le \delta \int_{Q_{\frac{3}{2}R}(x_0)} D^2 H(\varepsilon(u))(\partial_\alpha \varepsilon(u), \partial_\alpha \varepsilon(u))\, dx \\
&\quad + c(\delta) \int_{Q_{\frac{3}{2}R}(x_0)} \nabla w_\alpha : \nabla w_\alpha \, dx \\
&\overset{(3.11)}{\le} \delta \int_{Q_{\frac{3}{2}R}(x_0)} D^2 H(\varepsilon(u))(\partial_\alpha \varepsilon(u), \partial_\alpha \varepsilon(u))\, dx + c(\delta) \int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2 |\nabla \eta|^2 \, dx\,,
\end{aligned}
$$

hence

$$
(3.13) \qquad |S_2| \le \delta \int_{Q_{\frac{3}{2}R}(x_0)} b \, dx + c(\delta) \int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2 |\nabla \eta|^2 \, dx\,.
$$

Here we have abbreviated

$$b := D^2H\left(\varepsilon(u)\right)\left(\partial_\alpha\varepsilon(u), \partial_\alpha\varepsilon(u)\right) . \tag{3.14}$$

By applying exactly the same arguments to S_1 we see

$$|S_1| \leq \delta \int_{Q_{\frac{3}{2}R}(x_0)} b\,dx + c(\delta) \int_{Q_{\frac{3}{2}R}(x_0)} |\nabla\eta|^2|\nabla u|^2\,dx\,, \tag{3.15}$$

and (3.15) is valid for any choice of $\delta > 0$.

Next we look at S_3 : it holds

$$\begin{aligned} S_3 &= \int_{Q_{\frac{3}{2}R}(x_0)} u^k\partial_k u^i\partial_\alpha\left(\eta^2\partial_\alpha u^i\right)\,dx = -\int_{Q_{\frac{3}{2}R}(x_0)} \partial_\alpha(u^k\partial_k u^i)\eta^2\partial_\alpha u^i\,dx \\ &= -\int_{Q_{\frac{3}{2}R}(x_0)} \partial_\alpha u^k\partial_k u^i\partial_\alpha u^i\eta^2\,dx - \int_{Q_{\frac{3}{2}R}(x_0)} u^k\partial_\alpha\partial_k u^i\partial_\alpha u^i\eta^2\,dx\,, \end{aligned}$$

and since we are in the 2D-case, the first integral on the right-hand side vanishes. This shows

$$|S_3| = \frac{1}{2}\left|\int_{Q_{\frac{3}{2}R}(x_0)} u^k\partial_k|\nabla u|^2\eta^2\,dx\right| = \frac{1}{2}\left|\int_{Q_{\frac{3}{2}R}(x_0)} u\cdot\nabla\eta^2|\nabla u|^2\,dx\right|\,,$$

and we obtain

$$|S_3| \leq cR^{-1}\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\,dx \tag{3.16}$$

for a constant c depending on $\|u\|_{L^\infty(\mathbb{R}^2)}$. Finally we discuss S_4 again using the boundedness of the velocity field:

$$\begin{aligned} |S_4| &\leq c\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u||\partial_\alpha w_\alpha|\,dx \\ &\overset{(3.11)}{\leq} c\left(\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\,dx\right)^{1/2}\left(\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla\eta|^2|\nabla u|^2\,dx\right)^{1/2}\,, \end{aligned}$$

thus

$$|S_4| \leq cR^{-1}\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\,dx\,. \tag{3.17}$$

Putting together our estimates (3.13) - (3.17) and returning to (3.12) we have shown for any $\delta > 0$ the validity of the inequality

$$
\begin{aligned}
(3.18) \qquad \int_{Q_{\frac{3}{2}R}(x_0)} \eta^2 b\, dx \le \delta \int_{Q_{\frac{3}{2}R}(x_0)} b\, dx \\
+c(\delta)\left[R^{-2}\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\, dx + R^{-1}\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\, dx\right]
\end{aligned}
$$

