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A note on the Toeplitz projection associated with spherical isometries

Michael Didas

In [7], Prunaru constructed a projection onto the space of Toeplitz operators associated with arbitrary families of spherical isometries. We show that, in the case of a finite spherical isometry $T \in B(H)^n$ (or, more general, a regular A-isometry), this projection takes a particularly simple form. As a consequence we deduce that, in this case, the space of Toeplitz operators $\mathcal{T}(T)$ associated with T is 2-hyperreflexive, extending a result of Ptak (Theorem 4.1 in [8]) for the Hardy space $H^2(D)$ on the unit polydisc $D = \mathbb{D}^n$ to the case of strictly pseudoconvex or bounded symmetric domains $D \subset \mathbb{C}^n$. Another application concerns the exact sequence for the corresponding Toeplitz C^*-algebra, $0 \longrightarrow \mathcal{C} \hookrightarrow C^*(\mathcal{T}(T)) \xrightarrow{\pi} (U)' \longrightarrow 0$, which has been established by Prunaru in [7]. We are able to give a concrete formula for π based on the Toeplitz projection according to the minimal normal extension U of T.

The present paper is a continuation of [2] and [3]. In particular, the reader is assumed to be familiar with the definitions and notations concerning A-isometries and their Toeplitz operators determined in Chapter 2 of [3]. For those who are not, we just want to mention that every spherical isometry $T \in B(H)^n$ (i.e. a commuting tuple of Hilbert-space operators satisfying $\sum_{i=1}^n T_i^* T_i = 1_H$) is a regular A-isometry with respect to the ball algebra $A = A(\mathbb{B}_n)$, and that every commuting tuple of isometries is a regular A-isometry for the polydisc algebra $A = A(\mathbb{D}^n)$. In these special cases, the operator algebra $\mathscr{H}_T \subset B(H)$ associated with T (see [2] for the precise definition) coincides with the smallest weak* closed dual algebra $\mathcal{A}_T = \overline{\mathbb{C}[T_1, \ldots, T_n]}^{w^*}$ containing 1_H and the components of T.

An element $X \in B(H)$ belongs to the set $\mathcal{T}(T)$ of T-Toeplitz operators if it satisfies the generalized Brown-Halmos condition $J^* X J = X$ for every isometry $J \in \mathscr{H}_T$. In [7], Prunaru constructed a completely positive unital mapping $\Phi : B(H) \to B(H)$ with $\Phi^2 = \Phi$ and $\operatorname{ran}(\Phi) = \mathcal{T}(T)$ for every commuting family T of (spherical) isometries. For an A-isometry $T \in B(H)^n$, this construction takes a particularly simple form. To be more precise, let $\mathcal{I}_T$ denote the commutative semi-group of all isometries in $\mathscr{H}_T$. By a result of Dixmier [4], we may choose an invariant mean $m : \ell^\infty(\mathcal{I}_T) \to \mathbb{C}$, i.e. a state m on $\ell^\infty(\mathcal{I}_T)$ such that $m((a_J)_J) = m((a_{VJ})_J)$ whenever $(a_J)_J \in \ell^\infty(\mathcal{I}_T)$ and $V \in \mathcal{I}_T$. Motivated by Prunaru's approach (see the proof of Lemma 2.7 in [7]) we define a linear map $\Phi : B(H) \to B(H)$ by the formula

$$\langle \Phi(X)x, y\rangle = m_J(\langle J^* X J x, y\rangle) \qquad (X \in B(H),\, x, y \in H).$$

The following theorem summarizes some important properties of this map.

1 Theorem. *Let $T \in B(H)^n$ be a regular A-isometry. Then the map Φ defined above ...*

(a) is a self-adjoint, completely positive and unital projection ($\Phi^2 = \Phi$), with $\operatorname{ran}(\Phi) = \mathcal{T}(T)$,

(b) has the property that $A^* \Phi(X) B = \Phi(A^* X B)$ for all $A, B \in (T)'$ and $X \in B(H)$,

(c) and maps every operator $X \in B(H)$ into the WOT-closed convex hull of the set $\{J^* X J : J \in \mathcal{I}_T\} \subset B(H)$.

