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Quasicomplexes and Lefschetz numbers

Jörg Eschmeier

Dedicated to the memory of Béla Szőkefalvi-Nagy

In a recent paper of Tarkhanov and Wallenta [8] a definition of Lefschetz numbers for morphisms $a = (a^\bullet)$ of Fredholm quasicomplexes $E^\bullet = (E^\bullet, d^\bullet)$ with trace class curvature is proposed. In the present note we show that there always exist trace class perturbations of a and $E^\bullet$ to a cochain mapping $A = (A^\bullet)$ of a Fredholm complex $(E^\bullet, D^\bullet)$, and we clarify the relation between the Lefschetz number of A relative to the perturbed complex $(E^\bullet, D^\bullet)$ and the Lefschetz number of a relative to the original quasicomplex $(E^\bullet, d^\bullet)$. Furthermore, we prove that the Lefschetz numbers relative to $E^\bullet$ satisfy a natural commutativity property.

1 Quasicomplexes

For Banach spaces E and F and $1 \leq p < \infty$, we denote by $\mathcal{C}^p(E,F)$ the Schatten class consisting of all bounded operators $T \in L(E,F)$ for which the sequence $(\alpha_n(T))_n$ of approximation numbers

$$\alpha_n(T) = \inf\{\|T - S\|;\ S \in L(E,F) \text{ with } \dim(\mathrm{Im} S) < n\}$$

is p-summable. We write $\mathcal{C}^\infty(E,F)$ for the set of all compact operators from E to F.

Let $d \in L(E^0, E^1), d' \in L(F^0, F^1)$ and $a^i \in L(E^i, F^i) (i = 0, 1)$ be bounded linear operators between Banach spaces. We call

$$\begin{array}{ccc} E^0 & \xrightarrow{d} & E^1 \\ a^0 \downarrow & & \downarrow a^1 \\ F^0 & \xrightarrow{d'} & F^1 \end{array}$$

a commuting square if $a^1 d = d' a^0$. To indicate that the last identity only holds up to operators of Schatten class $\mathcal{C}^p$, that is, $a^1 d - d' a^0 \in \mathcal{C}^p(E^0, F^1)$, we say that the square is p-essentially commuting. A complex of Banach spaces is a sequence

$$E^\bullet : 0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{N-1}} E^N \to 0$$

of bounded linear operators between Banach spaces such that $d^{i+1} d^i = 0$ for all i. We denote by $H^i(E^\bullet) = \mathrm{Ker}\, d^i / \mathrm{Im}\, d^{i-1}$ $(i = 0, \ldots, N)$ the cohomology groups of a complex $E^\bullet$ as above and use the standard convention that non-defined spaces or maps have to be interpreted as the zero spaces or zero maps. Following [8] we call $E^\bullet$ a quasicomplex of curvature $\mathcal{C}^p$ or a p-quasicomplex if $d^{i+1}d^i \in \mathcal{C}^p(E^i, E^{i+2})$ for all i. The case of quasicomplexes with $\mathcal{C}^\infty$-curvature has also been studied in [2], where $\mathcal{C}^\infty$-quasicomplexes were called essential complexes.

A p-quasicomplex $E^\bullet$ as above is called Fredholm if there are bounded operators $\epsilon^i \in L(E^i, E^{i-1})$ such that

$$d^{i-1}\epsilon^i + \epsilon^{i+1}d^i \in 1_{E^i} + \mathcal{K}(E^i) \ (i = 0, \ldots, N).$$

1.1 Lemma. *Suppose that*

$$\begin{array}{ccc} E^0 & \xrightarrow{d} & E^1 \\ a^0 \downarrow & & \downarrow a^1 \\ F^0 & \xrightarrow{d'} & F^1 \end{array}$$

is a p-essentially commuting square of Banach spaces with $1 \leq p \leq \infty$ such that the operator d is Fredholm. Then there are operators $C^i \in \mathcal{C}^p(E^i, F^i)$ with the property that

$$(a^1 - C^1)d = d'(a^0 - C^0).$$

Proof. Since d is Fredholm, there is an operator $\epsilon \in L(E^1, E^0)$ such that

$$K^1 = d\epsilon - 1_{E^1}, \quad K^0 = 1_{E^0} - \epsilon d$$

are finite-rank operators. Then the operator C^1 defined by

$$C^1 = a^1 - d'a^0\epsilon = (a^1 d - d'a^0)\epsilon - a^1 K^1$$

belongs to $\mathcal{C}^p(E^1, F^1)$ and satisfies the identity $d'a^0\epsilon = a^1 - C^1$. Because of

