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# On some perturbations of the total variation image inpainting method. Part I: regularity theory

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**Abstract**

In this article we investigate some modifications of the total variation inpainting model and discuss the existence as well as the smoothness of solutions to these new variational problems. We emphasize that our variant of a linear growth regularization is completely discussed in the Sobolev space $W^1_1$ without passing to a relaxed version in the space of functions with bounded variation.

# 1 Introduction

Consider a black and white image described by a function $u\colon\,\Omega \to [0,1]$, where $\Omega$ denotes a bounded Lipschitz domain in $\mathbb{R}^2$, e.g. a rectangle, and where $u(x)$ measures the intensity of the grey level at $x \in \Omega$. Suppose that a certain part of this image is damaged, which means that there is a Lebesgue measurable subset $D$ of $\Omega$ satisfying ($\mathcal{L}^2$ denoting Lebesgue's measure on $\mathbb{R}^2$)

$$0 < \mathcal{L}^2(D) < \mathcal{L}^2(\Omega) \tag{1.1}$$

such that the intensity of the grey level is only known for points $x \in \Omega - D$. Let us represent the observed image through a (given) measurable function $f\colon\,\Omega - D \to [0,1]$.

Roughly speaking, image inpainting is the task to restore the missing part $D \to [0,1]$ of the image with the help of the given data.

There exists a variety of different image inpainting methods including theoretical and numerical aspects, we refer to the papers [ACS], [BHS], [BCMS], [CKS], [CS], [PSS], [Sh] and the references quoted therein. In the so-called variational approach the "reconstructed image" is found as a minimizer of the functional

$$J[u] = \int_\Omega \Psi\left(|\nabla u|\right)\,\mathrm{d}x + \frac{\lambda}{2}\int_{\Omega - D} (u-f)^2\,\mathrm{d}x\,, \tag{1.2}$$

in which $\Psi$ denotes a density being under our disposal and where $\lambda > 0$ is some parameter.

Thus the variational method is based on the joint minimization of the quadratic fidelity term calculated over the complement of the inpainting region $D$ and a suitable energy measured on the whole domain $\Omega$. In this setting a common choice is $\Psi\left(|\nabla u|\right) := |\nabla u|$ leading to the total variation (TV) inpainting model, see e.g. [ACS], [PSS], which has to be discussed in the space $\mathrm{BV}(\Omega)$ of functions $\Omega \to \mathbb{R}$ having bounded (= finite total) variation. Here, for $u \in \mathrm{BV}(\Omega)$, we denote by $\nabla u$ the distributional gradient being a vector valued Radon measure on $\Omega$ with finite total variation $\int_\Omega |\nabla u|$. For details we refer to [Gi].

Due to the lack of ellipticity variational problems involving the total variation in general admit only irregular solutions, and the purpose of our note is to replace the energy density $\Psi(|\nabla u|) = |\nabla u|$ by functions of nearly linear growth or even by a family of densities growing linear w.r.t. $|\nabla u|$, but with better ellipticity properties, trying to establish the smoothness of the corresponding minimizers.

As a model for the nearly linear growth case we propose the logarithmic density

$$\Psi(|\nabla u|) := |\nabla u| \ln(1 + |\nabla u|)$$

or any finite iteration of the logarithm, e.g.

$$\Psi(|\nabla u|) := |\nabla u| \ln(1 + \ln(1 + |\nabla u|)) \,, \dots .$$

Clearly these densities satisfy

$$\lim_{t\to\infty} \frac{\Psi(t)}{t} = \infty, \quad \lim_{t\to\infty} \frac{\Psi(t)}{t^p} = 0$$

for any $p > 1$. We have

**Theorem 1.1.** *Let $\Psi(t) := t\ln(1+t)$. Suppose that we have (1.1) and define $J$ according to (1.2). Then we have:*

*i) the variational problem $J \to \min$ in $W_1^1(\Omega)$ admits a unique solution $u$ in the space $W_1^1(\Omega)$;*

*ii) the function $u$ satisfies the inequality $0 \le u \le 1$, i.e. $u$ measures the intensity of the grey level.*

**Remark 1.1.** *For a definition of the Sobolev spaces $W_p^k(\Omega)$ we refer to [Ad].*

**Theorem 1.2.** *The solution $u$ from Theorem 1.1 is of class $C^{1,\alpha}$ in the interior of $\Omega$ for any exponent $\alpha < 1$.*