Proof. First observe that, for $X \in \mathcal{T}(T)$ and arbitrary vectors $x, y \in H$, we have $\langle \Phi(X)x, y \rangle = m_J(\langle Xx, y \rangle) = \langle Xx, y \rangle$. This proves that Φ is unital and $\mathcal{T}(T) \subset \operatorname{ran}(\Phi)$. From now on, let $X \in B(H)$ be an arbitrary operator. Making use of the invariance of m we immediately obtain the identity

$$\Phi(V^*XV) = \Phi(X) \qquad \text{(for every isometry } V \in \mathcal{I}_T \text{ and every } X \in B(H)).$$

Given $A, B \in (T)'$ and arbitrary vectors $x, y \in H$, the calculation

$$\langle A^*\Phi(X)Bx, y \rangle = m_J(\langle J^*XJBx, Ay \rangle) = m_J(\langle J^*A^*XBJx, y \rangle) = \langle \Phi(A^*XB)x, y \rangle$$

shows that the assertion of part (b) holds. Now applying these two relations to $A = B = V$ with an isometry $V \in \mathcal{I}_T$, we obtain $V^*\Phi(X)V = \Phi(V^*XV) = \Phi(X)$. This settles the inclusion $\operatorname{ran}(\Phi) \subset \mathcal{T}(T)$ and furthermore guarantees that $\Phi^2 = \Phi$, since $m_J(\langle J^*\Phi(X)Jx, y \rangle) = \langle \Phi(X)x, y \rangle$ for $x, y \in H$.

To finish the proof of part (a), it remains to show that Φ is completely positive. Towards this, fix an integer $n \geq 1$, a positive operator-matrix $X^{(n)} = (X_{ij}) \in M_n(B(H))$ as well as a vector $x^{(n)} = (x_1, \ldots, x_n) \in H^n$. We have to show that the n-th inflation $\Phi^{(n)} : M_n(B(H)) \to M_n(B(H))$ of Φ is positive. But this can be seen by the identity

$$\begin{aligned} \langle \Phi^{(n)}(X^{(n)})x^{(n)}, x^{(n)} \rangle &= \sum_{1 \leq i,j \leq n} \langle \Phi(X_{ij})x_j, x_i \rangle \\ &= \sum_{1 \leq i,j \leq n} m_J(\langle J^*X_{ij}Jx_j, x_i \rangle) \\ &= m_J(\langle (J^*X_{ij}J)_{ij}x^{(n)}, x^{(n)} \rangle) \end{aligned}$$

and the fact that the conjugation map $B(H) \to B(H)$, $X \mapsto J^*XJ$ is completely positive.

Towards a proof of part (c) suppose that, for some $X \in B(H)$, the image $\Phi(X)$ is not contained in the WOT-closed convex hull of the set $M = \{J^*XJ : J \in \mathcal{I}_T\}$. Then, by the Hahn-Banach theorem, there exists a WOT-continuous linear functional $\varphi : B(H) \to \mathbb{C}$ separating $\Phi(X)$ and M. This means that, for some $\varepsilon > 0$, we have

$$\operatorname{Re} \sum_{i=1}^{k} \langle \Phi(X)x_i, y_i \rangle + \varepsilon \leq \operatorname{Re} \sum_{i=1}^{k} \langle J^*XJx_i, y_i \rangle \qquad (J \in \mathcal{I}_T),$$

where the vectors $x_i, y_i \in H$ $(i = 1, \ldots, k)$ arise from a representation of φ as finite sum $\varphi = \sum_{i=1}^{k} x_i \otimes y_i$.

Applying the invariant mean m to both sides preserves the inequality (m is positive) and leaves the left-hand side unchanged, while on the right-hand side we obtain

$$m_J\left(\operatorname{Re} \sum_{i=1}^{k} \langle J^*XJx_i, y_i \rangle\right) = \operatorname{Re} \sum_{i=1}^{k} m_J(\langle J^*XJx_i, y_i \rangle) = \operatorname{Re} \sum_{i=1}^{k} \langle \Phi(X)x_i, y_i \rangle.$$

This yields a contradiction which finishes the proof. □

It should be mentioned that, as a consequence of part (a) of the above theorem, the map Φ is completely bounded and satisfies $\|\Phi\| = \|\Phi\|_{\mathrm{cb}} = 1$.