$$d'(a^0 - a^0 K^0) = d'a^0\epsilon d = (a^1 - C^1)d$$

the assertion holds with C^1 as defined above and with $C^0 = a^0 K^0$. □

Let us suppose that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^0 & \xrightarrow{d^0} & E^1 & \xrightarrow{d^1} & E^2 & \longrightarrow & 0 \\ & & a^0 \downarrow & & a^1 \downarrow & & a^2 \downarrow & & \\ 0 & \longrightarrow & F^0 & \xrightarrow{d'^0} & F^1 & \xrightarrow{d'^1} & F^2 & \longrightarrow & 0 \end{array}$$

is a diagram of bounded linear operators between Banach spaces such that the horizontal maps form complexes and such that the two squares in the diagram are p-essentially commuting.

1.2 Lemma. *In the setting explained above suppose further that* $\dim H^i(E^\bullet) < \infty$ *for $i = 1, 2$ and that there are topological direct complements N of the kernel of d^0 in E^0, M of the image of d^0 in E^1. Then there are operators $C^i \in \mathcal{C}^p(E^i, F^i)(i = 1, 2)$ such that*

$$(a^2 - C^2)d^1 = d'^1(a^1 - C^1).$$

Proof. Since the upper horizontal map in the p-essentially commuting square

$$\begin{array}{ccc} M & \xrightarrow{d^1} & E^2 \\ a^1 \downarrow & & \downarrow a^2 \\ F^1 & \xrightarrow{d'^1} & F^2 \end{array}$$

is Fredholm, Lemma 1.1 implies that there are operators $C^2 \in \mathcal{C}^p(E^2, F^2)$ and $K \in \mathcal{C}^p(M, F^1)$ such that $(a^2 - C^2)d^1 = d'^1(a^1 - K)$ on M. Since the upper horizontal map in the p-essentially commuting square

$$\begin{array}{ccc} N & \xrightarrow{d^0} & \text{Im } d^0 \\ a^0 \downarrow & & \downarrow a^1 \\ F^0 & \xrightarrow{d'^0} & F^1 \end{array}$$

is invertible, Lemma 1.1 shows that there are operators $K^1 \in \mathcal{C}^p$ (Im d^0, F^1) and $K^0 \in \mathcal{C}^p(N, F^0)$ with $(a^1 - K^1)d^0 = d'^0(a^0 - K^0)$ on N. In particular, it follows that $(a^1 - K^1)$ Im $d^0 \subset$ Im $d'^0 \subset$ Ker d'^1. But then the operator

$$C^1 = (K^1, K) : E^1 = \text{ Im } d^0 \oplus M \longrightarrow F^1$$

belongs to $\mathcal{C}^p(E^1, F^1)$ and satisfies

$$d'^1(a^1 - C^1)(x \oplus y) = d'^1(a^1 - K^1)x + d'^1(a^1 - K)y$$

$$= (a^2 - C^2)d^1 y = (a^2 - C^2)d^1(x \oplus y)$$

for all $x \in$ Im d^0 and $y \in M$. □

Let $E^\bullet = (E^n, d^n)_{n=0}^N$, $F^\bullet = (F^n, d'^n)_{n=0}^N$ be p-quasicomplexes of Banach spaces. A morphism between $E^\bullet$ and $F^\bullet$ is a sequence $a = (a^n)_{n=0}^N$ of bounded linear operators $a^n : E^n \to F^n$ such that $d'^n a^n - a^{n+1} d^n$ is of Schatten class $\mathcal{C}^p$ for every n. Let us suppose in addition that the p-quasicomplexes $E^\bullet$ and $F^\bullet$ are Fredholm. Our next aim is to show that there are perturbations of Schatten class $\mathcal{C}^p$ of $E^\bullet, F^\bullet$ and a which form a commuting diagram of Fredholm complexes of Banach spaces.