with $c(\delta)$ also depending on $\|u\|_{L^\infty(\mathbb{R}^2)}$. In order to control Dirichlet's integral on the right-hand side of (3.18) in an appropriate way, let us select $\varphi \in C_0^\infty(Q_{2R}(x_0))$, $0 \le \varphi \le 1$, $\varphi = 1$ on $Q_{\frac{3}{2}R}(x_0)$, $|\nabla\varphi| \le c/R$. We have by Korn's inequality

$$
\begin{aligned}
\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2\, dx &\le \int_{Q_{2R}(x_0)} \varphi^2 |\nabla u|^2\, dx \\
&\le c\left[\int_{Q_{2R}(x_0)} |\nabla(\varphi u)|^2\, dx + \int_{Q_{2R}(x_0)} |\nabla\varphi|^2 |u|^2\, dx\right] \\
&\le c\left[\int_{Q_{2R}(x_0)} |\varepsilon(\varphi u)|^2\, dx + \int_{Q_{2R}(x_0)} |\nabla\varphi|^2 |u|^2\, dx\right] \\
&\le c\left[\int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2\, dx + R^{-2}\int_{Q_{2R}(x_0)} |u|^2\, dx\right],
\end{aligned}
$$

and if we recall the support property of η, inequality (3.18) in combination with the above estimates implies

$$
\begin{aligned}
(3.19) \qquad \int_{Q_R(x_0)} b\, dx \le \delta \int_{Q_{2R}(x_0)} b\, dx \\
+c(\delta)\left[R^{-4}\int_{Q_{2R}(x_0)} |u|^2\, dx + R^{-3}\int_{Q_{2R}(x_0)} |u|^2\, dx\right. \\
\left. +R^{-2}\int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2\, dx + R^{-1}\int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2\, dx\right].
\end{aligned}
$$

We have by Hölder's and Young's inequality

$$
\begin{aligned}
\int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 \, dx &= \int_{Q_{2R}(x_0)} \varepsilon_{ij}(u)\varepsilon_{ij}(u)\varphi^2 \, dx \\
&= -\int_{Q_{2R}(x_0)} u^i \partial_j \left(\varepsilon_{ij}(u)\varphi^2\right) \, dx \\
&= -\int_{Q_{2R}(x_0)} u^i \partial_j \varepsilon_{ij}(u)\varphi^2 \, dx - \int_{Q_{2R}(x_0)} u^i \varepsilon_{ij}(u) \partial_j \varphi^2 \, dx \\
&\le c\left[\int_{Q_{2R}(x_0)} |\nabla\varepsilon(u)| \, dx + R^{-1}\int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx\right] \\
&= c\left[\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)^{-1/2} |\nabla\varepsilon(u)| (1+|\varepsilon(u)|)^{1/2} \, dx \right. \\
&\quad \left. + R^{-1}\int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx\right] \\
&\le \left[\left(\int_{Q_{2R}(x_0)} \frac{|\nabla\varepsilon(u)|^2}{1+|\varepsilon(u)|} \, dx\right)^{1/2} \left(\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|) \, dx\right)^{1/2} \right. \\
&\quad \left. + R^{-1}\int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx\right] \\
&\le \tau \int_{Q_{2R}(x_0)} b \, dx + c\tau^{-1}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|) \, dx \\
&\quad + cR^{-1}\int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx \, ,
\end{aligned}
$$

where τ is any positive number. During our calculation we have used that

$$D^2H(\varepsilon)(\sigma,\sigma) \ge \min\left\{\frac{h'(|\varepsilon|)}{|\varepsilon|}, h''(|\varepsilon|)\right\} |\sigma|^2 \, ,$$

hence

$$D^2H(\varepsilon)(\sigma,\sigma) \ge h''(|\varepsilon|)\, |\sigma|^2$$

on account of ($A3_{\mathrm{II}}$). Recalling our assumption

$$h''(t) \ge c(1+t)^{-1}$$

concerning the shear thinning case in Theorem 1.4, the above chain of inequalities is established. Choosing

$$\tau = \delta c(\delta)^{-1} R^2, \ c(\delta) \text{ from } (3.19) \, ,$$

we get with a new constant $\widetilde{c}(\delta)$ recalling also (3.14)