As a first application we prove that the space $\mathcal{T}(T)$ is 2-hyperreflexive. This can be done by a well-known argument (see Theorem 4.1 in Ptak [8]) which relies on the validity of condition (c) of the above theorem. First recall that a weak*-closed subset $\mathcal{S} \subset B(H)$ is k-hyperreflexive if there is a constant $a > 0$ such that the distance estimate

$$d(A, \mathcal{S}) \leq a \sup\{|\mathrm{trace}(Af)| : f \in \mathcal{S}_\perp,\ \mathrm{rank}(f) \leq k,\ \|f\|_1 \leq 1\} \qquad (A \in B(H))$$

holds, where $\|\cdot\|_1$ refers to the norm in the trace class. For fixed $k \in \mathbb{N}$, the infumum of all possible constants $a > 0$ occuring in this estimate is denoted by $\kappa_k(\mathcal{S})$. Now we can state the announced 2-hyperreflexivity result.

2 Corollary. *For every regular A-isometry $T \in B(H)^n$, the space $\mathcal{T}(T) \subset B(H)$ of all T-Toeplitz operators is 2-hyperreflexive with $\kappa_2(\mathcal{T}(T)) \leq 2$.*

Proof. It is an easy exercise to deduce that, for every subset $\mathcal{W} \subset B(H)$, the norm estimate

$$\|A\| \leq \sup_{C \in M} \|C\| \qquad (A \in \overline{\mathrm{conv}}^{\mathrm{WOT}}(M))$$

holds. Thus, using part (c) of the above theorem, we have , we have

$$d(A, \mathcal{T}(T)) \leq \|A - \Phi(A)\| \leq \sup_{J \in \mathcal{I}_T} \|A - J^*AJ\| \quad (A \in B(H)).$$

Since $\|x \otimes y\|_1 = \|x\| \cdot \|y\|$ $(x, y \in H)$, the latter norm can be computed as

$$\|A - J^*AJ\| = \sup\{|\langle (A - J^*AJ)x, y\rangle| : x, y \in H \text{ with } \|x \otimes y\|_1 = 1\}.$$

Now observe that the scalar product occuring in the supremum can be written as $\langle (A - J^*AJ)x, y\rangle = \mathrm{trace}(Af)$ with the rank 2-operator $f = x \otimes y - Jx \otimes Jy \in \mathcal{T}(T)_\perp$. Since J is an isometry, we have $\|f\|_1 \leq 2$ if $\|x \otimes y\|_1 = 1$, and hence

$$d(A, \mathcal{T}(T)) \leq 2 \cdot \sup\{|\mathrm{trace}(Af)| : f \in \mathcal{T}(T)_\perp, \quad \mathrm{rank}(f) \leq 2, \quad \|f\|_1 \leq 1\}$$

for every $A \in B(H)$, as we claimed. $\square$

Since the above corollary applies in particular to the tuple $T = (M_{z_1}, \ldots, M_{z_n}) \in B(H^2(D))^n$ of multiplication with the coordinate functions on the Hardy spaces over strictly pseudoconvex or bounded symmetric domains $D \subset \mathbb{C}^n$, it extends the cited result of Ptak for the polydisc-case $D = \mathbb{D}^n$.

Another consequence of the simple explicit formula for the Toeplitz projection Φ concerns compact perturbations of T-Toeplitz operators. It is known that there is no compact T-Toeplitz operator if $\sigma_p(T) = \emptyset$ (see [2]), so it seems natural to conjecture that Φ vanishes on the compact operators in this case. The following proposition settles this at least in the completely non-unitary case.