It was shown in Theorem 10.2.5 of [2] for the case $p = \infty$, and in Theorem 3.1 of [8] for the general case, that there are Schatten p-class perturbations D^n of d^n and D'^n of d'^n such that $(E^\bullet, D^\bullet)$ and $(F^\bullet, D'^\bullet)$ are Fredholm complexes of Banach spaces. Since the result in [8] was shown under the stronger Fredholm condition that there are operators $\epsilon^i \in L(E^i, E^{i-1})$ such that

$$d^{i-1}\epsilon^i + \epsilon^{i+1}d^i \in 1_{E^i} + \mathcal{C}^p(E^i) \quad (i = 0, \ldots, N),$$

we include a proof which shows that the latter condition holds automatically.

1.3 Theorem. *Let $E^\bullet = (E^n, d^n)_{n=0}^N$ be a p-quasicomplex with $1 \leq p \leq \infty$. Suppose that $E^\bullet$ is Fredholm, that is, there are operators $\epsilon^i \in L(E^i, E^{i-1})$ such that*

$$d^{i-1}\epsilon^i + \epsilon^{i+1}d^i \in 1_{E^i} + \mathcal{K}(E^i) \quad (i = 0, \ldots, N).$$

Then there are operators $\tau^i \in \mathcal{C}^p(E^i, E^{i+1})$ *with*

$$(d^{i+1} - \tau^{i+1})(d^i - \tau^i) = 0 \qquad (i = 0, ..., N-1)$$

and operators $h^i \in L(E^i, E^{i-1})$ *such that*

$$d^{i-1}h^i + h^{i+1}d^i \in 1_{E^i} + \mathcal{C}^p(E^i) \quad (i = 0, \ldots, N).$$

Proof. The existence of the operators τ^i can be proved by induction on N. For $N = 1$, nothing has to be shown. Let $E^\bullet = (E^n, d^n)_{n=0}^N$ be a Fredholm p-quasicomplex with $N > 1$. Set $K = \ker d^{N-1}$. By Lemma 2.6.13 in [2] the operator d^{N-1} has finite-codimensional range and there is a topological direct complement L of K in E^{N-1}. Then the operator

$$\delta : L \to \mathrm{Im} d^{N-1}, x \mapsto d^{N-1}x$$

is a topological isomorphism and the composition

$$\tau : \; E^{N-2} \xrightarrow{d^{N-2}} E^{N-1} \xrightarrow{d^{N-1}} \mathrm{Im} d^{N-1} \xrightarrow{\delta^{-1}} L \hookrightarrow E^{N-1}$$

defines an operator $\tau \in \mathcal{C}^p(E^{N-2}, E^{N-1})$. Obviously the operator defined as $P = \delta^{-1}d^{N-1}$ is the projection of E^{N-1} onto L with kernel K. Therefore we obtain that

$$d^{N-1}(d^{N-2} - \tau) = d^{N-1}d^{N-2} - d^{N-1}\tau = d^{N-1}(1 - P)d^{N-2} = 0.$$

Define $D^{N-2} = d^{N-2} - \tau \in L(E^{N-2}, K)$. Then it is elementary to check that

$$E^\bullet : 0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{N-3}} E^{N-2} \xrightarrow{D^{N-2}} K \to 0$$

is a Fredholm p-quasicomplex again. Hence a straightforward inductive argument completes the proof of the existence of the operators τ^i.

Define $D^i = d^i - \tau^i$ $(i = 0, \ldots, N-1)$. Then $(E^n, D^n)_{n=0}^N$ is a Fredholm complex of Banach spaces. It is well known (see the proof of part (b) of Theorem 2.6.13 in [2]) that in this case there are operators $h^i \in L(E^i, E^{i-1})$ such that

$$1_{E^i} - (D^{i-1}h^i + h^{i+1}D^i) \in L(E^i)$$

are finite-rank projections. Clearly this observation completes the proof. □

To prove that a morphism of Fredholm p-quasicomplexes $E^\bullet = (E^n, d^n)_{n=0}^N$ and $F^\bullet = (F^n, d'^n)_{n=0}^N$ admits perturbations of Schatten class $\mathcal{C}^p$ to a cochain mapping of Fredholm complexes, we proceed in two steps. We first replace $E^\bullet$ and $F^\bullet$ by Fredholm complexes $\tilde{E}^\bullet$ and $\tilde{F}^\bullet$ using the previous result. Then the following result will allow us to replace the morphism by a cochain mapping of the complexes $\tilde{E}^\bullet$ and $\tilde{F}^\bullet$.

