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A result of Davidson from 1977 says that a derivation D of the Toeplitz algebra $\mathcal{T}(H^\infty + C(\mathbb{T}))$ on the Hardy space H^2 of the unit disc is inner if the range of D is contained in the compact operators $\mathcal{K}(H^2)$ (see Corollary 4 in [2]). We show that the range of every continuous derivation of $\mathcal{T}(H^\infty + C(\mathbb{T}))$ is contained in $\mathcal{K}(H^2)$. Hence each such derivation is inner.

Let $\mathbb{T}$ denote the unit circle equipped with the normalized Lebesgue measure. Recall that, on the classical Hardy space $H^2 \subset L^2(\mathbb{T})$, the Toeplitz operator T_f with symbol $f \in L^\infty(\mathbb{T})$ is defined by

$$T_f h = P_{H^2}(f \cdot h) \qquad (h \in H^2),$$

where P_{H^2} denotes the orthogonal projection from $L^2(\mathbb{T})$ onto H^2. For a subset $S \subset L^\infty(\mathbb{T})$, the corresponding Toeplitz algebra $\mathcal{T}(S) \subset B(H^2)$ is defined as the smallest closed subalgebra of $B(H^2)$ containing the set $\{T_f : f \in S\}$.

For various symbol classes S, the structure of the corresponding Toeplitz operators and associated Toeplitz algebras is well understood (cp. Chapter 7 of Douglas' book [3] and the references therein). Among them are H^∞ (i.e. the set of all radial boundary values of bounded holomorphic functions on the unit disc), $C(\mathbb{T})$ (the continuous functions on $\mathbb{T}$) and their sum $H^\infty + C(\mathbb{T}) \subset L^\infty(\mathbb{T})$. The latter one is in fact a closed subalgebra of $L^\infty(\mathbb{T})$ containing $\{\psi\overline{z}^n : \psi \in H^\infty, n \in \mathbb{N}_0\}$ as a dense subset (see Proposition 6.36 in [3]). The Toeplitz algebras induced by the above-mentioned symbol classes are the algebra of all analytic Toeplitz operators, $\mathcal{T}(H^\infty) = \{T_\psi : \psi \in H^\infty\}$, the Toeplitz C^*-algebra $\mathcal{T}(C(\mathbb{T})) = C^*(1, T_z)$ and the algebra $\mathcal{T}(H^\infty + C(\mathbb{T}))$ appearing in the title, respectively.

Let $\mathcal{K}(H^2) \subset B(H^2)$ denote the ideal of all compact operators. In 1977, K. R. Davidson proved that an operator S on H^2 commutes modulo the compact operators with all operators in $\mathcal{T}(H^\infty)$ if and only if $S = T_g + K$ with $g \in H^\infty + C(\mathbb{T})$ and $K \in \mathcal{K}(H^2)$ (Theorem 1 in [2]). As a corollary he obtained a result on derivations of $\mathcal{T}(H^\infty + C(\mathbb{T}))$. Recall that a derivation D on an algebra $\mathcal{A}$ is a (not necessarily continuous) linear map

$$D : \mathcal{A} \to \mathcal{A} \quad \text{satisfying} \quad D(XY) = D(X)Y + XD(Y) \text{ for all } X, Y \in \mathcal{A},$$

and that D is called inner, if $D(X) = XS - SX$ (for all $X \in \mathcal{A}$) with some fixed element $S \in \mathcal{A}$. The aforementioned corollary (Corollary 4 in [2]) says the following:

1 Proposition. (Davidson, 1977) *Every (not necessarily continuous) derivation D of $\mathcal{T}(H^\infty + C(\mathbb{T}))$ with range in $\mathcal{K}(H^2)$ is inner.* □

It seems natural to ask whether the range condition is always satisfied. In this paper we settle at least the continuous case.

Towards this end, let us first recall Theorem 7.29 from [3] which asserts that the commutator ideal of $\mathcal{T}(H^\infty + C(\mathbb{T}))$ is $\mathcal{K}(H^2)$ and that the map

$$\xi : H^\infty + C(\mathbb{T}) \longrightarrow \mathcal{T}(H^\infty + C(\mathbb{T}))/\mathcal{K}(H^2), \quad f \mapsto [T_f]$$

is an isometric algebra isomorphism. This implies in particular, that all elements of $\mathcal{T}(H^\infty + C(\mathbb{T}))$ commute with each other modulo the compact operators or, equivalently, that they commute in the Calkin algebra $\mathcal{C}(H^2) = B(H^2)/\mathcal{K}(H^2)$.

Our proof relies on the following simple observation:

2 Lemma. *Suppose that the function $g \in H^\infty + C(\mathbb{T})$ is either*

(a) an inner function, i.e. $g = \theta \in H^\infty$ with $|\theta|^2 = 1$ a.e. on $\mathbb{T}$, or

(b) $g = \overline{z}^n$ for some $n \in \mathbb{N}_0$.

Then the multiplication operator $\mathcal{C}(H^2) \to \mathcal{C}(H^2)$, $[X] \mapsto [T_g X]$ induced by the Toeplitz operator T_g on the Calkin algebra is an isometry.