$$
\begin{aligned}
(3.20) \qquad c(\delta) R^{-2}\int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 \, dx &\le \delta \int_{Q_{2R}(x_0)} b \, dx \\
+\widetilde{c}(\delta)\left[R^{-4}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|) \, dx \right. &+ \left. R^{-3}\int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx\right] ,
\end{aligned}
$$

whereas the choice

$$\tau := \delta c(\delta)^{-1} R$$

leads to

$$\begin{aligned}(3.21) \qquad c(\delta)R^{-1}\int_{Q_{2R}(x_0)} \varphi^2 \,|\varepsilon(u)|^2 \; dx \le \delta \int_{Q_{2R}(x_0)} b\, dx \\ +\widetilde{c}(\delta)\left[R^{-2}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)\; dx + R^{-2}\int_{Q_{2R}(x_0)} |\varepsilon(u)|\, dx\right].\end{aligned}$$

With (3.20) and (3.21) we return to (3.19) writing again $c(\delta)$ for constants depending on δ (and $\|u\|_{L^\infty(\mathbb{R}^2)}$) and replacing the parameter δ by $\delta/3$. We obtain:

$$\begin{aligned}(3.22) \qquad \int_{Q_R(x_0)} b\, dx \le \delta \int_{Q_{2R}(x_0)} b\, dx + c(\delta)\Bigg[R^{-4}\int_{Q_{2R}(x_0)} |u|^2\, dx \\ +R^{-3}\int_{Q_{2R}(x_0)} |u|^2\, dx + R^{-4}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)\; dx \\ +R^{-3}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)\; dx + R^{-2}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)\; dx\Bigg].\end{aligned}$$

To estimate (3.22) we can apply Lemma 3.1 from [FZ] and get for all squares $Q_R(x_0)$ with $c = c(\|u\|_{L^\infty(\mathbb{R}^2)})$

$$\begin{aligned}(3.23) \qquad \int_{Q_R(x_0)} b\, dx \le c\Bigg[R^{-4}\int_{Q_{2R}(x_0)} |u|^2\, dx \\ +R^{-3}\int_{Q_{2R}(x_0)} |u|^2\, dx + R^{-4}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)\; dx \\ +R^{-3}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)\; dx + R^{-2}\int_{Q_{2R}(x_0)} (1+|\varepsilon(u)|)\; dx\Bigg].\end{aligned}$$

Now, if the case $R \ge 1$ is considered, inequality (3.23) implies the bound

$$(3.24) \qquad \int_{Q_R(x_0)} b\, dx \le c\left[1 + R^{-2}\int_{Q_{2R}(x_0)} |\varepsilon(u)|\, dx\right].$$

Clearly the assumptions imposed on h yield ($Q^\pm := Q_{2R}(x_0) \cap \left[|\varepsilon(u)| \substack{\ge \\ \le} 1\right]$)

$$\begin{aligned}\int_{Q_{2R}(x_0)} |\varepsilon(u)|\, dx &= \int_{Q^-} |\varepsilon(u)|\, dx + \int_{Q^+} |\varepsilon(u)|\, dx \\ &\le \left(\int_{Q^-} 1\, dx\right)^{1/2}\left(\int_{Q^-} |\varepsilon(u)|^2\, dx\right)^{1/2} + c\int_{Q^+} H\left(\varepsilon(u)\right)\, dx \\ &\le cR\left(\int_{Q_{2R}(x_0)} H\left(\varepsilon(u)\right)\, dx\right)^{1/2} + c\int_{Q_{2R}(x_0)} H\left(\varepsilon(u)\right)\, dx\end{aligned}$$

and since we still assume that $R \geq 1$, we get from (3.1) the bound

$$\int_{Q_{2R}(x_0)} |\varepsilon(u)|\, dx \leq cR^{3/2}\,. \tag{3.25}$$

Now, if we insert (3.25) into (3.24), our claims (3.8) and (3.9) are clearly established.