3 Proposition. *Let $T \in B(H)^n$ be a completely non-unitary regular A-isometry. Then the Toeplitz projection Φ defined above satisfies $\Phi(K) = 0$ for every compact operator $K \in B(H)$.*

Proof. Fix $x, y \in H$ with $\|x\|, \|y\| \leq 1$, a compact operator $K \in B(H)$ and a sequence

$$(V_k)_{k \geq 1} \text{ of isometries in } \mathscr{H}_T \text{ such that } V_k^* \longrightarrow 0 \text{ (SOT) if } k \to \infty$$

which exists by hypothesis according to Proposition 3.13 and Lemma 3.12 in [2].

From the invariance of m we deduce that

$$\langle \Phi(K)x, y\rangle = m_J(\langle J^* V_k^* K V_k J x, y\rangle) \quad (k \geq 1).$$

Since the right-hand side is constant in k, we may add $\lim_{k\to\infty}$ in front of m_J. To prove the proposition, it suffices to show that the argument of m is a zero-sequence in $\ell^\infty(\mathcal{I}_T)$.

Towards this aim, fix an arbitrary real number $\varepsilon > 0$ and an arbitrary isometry $J \in \mathscr{H}_T$. Since K is compact, there exist finitely many vectors $x_1, \ldots, x_m \in H$ such that

$$KB_1(0) \subset \bigcup_{j=1}^{n} B_\varepsilon(x_i),$$

where $B_r(p) \subset H$ denotes the closed ball of radius r centered at p. Since $V_k^* \to 0$ (SOT), we can find an index N_ε such that

$$\|V_k^* x_i\| < \varepsilon \qquad (i = 1, \ldots, m) \quad \text{whenever } k \geq N_\varepsilon.$$

Now fix any $k \geq N_\varepsilon$. Then the vector $z = KV_k J x \subset KB_1(0)$ lies in some $B_\varepsilon(x_i)$ for $1 \leq i \leq m$. Thus

$$|\langle J^* V_k^* K V_k J x, y\rangle| \leq \|V_k^* z\| \leq \|V_k^*(z - x_i)\| + \|V_k^* x_i\| < 2\varepsilon.$$

Note that N_ε does not depend on the choice of J. Thus we have shown that

$$\sup_{j\in\mathcal{I}_T} |\langle J^* V_k^* K V_k J x, y\rangle| \xrightarrow{k\to\infty} 0,$$

and the proof is complete. $\square$

Combining the above proposition with Theorem 4.6 in [3] we obtain the following consequence.

4 Corollary. *If $T \in B(H)^n$ is an essentially normal, completely non-unitary, regular A-isometry and $S \in B(H)$ belongs to the essential commutant of the dual algebra $\mathscr{H}_T$, then $S - \Phi(S)$ is compact.* $\square$

From now on, fix a regular A-isometry $T \in B(H)^n$ together with a minimal normal extension $U \in B(K)^n$. Let us write $C^*(\mathcal{T}(T))$ for the C^*-subalgebra of $B(H)$ generated by all T-Toeplitz operators. In [7], Prunaru proved the existence of a generalized symbol map for this Toeplitz C^*-algebra. More precisely, he showed that there is a (unique, see below) $*$-homomorphism

$$\pi : C^*(\mathcal{T}(T)) \to (U)' \quad \text{which satisfies} \quad \pi(P_H Y|H) = Y \text{ for every } Y \in (U)',$$

and that this map π yields an exact sequence of the form

$$0 \longrightarrow \mathcal{C} \hookrightarrow C^*(\mathcal{T}(T))) \xrightarrow{\pi} (U)' \longrightarrow 0,$$

where $\mathcal{C}$ denotes the commutator ideal of $C^*(\mathcal{T}(T))$. In view of the identity

$$\mathcal{T}(T) = \{P_H Y|H : Y \in (U)'\} \subset B(H)$$

(see Prunaru [7] or, for the case of A-isometries, Proposition 3.2 in [3]), such a map π is unique. Its existence has been shown in [7] by making use of Stinespring's dilation theorem. Following ideas of Mancera and Paul (see [6]), we give an explicit formula for π which is based on the Toeplitz projection $\Phi_U : B(K) \to B(K)$ associated with the minimal normal extension U of T. (Note that if T is a regular A-isometry, then so is U.) Remember that Φ_U itself is defined by the formula $\langle \Phi_U(X)x, y\rangle = m_{\hat{J}}(\langle \hat{J}^* X \hat{J} x, y\rangle)$ $(X \in B(K),\, x, y \in K)$, where $\hat{J}$ ranges over the set $\mathcal{I}_U$ of all isometries contained in the dual algebra $\mathscr{H}_U$. The following simple observations will play an important role in the sequel. We will use them mostly without further comment.