1.4 Theorem. *Let* $E^\bullet = (E^n, d^n)_{n=0}^N$ *and* $F^\bullet = (F^n, d'^n)_{n=0}^N$ *be Fredholm complexes of Banach spaces and let* $a = (a^n)_{n=0}^N$ *be a sequence of bounded linear operators* $a^n \in L(E^n, F^n)$ *such that*

$$d'^n a^n - a^{n+1}d^n \in \mathcal{C}^p(E^n, F^{n+1})$$

for all n. *Then there are operators* $C^n \in \mathcal{C}^p(E^n, F^n)$ *such that*

$$d'^n(a^n - C^n) = (a^{n+1} - C^{n+1})d^n \quad (n = 0, \ldots, N-1).$$

Proof. Since $E^\bullet$ is a Fredholm complex, the closed subspaces Ker $d^n \subset E^n$ and Im $d^n \subset E^{n+1}$ possess topological direct complements and the cohomology groups $H^n(E^\bullet) = \text{Ker } d^n/\text{Im } d^{n-1}$ are finite dimensional for all n (Lemma 2.6.13 in [2]).

By Lemma 1.2 there are operators $C^N \in \mathcal{C}^p(E^N, F^N), \tilde{C}^{N-1} \in C^p(E^{N-1}, F^{N-1})$ such that

$$d'^{N-1}(a^{N-1} - \tilde{C}^{N-1}) = (a^N - C^N)d^{N-1}.$$

Define $\tilde{a}^{N-1} = a^{N-1} - \tilde{C}^{N-1}$. Then the horizontal lines in the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & E^0 & \xrightarrow{d^0} & E^1 & \xrightarrow{d^1} & \dots & \xrightarrow{d^{N-3}} E^{N-2} & \xrightarrow{d^{N-2}} & \text{Ker}\, d^{N-1} & \longrightarrow & 0 \\ & & a^0 \downarrow & & a^1 \downarrow & & & a^{N-2} \downarrow & & \tilde{a}^{N-1} \downarrow & & \\ 0 & \longrightarrow & F^0 & \xrightarrow{d'^0} & F^1 & \xrightarrow{d'^1} & \dots & \xrightarrow{d'^{N-3}} F^{N-2} & \xrightarrow{d'^{N-2}} & \text{Ker}\, d'^{N-1} & \longrightarrow & 0 \end{array}$$

are Fredholm complexes of Banach spaces such that all squares are p-essentially commuting. Again as an application of Lemma 1.2 we obtain the existence of operators $\hat{C}^{N-1} \in \mathcal{C}^p(\text{Ker}\, d^{N-1}, \text{Ker}\, d'^{N-1}), \tilde{C}^{N-2} \in \mathcal{C}^p(E^{N-2}, F^{N-2})$ such that

$$d'^{N-2}(a^{N-2} - \tilde{C}^{N-2}) = (\tilde{a}^{N-1} - \hat{C}^{N-1})d^{N-2}.$$

Choose a closed subspace $L \subset E^{N-1}$ with $E^{N-1} = \text{Ker}\, d^{N-1} \oplus L$. Then the operator

$$C^{N-1} : E^{N-1} = \text{Ker}\, d^{N-1} \oplus L \to F^{N-1}, x \oplus y \mapsto \tilde{C}^{N-1}(x \oplus y) + \hat{C}^{N-1}x$$

belongs to $\mathcal{C}^p(E^{N-1}, F^{N-1})$ and satisfies the identities

$$d'^{N-1}(a^{N-1} - C^{N-1}) = (a^N - C^N)d^{N-1}$$

and

$$d'^{N-2}(a^{N-2} - \tilde{C}^{N-2}) = (a^{N-1} - C^{N-1})d^{N-2}.$$

Continuing in this way, we find operators $C^n \in \mathcal{C}^p(E^n, F^n)(0 \leq n \leq N)$ which satisfy the required intertwining relations. □

Combining Theorem 1.3 and Theorem 1.4 we obtain our main perturbation result for morphisms of quasicomplexes with Schatten class curvature.

1.5 Corollary. *Let $E^\bullet = (E^n, d^n)_{n=0}^N, F^\bullet = (F^n, d'^n)_{n=0}^N$ be Fredholm p-quasicomplexes of Banach spaces with $1 \leq p \leq \infty$ and let $a = (a^n)_{n=0}^N$ be a sequence of bounded linear operators $a^n : E^n \to F^n$ with*

$$d'^n a^n - a^{n+1} d^n \in \mathcal{C}^p(E^n, F^{n+1})$$

for all n. Then there are perturbations D^n of d^n, D'^n of d'^n and A^n of a^n of Schatten class $\mathcal{C}^p$ such that $(E^\bullet, D^\bullet)$, $(F^\bullet, D'^\bullet)$ are Fredholm complexes and

$$D'^n A^n = A^{n+1} D^n$$

holds for all n. In the case that $E^\bullet = F^\bullet$ one can choose $D^\bullet = D'^\bullet$.