In a final step we show

$$\int_{\mathbb{R}^2} b\, dx = 0\,. \tag{3.26}$$

Obviously (recall (3.14)) equation (3.26) gives $\nabla\varepsilon(u) = 0$, hence $\nabla^2 u = 0$ so that u must be affine. But since we assume $u \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$, the constancy of u clearly follows.
It remains to prove (3.26): let

$$b_\infty := \int_{\mathbb{R}^2} b\, dx\,.$$

Going through the calculations leading to (3.18) with the choice $x_0 = 0$, a closer look at the quantities $S_i, i = 1, \dots, 4$, implies the inequality

$$\begin{aligned}\int_{Q_R} b\, dx \leq \delta \int_{Q_{\frac{3}{2}R}} b\, dx + c(\delta) \Bigg[R^{-2} \int_{T_{\frac{3}{2}R}} |\nabla u|^2\, dx \\ + R^{-1} \int_{T_{\frac{3}{2}R}} |\nabla u|^2\, dx + R^{-1} \left(\int_{Q_{\frac{3}{2}R}} |\nabla u|^2\, dx \right)^{1/2} \left(\int_{T_{\frac{3}{2}R}} |\nabla u|^2\, dx \right)^{1/2} \Bigg]\,,\end{aligned} \tag{3.27}$$

where we have abbreviated $T_{\frac{3}{2}R} := Q_{\frac{3}{2}R} - \overline{Q}_R$ and where on the right-hand side of (3.27) the integration over $T_{\frac{3}{2}R}$ has to be performed in appropriate places due to the support properties of $\nabla\eta$. In the calculations after (3.18) we estimated $\int_{Q_{\frac{3}{2}R}} |\nabla u|^2\, dx$, but of course we can bound $\int_{T_{\frac{3}{2}R}} |\nabla u|^2\, dx$ in the same way by choosing $\varphi \equiv 1$ on $T_{\frac{3}{2}R}$, $0 \leq \varphi \leq 1$, $|\nabla\varphi| \leq c/R$ and $\operatorname{spt}\varphi \subset Q_{2R} - \overline{Q}_{R/2} =: T_{2R}$. This yields

$$\int_{T_{\frac{3}{2}R}} |\nabla u|^2\, dx \leq c \left[\int_{T_{2R}} \varphi^2 |\varepsilon(u)|^2\, dx + R^{-2} \int_{T_{2R}} |u|^2\, dx \right] \tag{3.28}$$

and from the arguments used after (3.19) we deduce

$$\begin{aligned}\int_{T_{2R}} \varphi^2 |\varepsilon(u)|^2\, dx \;\leq\; c \Bigg[\left(\int_{T_{2R}} b\, dx \right)^{1/2} \left(\int_{T_{2R}} (1 + |\varepsilon(u)|)\, dx \right)^{1/2} \\ + R^{-1} \int_{T_{2R}} |\varepsilon(u)|\, dx \Bigg] =: \Phi(R)\,.\end{aligned} \tag{3.29}$$

Putting together (3.28) and (3.29) and going back to (3.27) we obtain choosing $\delta = 1/2$

$$
\begin{aligned}
(3.30)\quad \int_{Q_R} b\,dx \le \frac{1}{2} b_\infty \quad + \quad & c \Bigg\{ R^{-4} \int_{Q_{2R}} |u|^2\,dx + R^{-3} \int_{Q_{2R}} |u|^2\,dx \\
& + R^{-2}\Phi(R) + R^{-1}\Phi(R) \\
& + R^{-1} \left(\int_{Q_{\frac{3}{2}R}} |\nabla u|^2\,dx \right)^{1/2} \left(\int_{T_{\frac{3}{2}R}} |\nabla u|^2\,dx \right)^{1/2} \Bigg\} .
\end{aligned}
$$

Clearly $R^{-4} \int_{Q_{2R}} |u|^2\,dx + R^{-3} \int_{Q_{2R}} |u|^2\,dx \longrightarrow 0$ as $R \to \infty$, and from (3.25) we obtain ($R \ge 1$)

$$
\Phi(R) \le c \left[R \left(\int_{T_{2R}} b\,dx \right)^{1/2} + R^{1/2} \right] ,
$$

hence $R^{-2}\Phi(R) + R^{-1}\Phi(R) \longrightarrow 0$ as $R \to \infty$ on account of (3.8). At this stage we like to remark that here it is essential to integrate b just over the set T_{2R}. Let us finally look at the quantity

$$
\Psi(R) := R^{-1} \left(\int_{Q_{\frac{3}{2}R}} |\nabla u|^2\,dx \right)^{1/2} \left(\int_{T_{\frac{3}{2}R}} |\nabla u|^2\,dx \right)^{1/2} .
$$