5 Lemma. *For a regular A-isometry $T \in B(H)^n$ with minimal normal extension $U \in B(K)^n$, the following assertions hold:*

(a) *The restriction $\mathscr{H}_U \to \mathscr{H}_T$, $Y \mapsto Y|H$ is an isomorphism of dual algebras. Its inverse gives a canonical extension map*

$$\mathscr{H}_T \to \mathscr{H}_U, \quad X \mapsto \hat{X},$$

which yields a bijection between the sets $\mathcal{I}_T$ and $\mathcal{I}_U$.

(b) *The set of U-Toeplitz operators $\mathcal{T}(U) = \mathrm{ran}(\Phi_U)$ coincides with the commutant $(U)' \subset B(K)$.*

(c) *The space K is the closed linear span of the set $\{\hat{J}^* x : \hat{J} \in \mathcal{I}_U, x \in H\}$.*

Proof. Let $\Psi_U : L^\infty(\mu) \to B(K)$ denote the L^∞-functional calculus of U associated with a fixed scalar-spectral measure $\mu \in M_1^+(K)$. Then it is well known that Ψ_U induces a dual algebra isomorphism $H^\infty(\mu) \xrightarrow{\Psi_U} \mathscr{H}_U$ as well as an isomorphism of dual algebras $\gamma_T : H^\infty(\mu) \to \mathscr{H}_T$, $f \mapsto \Psi_U(f)|H$ (see [1], Proposition 1.1). Moreover, by Lemma 1.1 in [2], the isometries in $\mathscr{H}_T$ ($\mathscr{H}_U$, respectively) are precisely the images of elements $\theta \in H^\infty(\mu)$ with $|\theta| = 1$ (μ-a.e.) under the map γ_T (Ψ_U, respectively). This proves part (a) and, moreover, shows that every $\hat{J} \in \mathcal{I}_U$ is even unitary. Using this and the generalized Brown-Halmos condition $\hat{J}^* Y \hat{J} = Y$ ($\hat{J} \in \mathcal{I}_U$) for a U-Toeplitz operator Y, we derive that an element $Y \in B(K)$ belongs to $\mathcal{T}(U)$ if and only if it belongs to the commutant $(\mathcal{I}_U)'$ which is known to be equal to $(U)'$ (see e.g. the proof of Proposition 3.7 in [2]). Finally consider the closed subspace $M = \overline{\mathrm{LH}}\{\hat{J}^* x : \hat{J} \in \mathcal{I}_U, x \in H\} \subset K$. Then M clearly contains H and is reducing for $\mathcal{I}_U$ (and hence U, by Proposition 2.8 (a) in [3]) and thus coincides with K, since K is minimal with these properties. $\square$

The generalized symbol-homomorphism π that we are looking for will be obtained as a suitable restriction of the map

$$\widehat{\pi} : B(H) \to B(K) \quad \text{given by} \quad \widehat{\pi}(X) = \Phi_U(i_H X P_H),$$

where $i_H : H \hookrightarrow K$ denotes the inclusion of H into K and $P_H : K \to H$ the corresponding orthogonal projection. In Proposition 7 below, we give a collection of some important properties of this map $\widehat{\pi}$. As a preparation, we first show a few auxiliary results making use of the explicit formula

$$\langle \widehat{\pi}(X)x, y\rangle = m_{\hat{J}}(\langle \hat{J}^* X P_H \hat{J} x, y\rangle) \qquad (x, y \in K,\ X \in B(H)),$$

where $\hat{J}$ runs through the set $\mathcal{I}_U$ of all isometries contained in $\mathscr{H}_U$. Recall that, if $X \in \mathscr{H}_T$, then the unique element $Y \in \mathscr{H}_U$ with $Y|H = X$ will be denoted by $\hat{X}$.