2 Lefschetz numbers

Let $E^\bullet = (E^n, d^n)_{n=0}^N$ be a Fredholm quasicomplex of trace class curvature consisting of Hilbert spaces E^n and let $a = (a^n)_{n=0}^N$ be a morphism of $E^\bullet$, that is, a sequence of bounded operators $a^n : E^n \to E^n$ such that

$$d^n a^n - a^{n+1} d^n \in \mathcal{C}^1(E^n, E^{n+1})$$

for all n. By Theorem 1.3 there exist operators $\epsilon^i \in L(E^i, E^{i-1})$ with

$$d^{i-1}\epsilon^i + \epsilon^{i+1} d^i \in 1_{E^i} + \mathcal{C}^1(E^i) \quad (i = 0, \dots, N).$$

According to Corollary 1.5 we can choose trace class perturbations $D^n$of d^n, A^n of a^n such that $(E^n, D^n)_{n=0}^N$ is a complex and $A = (A^n)_{n=0}^N$ is a cochain mapping of $(E^n, D^n)_{n=0}^N$ into itself. Then the relations

$$D^{i-1}\epsilon^i + \epsilon^{i+1} D^i = 1_{E^i} - r^i \quad (i = 0, \dots, N)$$

define trace class operators r^i on E^i. In Theorem 4.2 of [8] it was shown that

$$\sum_{i=0}^N (-1)^i \operatorname{tr}(A^i, H^i(D^\bullet)) = \sum_{i=0}^N (-1)^i \operatorname{tr}(A^i - (A^i \epsilon^{i+1}) d^i - d^{i-1}(A^{i-1}\epsilon^i)).$$

To see what happens if the operators A_i on the right-hand side are replaced by the operators a_i, we recall the arguments from [8]. The relations

$$(A^i \epsilon^{i+1}) D^i + D^{i-1}(A^{i-1}\epsilon^i) = A^i - A^i r^i \quad (i = 0, \dots, N)$$

show that the cochain mappings $(A^n)_{n=0}^N$ and $(A^n r^n)_{n=0}^N$ of the complex $(E^\bullet, D^\bullet)$ are homotopic, and hence induce the same cohomology maps. Using Theorem 19.1.5 from [4], we find that

$$\sum_{i=0}^N (-1)^i \operatorname{tr}(A^i, H^i(D^\bullet)) = \sum_{i=0}^N (-1)^i \operatorname{tr}(A^i r^i, H^i(D^\bullet))$$

$$= \sum_{i=0}^N (-1)^i \operatorname{tr}(A^i r^i) = \sum_{i=0}^N (-1)^i \operatorname{tr}(A^i - A^i \epsilon^{i+1} D^i - D^{i-1} A^{i-1} \epsilon^i).$$

Since the alternating sum of the traces of the operators

$$a^i \epsilon^{i+1} d^i - A^i \epsilon^{i+1} D^i + d^{i-1} a^{i-1} \epsilon^i - D^{i-1} A^{i-1} \epsilon^i$$

$$= a^i \epsilon^{i+1}(d^i - D^i) + (d^{i-1} - D^{i-1}) a^{i-1}\epsilon^i + (a^i - A^i)\epsilon^{i+1} D^i + D^{i-1}(a^{i-1} - A^{i-1})\epsilon^i$$

is zero, it follows that

$$\sum_{i=0}^N (-1)^i \operatorname{tr}(A^i, H^i(D^\bullet)) - \sum_{i=0}^N (-1)^i \operatorname{tr}(A^i - a^i) = \sum_{i=0}^N (-1)^i \operatorname{tr}(a^i - a^i \epsilon^{i+1} d^i - d^{i-1} a^{i-1} \epsilon^i).$$

As proposed in [8], we call the number occurring on the right-hand side of the last equation, the Lefschetz number of the morphism a relative to $(E^\bullet, d^\bullet)$.