By (3.28) and (3.29) we have

$$
\int_{T_{\frac{3}{2}R}} |\nabla u|^2\,dx \le c \left[R^{-2} \int_{T_{2R}} |u|^2\,dx + \Phi(R) \right] ,
$$

thus

$$
\Psi(R) \le c \left(R^{-1} \int_{Q_{\frac{3}{2}R}} |\nabla u|^2\,dx \right)^{1/2} \left(R^{-3} \int_{T_{2R}} |u|^2\,dx + R^{-1}\Phi(R) \right)^{1/2}
$$

and the second factor on the right-hand side goes to zero as $R \to \infty$ as observed earlier. Returning to our previous bound

$$
\begin{aligned}
\int_{Q_{\frac{3}{2}R}} |\nabla u|^2\,dx \quad \le \quad & c \Bigg[R^{-2} \int_{Q_{2R}} |u|^2\,dx \\
& + \left(\int_{Q_{2R}} b\,dx \right)^{1/2} \left(\int_{Q_{2R}} (1 + |\varepsilon(u)|)\,dx \right)^{1/2} + R^{-1} \int_{Q_{2R}} |\varepsilon(u)|\,dx \Bigg]
\end{aligned}
$$

we see in combination with (3.25) and (3.8) that $R^{-1} \int_{Q_{\frac{3}{2}R}} |\nabla u|^2\,dx$ stays bounded, which means that also $\Psi(R) \longrightarrow 0$ as $R \to \infty$. Therefore the passage to the limit in (3.30) finally yields our claim $b_\infty = 0$. □

4 Some related problems

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, denote a bounded domain having a smooth boundary $\partial\Omega$. In 1933 J. Leray [Le] investigated the solvability of an exterior problem for (NSE) which reads as follows (w.l.o.g. $\mu = 1$)

$$(4.1) \qquad \begin{cases} -\Delta u + u^k \partial_k u + \nabla \pi = 0\,, \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^d - \overline{\Omega}\,, \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

together with the asymptotic condition

$$(4.2) \qquad \lim_{|x| \to \infty} u(x) = u_\infty\,.$$

Here u_∞ is a prescribed vector from $\mathbb{R}^d$, and (4.2) has to be interpreted as uniform convergence. The threedimensional case of (4.1) along with (4.2) is well understood and different from the $2D$-situation. A nice explanation together with the historical background is provided in Galdi's paper [Ga3]. There it is also outlined how in the twodimensional case the solvability of (4.1) and (4.2) with arbitrary vector $u_\infty \in \mathbb{R}^2$ is related to the up to now open question if for $u_\infty = 0$ the equations (4.1) and (4.2) admit only the trivial solution $u \equiv 0$. Of course we could not prove or disprove the conjecture that u must be identically zero, however it holds (see [BF2])

THEOREM 4.1. *Let $d = 2$ and consider a weak solution $u \in C^1(\mathbb{R}^2 - \Omega, \mathbb{R}^2)$ (cf. equation (1.15)) of (1.7) and (1.2) on $\mathbb{R}^2 - \overline{\Omega}$ with potential H given by (1.8) and (1.9). Let h satisfy (A1,2,3_I) or (A1,2,3_{II}). If u vanishes on $\partial\Omega$, then u is identically zero in each of the following cases:*

(i) $u(x) \to 0$ uniformly as $|x| \to \infty$ and $u^k \partial_k u$ is neglected,

(ii) $|x|^{1/3}\,|u(x)| \to 0$ uniformly as $|x| \to \infty$.

These results apply to the non-degenerate p-case, $1 < p < \infty$, as well as to the Prandtl-Eyring fluid. The proof of Theorem 4.1 is based on energy estimates in the spirit of Lemma 3.1, where now one has to work on the exterior domain $\mathbb{R}^2 - \overline{\Omega}$. Note that in (ii) actually a rather strong decay of u at infinity is required.