6 Lemma. *The map $\widehat{\pi} : B(H) \to B(K)$ has the following properties:*

(a) *The range of $\widehat{\pi}$ is contained in the commutant $(U)' \subset B(K)$.*

(b) *If $X \in (T)'$, then $\widehat{\pi}(X)|H = X$. Moreover, $\widehat{\pi}(W) = \hat{W}$ whenever $W \in \mathcal{I}_T$.*

(c) *If $Y \in B(K)$ with $YH \subset H$, then $\widehat{\pi}(Y|H)x = \Phi_U(Y)x$ holds for every $x \in H$.*

(d) *Given $x, y \in H$ and $Y \in (U)'$, the equality $\langle \widehat{\pi}(P_H Y|H)x, y\rangle = \langle Yx, y\rangle$ holds.*

Proof. By a look at the definition of the map $\widehat{\pi}$ and the preceding remarks we see that part (a) is obviously true and that $\langle \widehat{\pi}(X)x, y\rangle = m_{\hat{J}}(\langle \hat{J}^* XJx, y\rangle)$ holds for every choice of vectors $x \in H$ and $y \in K$. Now, given $X \in (T)'$, we may replace XJx by $\hat{J}Xx$ in the last identity, proving the first part of assertion (b). If $W \in \mathscr{H}_T$ and $\hat{V} \in \mathscr{H}_U$ denote arbitrary isometries, then we deduce from the above that

$$\widehat{\pi}(W)\hat{V}^* x = \hat{V}^* W x = \hat{W}\hat{V}^* x \quad (x \in H).$$

In view of Lemma 5 (c), thus the rest of part (b) follows. To establish part (c), fix an operator $Y \in B(K)$ leaving H invariant, as well as arbitrary vectors $x \in H$ and $y \in K$. Then

$$\langle \widehat{\pi}(Y|H)x, y\rangle = m_{\hat{J}}(\langle \hat{J}^* Y P_H \hat{J}x, y\rangle) = m_{\hat{J}}(\langle \hat{J}^* Y \hat{J}x, y\rangle) = \langle \Phi_U(Y)x, y\rangle$$

holds, as desired. Finally, fix $x, y \in H$ and $Y \in (U)'$. Then we have

$$\begin{aligned} \langle \widehat{\pi}(P_H Y|H)x, y\rangle &= m_{\hat{J}}(\langle \hat{J}^* P_H Y P_H \hat{J}x, y\rangle) \\ &= m_{\hat{J}}(\langle \hat{J}^* P_H \hat{J} Y x, y\rangle) \\ &= m_{\hat{J}}(\langle P_H \hat{J} Y x, \hat{J}y\rangle) \\ &= m_{\hat{J}}(\langle \hat{J} Y x, \hat{J}y\rangle) = \langle Yx, y\rangle \end{aligned}$$

and the proof is complete. □

Now we are able to prove the relevant properties turning $\widehat{\pi}$ into a generalized symbol map.

7 Proposition. *The map $\widehat{\pi} : B(H) \to B(K)$ defined above ...*

(a) *is a completely positive and completely contractive mapping with $\mathrm{ran}(\widehat{\pi}) = (U)'$;*

(b) *satisfies the identity $P_H \widehat{\pi}(X)|H = X$ for every $X \in \mathcal{T}(T)$;*

(c) *is multiplicative in the sense that $\widehat{\pi}(X_1 X_2) = \widehat{\pi}(X_1)\widehat{\pi}(X_2)$ holds, whenever $X_1 \in B(H)$ and $X_2 \in \mathcal{T}(T)$;*