2.1 Definition. *Let $E^\bullet = (E^n, d^n)_{n=0}^N$ be a Fredholm quasicomplex of Hilbert spaces with trace class curvature and let $a = (a^n)_{n=0}^N$ be a morphism of $E^\bullet$. Then the Lefschetz number of a relative to $E^\bullet$ is defined as*

$$L_{E^\bullet}(a) = \sum_{i=0}^{N} (-1)^i \mathrm{tr}(a^i - a^i \epsilon^{i+1} d^i - d^{i-1} a^{i-1} \epsilon^i),$$

where $\epsilon^i \in L(E^i, E^{i-1})$ are arbitrary operators with $d^{i-1}\epsilon^i + \epsilon^{i+1} d^i \in 1_{E^i} + \mathcal{C}^1(E^i)$ for all i.

Note that the remarks leading to the above definition show that the alternating sum of traces defining $L_{E^\bullet}(a)$ is independent of the particular choice of the operators ϵ^i. An inspection of the proofs of Theorem 1.3 and Theorem 1.4 shows that one can always choose trace class perturbations D^n of d^n and A^n of a^n in such a way that $A = (A^n)$ is a cochain mapping of the Fredholm complex $(E^\bullet, D^\bullet)$ with

$$L_{E^\bullet}(a) = \sum_{i=0}^{N} (-1)^i \mathrm{tr}(A^i, H^i(D^\bullet)).$$

As in the classical situation the Lefschetz numbers possess a certain commutativity property.

2.2 Theorem. *Let $E^\bullet = (E^n, d^n)_{n=0}^N$ be a Fredholm quasicomplex of Hilbert spaces with trace class curvature and let $a = (a^n)_{n=0}^N$, $b = (b^n)_{n=0}^N$ be morphisms of $E^\bullet$. Then $ab = (a^n b^n)_{n=0}^N$ and $ba = (b^n a^n)_{n=0}^N$ are morphisms of $E^\bullet$ and*

$$L_{E^\bullet}(ab) = L_{E^\bullet}(ba).$$

Proof. By Corollary 1.5 and its proof, there are trace class perturbations D^n of d^n, A^n of a^n and B^n of b^n such that $(E^\bullet, D^\bullet)$ is a complex and such that $D^n A^n = A^{n+1} D^n$ and $D^n B^n = B^{n+1} D^n$ for all n. Since ab and ba are morphisms of $E^\bullet$ and since

$$A^n B^n - a^n b^n, B^n A^n - b^n a^n \in \mathcal{C}^1(E^n)$$

for all n, we find that

$$L_{E^\bullet}(ab) = \sum_{i=0}^{N} (-1)^i \mathrm{tr}(a^i b^i - a^i b^i \epsilon^{i+1} d^i - d^{i-1} a^{i-1} b^{i-1} \epsilon^i)$$

$$= \sum_{i=0}^{N} (-1)^i \mathrm{tr}(A^i B^i, H^i(D^\bullet))) - \sum_{i=0}^{N} (-1)^i \mathrm{tr}(A^i B^i - a^i b^i)$$

$$= \sum_{i=0}^{N} (-1)^i \mathrm{tr}(B^i A^i, H^i(D^\bullet)) - \sum_{i=0}^{N} (-1)^i \mathrm{tr}(B^i A^i - b^i a^i)$$

$$+ \sum_{i=0}^{N} (-1)^i \mathrm{tr}(B^i A^i - b^i a^i - A^i B^i + a^i b^i)$$

$$= L_{E^\bullet}(ba) + \sum_{i=0}^{N} (-1)^i \mathrm{tr}((B^i - b^i)A^i + b^i(A^i - a^i) - (A^i - a^i)B^i - a^i(B^i - b^i))$$

$$= L_{E^\bullet}(ba) + \sum_{i=0}^{N} (-1)^i \mathrm{tr}((B^i - b^i)(A^i - a^i) - (A^i - a^i)(B^i - b^i)) = L_{E^\bullet}(ba),$$

where the operators ϵ^i are chosen as in Definition 2.1. $\square$

If $E^\bullet$ and $F^\bullet$ are Fredholm quasicomplexes of Hilbert spaces with trace class curvature, a is a morphism from $E^\bullet$ into $F^\bullet$ and b is a morphism of $F^\bullet$ into $E^\bullet$, then exactly as in the proof of Theorem 2.2 it follows that $L_{F^\bullet}(ab) = L_{E^\bullet}(ba)$.