Next recall that in Theorem 1.4 the Prandtl-Eyring fluid model is included with potential H given by $H(\varepsilon) = |\varepsilon| \ln(1 + |\varepsilon|)$. However, the applicability of Theorem 1.4 is limited to solutions u of class C^1, and in order to motivate Theorem 1.4 for this particular fluid model one should firstly discuss the (weak) solvability of the e.g. homogeneous boundary value problem on some bounded smooth domain $\Omega \subset \mathbb{R}^2$ with given volume forces $f : \Omega \to \mathbb{R}^2$ and secondly investigate the differentiability properties of such weak solutions. The first question was recently answered in the paper [BDF]. Letting $h(t) = t\ln(1+t)$ we introduce the space

$$V^{1,h}_{0,\mathrm{div}} := \left\{ v \in L^1(\Omega, \mathbb{R}^2) : \int_\Omega h\left(|\varepsilon(v)|\right)\, dx < \infty,\ \operatorname{div} v = 0, v|_{\partial\Omega} = 0 \right\}$$

and consider the problem (cf. (1.15)) to find $u \in V^{1,h}_{0,\mathrm{div}}$ such that

$$\int_\Omega DH\left(\varepsilon(u)\right) : \varepsilon(\varphi)\, dx = \int_\Omega f \cdot \varphi \, dx + \int_\Omega u \otimes u : \varepsilon(\varphi)\, dx \tag{4.3}$$

holds for all $\varphi \in C^\infty_0(\Omega, \mathbb{R}^2)$, $\operatorname{div} \varphi = 0$. Note that $V^{1,h}_{0,\mathrm{div}}$ is a subspace of the class $BD(\Omega)$ (= fields of bounded deformation) so that

$$V^{1,h}_{0,\mathrm{div}} \hookrightarrow L^2(\Omega, \mathbb{R}^2) \text{ continuously},$$

which means that the second integral on the r.h.s. is well defined. In their paper [FMS] Frehse, Málek and Steinhauer investigated the existence problem for the model case $H_p(\varepsilon) = |\varepsilon|^p$ and proved existence in $\overset{\circ}{W}{}^1_p(\Omega, \mathbb{R}^2)$ of weak solutions for any $p > 1$. Since they work in the Sobolev space $\overset{\circ}{W}{}^1_p(\Omega, \mathbb{R}^2)$, it is possible to adjust the Lipschitz truncation technique, which roughly speaking replaces a function u through a Lipschitz function $\widetilde{u}$ different from u only on a small set. In [BDF] we provide a similar procedure where now a field w from $V^{1,h}_{0,\mathrm{div}}$ is replaced through a solenoidal one having bounded symmetric gradient and being different from w again on a set with small measure. This yields

THEOREM 4.2. *Let $\Omega \subset \mathbb{R}^2$ denote a bounded smooth domain and let $f \in L^{p_o}(\Omega, \mathbb{R}^2)$ for some $p_o > 1$. Then (4.3) has at least one solution $u \in V^{1,h}_{0,\mathrm{div}}$.*

However, we know nothing about the regularity properties of this solution u, and up to now there are no existence results in the $3D$-case.

References

[BDF] Breit, D., Diening, L., Fuchs, M., Solenoidal Lipschitz truncation and applications in fluid mechanics. J. Diff. Equ. 253, 1910–1942 (2012).

[BF1] Bildhauer, M., Fuchs, M., Variational integrals of splitting type: higher integrability under general growth conditions. Annali Mat. Pura Appl. 188, 467–496 (2009).

[BF2] Bildhauer, M., Fuchs, M., On the exterior problem in $2D$ for stationary flows of fluids with shear dependent viscosity. Comm. M. U. C. 53,2, 221–236 (2012)

[BFZ] Bildhauer, M., Fuchs, M., Zhang, G., Liouville type theorems for stationary power law fluids in the plane.

[FMS] Frehse, J., Málek, J., Steinhauer, M., On analysis of steady flows of fluids with shear dependent viscosity based on the Lipschitz truncation method. SIAM J. Math. Anal. 34(5), 1064–1083 (2003).

[Fu] Fuchs, M., Liouville theorems for stationary flows of shear thickening fluids in the plane. J. Math. Fluid Mech. DOI 10.1007/s00021-011-0070-1.

[FS] Fuchs, M., Seregin, G.A., Variational methods for problems from plasticity theory and for generalized Newtonian fluids. Lect. Notes Math. 1749, Springer-Verlag, Berlin-Heidelberg-New York 2000.