Proof. If $\theta \in H^\infty$ is an inner function, then the algebraic identity

$$T_\theta^* T_\theta = T_{\overline{\theta}} T_\theta = T_{|\theta|^2} = 1_{H^2}$$

shows that T_θ is an isometry. Hence we have the estimate

$$\|[X]\| = \|[T_\theta^*][T_\theta X]\| \leq \|[T_\theta X]\| \leq \|[X]\| \quad (X \in B(H^2)),$$

which guarantees that the induced multiplication operator on the Calkin algebra is also isometric. In the case where $g = \overline{z}^n$, we also have $|g|^2 = 1$ and, moreover, by Proposition 7.22 in [3], both operators $T_g T_{\overline{g}} - 1_{H^2}$ and $T_{\overline{g}} T_g - 1_{H^2}$ are compact. Consequently, $[T_g]$ is a unitary element in the Calkin algebra. This shows that the above estimate also holds for $[T_g]$ instead of $[T_\theta]$, and the proof is complete. $\square$

By modifying an idea of Cao (see the proof of Proposition 7 in [1]) we are now able to prove the result announced in the title.

3 Theorem. *Every continuous derivation of $\mathcal{T}(H^\infty + C(\mathbb{T}))$ is inner.*

Proof. We show that the range of every bounded derivation D of $\mathcal{T}(H^\infty + C(\mathbb{T}))$ is contained in $\mathcal{K}(H^2)$. Then, by the cited result of Davidson (see Proposition 1 above), the theorem follows.

Since $\mathcal{T}(H^\infty + C(\mathbb{T})) = \overline{LH}(\mathcal{S})$ with

$$\mathcal{S} = \{T_{g_1} \cdots T_{g_n} : n \in \mathbb{N}, g_i \in H^\infty + C(\mathbb{T}) \ (1 \leq i \leq n)\}$$

it suffices by continuity and linearity to show that $[D(X)] = 0$ in the Calkin algebra for $X \in \mathcal{S}$. But if $X = T_{g_1} \cdots T_{g_n}$ is an operator as in the definition of $\mathcal{S}$, then we have

$$D(X) = \sum_{i=1}^{n} T_{g_1} \cdots T_{g_{i-1}} D(T_{g_i}) T_{g_{i+1}} \cdots T_{g_n}.$$

Since $\mathcal{K}(H^2)$ is an ideal, it suffices to prove that

$$D(T_g) \in \mathcal{K}(H^2) \quad \text{for every } g \in H^\infty + C(\mathbb{T}).$$

By Proposition 6.36 in [3], the set of all products $\overline{z}^n \psi$ with $\psi \in H^\infty$ and $n \in \mathbb{N}_0$ is norm dense in $H^\infty + C(\mathbb{T})$. Hence, by continuity, it remains to check that

$$D(T_{\overline{z}^n \psi}) \in \mathcal{K}(H^2) \qquad \text{for every } \psi \in H^\infty \text{ and every } n \in \mathbb{N}_0.$$

For the rest of the proof, fix arbitrary elements $\psi \in H^\infty$ and $n \in \mathbb{N}_0$. Then the identity

$$D(T_{\overline{z}^n\psi}) = D(T_{\overline{z}^n}T_\psi) = D(T_{\overline{z}^n})T_\psi + T_{\overline{z}^n}D(T_\psi)$$

allows us to finish the proof by showing that

$$[D(T_{\overline{z}^n})] = 0 \quad \text{and} \quad [D(T_\psi)] = 0$$

separately. Recall from the remarks preceding Lemma 2 that $\mathcal{T}(H^\infty + C(\mathbb{T}))$ is essentially commutative. Hence, for every operator $W \in \mathcal{T}(H^\infty + C(\mathbb{T}))$ and every $m \in \mathbb{N}$, we have

$$[D(W^m)] = [\sum_{i=1}^{m} W^{i-1}D(W)W^{m-i}] = m[W^{m-1}D(W)].$$

If we apply this identity to a Toeplitz operator $W = T_g$ whose symbol g is either an inner function ($g \in H^\infty$, $|g| = 1$) or $g = \overline{z}^n$, then a look at Lemma 2 yields

$$\|[D(T_g^m)]\| = m \cdot \|[T_g^{m-1}D(T_g)]\| = m \cdot \|[D(T_g)]\|.$$

Combining this with $\|[D(T_g^m)]\| \leq \|D\|$ we end up with the estimate

$$m \cdot \|[D(T_g)]\| \leq \|D\| \quad \text{(for all } m \in \mathbb{N}).$$

Clearly, this can only happen if $\|[D(T_g)]\| = 0$.

While this directly shows that $[D(T_{\overline{z}^n})] = 0$, another density argument is needed to show that $[D(T_\psi)] = 0$. To be more specific, an application of Marshall's theorem (see [4]) which says that the norm-closed linear span of all inner functions is dense in H^∞ finishes the proof. □

One can prove a generalized version of the above result for the Hardy space over the unit ball in $\mathbb{C}^n$. The details will be presented in a subsequent paper.

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