In [8] the question arose whether, for every morphism $a = (a^n)_{n=0}^N$ of a Fredholm quasicomplex $E^\bullet = (E^n, d^n)_{n=0}^N$ of Hilbert spaces with trace class curvature, there are trace class perturbations D^n of d^n such that $(E^\bullet, D^\bullet)$ is a complex and a is a cochain mapping of $(E^\bullet, D^\bullet)$. We give an elementary counterexample.

Assume that there were a positive answer. Denote by $H^2 = H^2(\mathbb{T})$ and $H^\infty = H^\infty(\mathbb{T})$ the Hardy space on the unit circle and its multiplier space. For $f \in L^\infty(\mathbb{T})$, let $T_f \in L(H^2)$ be the Toeplitz operator with symbol f. Since $T_{\overline{z}}$ is Fredholm with $[T_{\overline{z}}, T_z] \in \mathcal{C}^1(H^2)$, there would have to be an operator $C \in \mathcal{C}^1(H^2)$ with

$$T_z(T_{\overline{z}} + C) = (T_{\overline{z}} + C)T_z.$$

Since the commutant of T_z consists of all Toeplitz operators with symbol in H^∞, there would be a function $g \in H^\infty$ with $T_{\overline{z}} + C = T_g$. But it is well known that there are no non-zero compact Toeplitz operators on H^2. Thus we obtain the contradiction that $\overline{z} = g \in H^\infty$.

We conclude the paper with an elementary one-dimensional example in which an integral formula for the Lefschetz number can be given.

2.3 Example. *Let $f, g \in C^\infty(\mathbb{T})$ be smooth functions on the unit circle. It is well known that the essential spectrum of T_g is given by $\sigma_e(T_g) = g(\mathbb{T})$, that $T_{fg} - T_f T_g \in \mathcal{C}^1(H^2)$ and that*

$$\mathrm{tr}\,[T_f, T_g] = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} f(d_z g) dz,$$

where the integral is the contour integral along the unit circle and $d_z g \in C^\infty(\mathbb{T})$ is given by $(z d_z g)(e^{it}) = -i \frac{d}{dt}(g(e^{it}))$ (see e.g. Section 1 in [3]).

Suppose that $0 \notin g(\mathbb{T})$. Then T_g is Fredholm and $1 - T_g T_{g^{-1}}$, $1 - T_{g^{-1}} T_g$ are both trace class. Hence the Lefschetz number of T_f relative to T_g can be calculated as

$$L_{T_g}(T_f) = \mathrm{tr}\,(T_f - T_f T_{g^{-1}} T_g) - \mathrm{tr}(T_f - T_g T_f T_{g^{-1}})$$

$$= \mathrm{tr}\,[T_g, T_f T_{g^{-1}}] = \mathrm{tr}\,[T_g, T_{fg^{-1}}]$$

$$= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} g d_z(fg^{-1}) dz = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \big(d_z(f) + g f d_z(g^{-1})\big) dz = -\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(d_z g)}{g}\, dz.$$

By choosing $f = 1$, we obtain the well known index formula

$$\mathrm{ind}(T_g) = L_{T_g}(1) = -\frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{d_z g}{g} \, dz.$$

References

[1] C. Ambrozie, F.-H. Vasilescu, Banach space complexes, Kluwer Acacemic Publishers, Dordrecht, 1995.

[2] J. Eschmeier, M. Putinar, Spectral decompositions and analytic sheaves, LMS Monograph Series, Vol. 10, Clarendon Press, Oxford 1996.

[3] J.W. Helton, R. Howe, Integral operators, commutator traces, index and homology, in: Proceedings of a conference on operator theory, Springer Lecture Notes 345, Springer, Berlin, 1973.

[4] L. Hörmander, The analysis of linear partial differential operators III, Springer, Berlin 1994.

[5] R. Levy, Algebraic and topological K-functors of commuting n-tuples of operators, J. Operator Theory 21 (1989), 219-254.

[6] M. Putinar, Some invariants for semi-Fredholm systems of essentially commuting operators, J. Operator Theory 8 (1982), 65-90.

[7] M. Putinar, F.-H. Vasilescu, Continuous and analytic invariants for deformations of Fredholm complexes, J.Operator Theory 9 (1983), 3-26.

[8] N. Tarkhanov, D. Wallenta, The Lefschetz number of sequences of trace class curvature, Preprint 2012, Universität Potsdam.

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