[FZ] Fuchs, M., Zhang, G., Liouville theorems for entire local minimizers of energies defined on the class $L \log L$ and for entire solutions of the stationary Prandtl-Eyring fluid model. Calc. Var. 44(1-2), 271–295 (2012).

[Ga1] Galdi, G., An introduction to the mathematical theory of the Navier-Stokes equations Vol.I, Springer Tracts in Natural Philosophy Vol.38, Springer, Berlin-Heidelberg-New York 1994.

[Ga2] Galdi, G., An introduction to the mathematical theory of the Navier-Stokes equations Vol.II, Springer Tracts in Natural Philosophy Vol.39, Springer, Berlin-Heidelberg-New York 1994.

[Ga3] Galdi, G., On the existence of symmetric steady-state solutions to the plane exterior Navier–Stokes problem for arbitrary large Reynolds number. Advances in fluid dynamics, Quad. Mat.,4, Aracne, Rome, 1–25 (1999).

[GM] Giaquinta, M., Modica, G., Nonlinear systems of the type of stationary Navier-Stokes system. J. Reine Angew. Math. 330, 173–214 (1982).

[GW] Gilbarg, D., Weinberger, H.F., Asymptotic properties of steady plane solutions of the Navier-Stokes equations with bounded Dirichlet integral. Ann. S.N.S. Pisa (4), 5, 381–404 (1978).

[KNSS] Koch, G., Nadirashvili, N., Seregin, G., Sverǎk, V., Liouville theorems for the Navier-Stokes equations and applications. Acta Math.203, 83–105 (2009).

[La1] Ladyzhenskaya, O. A., The mathematical theory of viscous incompressible flow. Gordon and Breach, 1969.

[La2] Ladyzhenskaya, O. A., Solution "in the large" of boundary value problems for the Navier-Stokes equations in two space variables. Dokl. Akad. Nauk SSSR 123 (1958), 427429; English transl., Soviet Phys. Dokl. 3 (1959), 11281131 and Comm. Pure App. Math. 12 (1959), 427433.

[La3] Ladyzhenskaya, O. A., Uniqueness and smoothness of generalized solutions of the Navier-Stokes equations. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 5 (1967), 169-185; English transl., Sem. Math. V. A. Steklov Math. Inst. Leningrad 5 (1969), 60-66.

[La4] Ladyzhenskaya, O. A., Unique global solvability of the three-dimensional Cauchy problem for the Navier-Stokes equations in the presence of axial symmetry. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 7 (1968), 155-177; English transl., Sem. Math. V. A. Steklov Math. Inst. Leningrad 7 (1970), 70-79.

[La5] Ladyzhenskaya, O. A., A dynamical system generated by the Navier- Stokes equations. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 27 (1972), 91115; English transl., J. Soviet. Math. 3 (1975), 458479.

[LK] Ladyzhenskaya, O. A., Kiselev, A. A., On the existence and uniqueness of the solution of the non-stationary problem for a viscous incompressible fluid. Izv. Akad. Nauk SSSR Ser. Mat. 21 (1957), 665-680; English transl., Amer. Math. Soc. Transl. (2) 24 (1963), 79106.

[Le] Leray, J., Etude de diverses équations intégrals non linéaires et de quelques problémes que pose l'hydrodynamique. J. Math. Pures Appl. 12, 1–82 (1933).

[MNRR] Málek, J., Necǎs, J., Rokyta, M., Růžička, M., Weak and measure valued solutions to evolutionary PDEs. Chapman & Hall, London–Weinheim–New York, 1996.

[Na] Naumann, J., On the differentiability of weak solutions of a degenerate system of PDE's in fluid mechanics. Annali Mat. Pura Appl. (IV)151, 225–238 (1988).

[Wo] Wolf, J., Interior $C^{1,\alpha}$-regularity of weak solutions to the equations of stationary motions of certain non-Newtonian fluids in two dimensions. Boll. U.M.I. (8)10-B, 317–340 (2007).

[Zh] Zhang, G., Liouville theorems for stationary flows of generalized Newtonian fluids. PhD thesis. University of Jyväskylä 2012.