(d) *fulfills $\widehat{\pi}(P_H Y|H) = Y$ whenever $Y \in (U)'$.*

Proof. Part (a) is obvious from the definition of the map $\widehat{\pi}$ except the inclusion $\mathrm{ran}(\widehat{\pi}) \supset (U)'$ which will be proved later. Reformulating part (d) of the above lemma, we obtain that $P_H \widehat{\pi}(P_H Y|H)|H = P_H Y|H$ for every $Y \in (U)'$. Since every $X \in \mathcal{T}(T)$ has the form $X = P_H Y|H$ for some $Y \in (U)'$, the assertion of part (b) follows. Towards a proof of part (c), fix arbitrary operators $X_1 \in B(H)$, $X_2 \in \mathcal{T}(T)$

and $\hat{V} \in \mathcal{I}_U$, as well as some vector $x \in H$. The desired multiplicativity then follows from the calculation

$$\begin{aligned}
\widehat{\pi}(X_1X_2)\hat{V}^*x &= \hat{V}^*\widehat{\pi}(X_1X_2)x \\
&= \hat{V}^*\widehat{\pi}(X_1P_H\widehat{\pi}(X_2)|H)x \qquad \text{(by part (b))} \\
&= \hat{V}^*\Phi_U(i_HX_1P_H\widehat{\pi}(X_2))x \qquad \text{(by part(c) of Lemma 6)} \\
&= \hat{V}^*\Phi_U(i_HX_1P_H)\widehat{\pi}(X_2)x \qquad \text{(by Theorem 1 (c))} \\
&= \widehat{\pi}(X_1)\widehat{\pi}(X_2)\hat{V}^*x
\end{aligned}$$

combined with the fact that elements of the form $\hat{V}^*x$ with $\hat{V} \in \mathcal{I}_U$ and $x \in H$ span K by Lemma 5. Finally, fix isometries $\hat{V}, \hat{W} \in \mathcal{I}_U$ and vectors $x, y \in H$. Then we have

$$\begin{aligned}
\langle \widehat{\pi}(P_HY|H)\hat{V}^*x, \hat{W}^*y\rangle &= \langle \hat{W}\widehat{\pi}(P_HY|H)x, \hat{V}y\rangle \\
&= \langle \widehat{\pi}(P_H(Y|H)W)x, \hat{V}y\rangle \quad \text{(by (c) and Lemma 6 (b))} \\
&= \langle \widehat{\pi}(P_HY\hat{W}|H)x, \hat{V}y\rangle \\
&= \langle Y\hat{W}x, \hat{V}y\rangle \qquad \text{(by Lemma 6 (d))} \\
&= \langle Y\hat{V}^*x, \hat{W}^*y\rangle.
\end{aligned}$$

This proves part (d) which also implies the remaining inclusion of part (a). $\square$

Note that $C^*(\mathcal{T}(T)) = \overline{LH}\{T_1 \cdots T_n : n \in \mathbb{N}, T_i \in \mathcal{T}(T) \text{ for } 1 \le i \le n\}$ since $\mathcal{T}(T)$ is self-adjoint. Thus part (c) of the preceding proposition says that the restriciton $\pi = \widehat{\pi}|C^*(\mathcal{T}(T))$ is multiplicative and hence an algebra homomorphism. Moreover, if $X = P_HY|H$ with $Y \in (U)'$ is an arbitrary T-Toeplitz operator, then $\pi(X) = Y$ and $\pi(X^*) = \pi(P_HY^*|H) = Y^*$ by part (d). This proves that π is actually a $*$-homomorphism and thus coincides with the symbol homomorphism defined by Prunaru in [7] (see [3] for the case of A-isometries).

8 Theorem. *Let $T \in B(H)^n$ be a regular A-isometry and let Φ_U denote the Toeplitz projection according to the minimal normal extension $U \in B(K)^n$. If $i_H : H \hookrightarrow K$ denotes the inclusion and $P_H : K \to H$ the corresponding orthogonal projection, then the formula*

$$\pi(X) = \Phi_U(i_HXP_H) \qquad (X \in C^*(\mathcal{T}(T)))$$

defines the unique $*$-homomorphism $\pi : C^*(\mathcal{T}(T)) \to B(K)$ with $\pi(P_HY|H) = Y$ for $Y \in (U)'$. Moreover, the sequence

$$0 \longrightarrow \mathcal{C} \hookrightarrow C^*(\mathcal{T}(T))) \xrightarrow{\pi} (U)' \longrightarrow 0$$

is exact, where $\mathcal{C}$ denotes the commutator ideal of $C^*(\mathcal{T}(T))$. $\square$

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