Let us pass to the linear growth case: we introduce the energy

$$I[w] := \int_\Omega F(\nabla w)\, \mathrm{d}x + \frac{\lambda}{2} \int_{\Omega - D} (w - f)^2 \, \mathrm{d}x \tag{1.3}$$

for functions $w$ from the space $W_1^1(\Omega)$, where the density $F\colon \mathbb{R}^2 \to [0,\infty)$ is of class $C^2$ with $F(0) = 0$, $DF(0) = 0$ and satisfies the following set of assumptions:

there exist positive constants $\nu_1$, $\nu_2$, $\nu_3$ and a real number $\mu > 1$ such that for any $Y$, $Z \in \mathbb{R}^2$ we have

$$|DF(Z)| \le \nu_1 \tag{1.4}$$

and

$$\nu_2 \frac{1}{(1+|Z|)^\mu} |Y|^2 \le D^2F(Z)(Y,Y) \le \nu_3 \frac{1}{1+|Z|} |Y|^2 \,. \tag{1.5}$$

Of course these hypotheses look rather technical and therefore need some comments:

**Remark 1.2.** *It is easy to show that $F$ is of linear growth in the sense that*

$$a|Z| - b \le F(Z) \le A|Z| + B$$

*holds for all $Z \in \mathbb{R}^2$ with suitable constants $a$, $A > 0$, $b$, $B \in \mathbb{R}$. We refer to [Bi], Remark 4.2, p.97.*

**Remark 1.3.** *Using (1.5) we obtain the inequalities*

$$\begin{aligned} &DF(Z)\cdot Z \ge \nu_4|Z| - \nu_5\,,\\ &|D^2F(Z)|\,|Z|^2 \le \nu_6\,(1+F(Z)) \end{aligned}$$

*again for arbitrary vectors $Z \in \mathbb{R}^2$ and with constants $\nu_4$, $\nu_5$ and $\nu_6$ being positive. Since we suppose $DF(0) = 0$, we may even choose $\nu_5 = 0$ (see, e.g., [Bi], p. 98). The second estimate shows that $F$ automatically satisfies a "balancing condition".*

**Remark 1.4.** *i) The most prominent example for which we have (1.4) and (1.5) is the minimal surface integrand given by $F(Z) := \sqrt{1+|Z|^2}$. In this case (1.5) holds for the optimal choice $\mu = 3$.*

*ii) Suppose next that we are given a number $\mu > 1$ and let*

$$\Phi_\mu(t) := \int_0^t \int_0^s (1+r)^{-\mu}\,\mathrm{d}r\,\mathrm{d}s,\ t \ge 0\,, \tag{1.6}$$

*together with*

$$F_\mu(Z) := \Phi_\mu(|Z|),\ Z \in \mathbb{R}^2\,. \tag{1.7}$$

*Then it holds: the density $F_\mu$ from (1.7) with $\Phi_\mu$ defined in (1.6) satisfies (1.4) and (1.5) exactly with the prescribed parameter $\mu$.*

*For $\mu \neq 2$ we have the formula*

$$\Phi_\mu(t) = \frac{t}{\mu-1} + \frac{1}{\mu-1}\frac{1}{\mu-2}(t+1)^{-\mu+2} - \frac{1}{\mu-1}\frac{1}{\mu-2}\,, \tag{1.8}$$

*whereas*

$$\Phi_2(t) = t - \ln(1+t)\,,$$

*and we see that*

$$(\mu-1)F_\mu(Z) \longrightarrow |Z|,\ \mu\to\infty\,, \tag{1.9}$$

*for vectors $Z \in \mathbb{R}^2$. For this reason and also with respect to the explicit formula (1.8) the density $F_\mu(\nabla u)$ serves as a very good candidate for an approximation of $|\nabla u|$ by more regular integrands of linear growth.*

*iii) There is another interesting feature of the functions $\Phi_\mu$: if we formally let $\mu = 1$ in (1.6), then we obtain*

$$\Phi_1(t) = t\ln(1+t) + \ln(1+t) - t\,,$$

*which means that up to lower order terms the function $\Phi_1$ coincides with the logarithmic density $t\ln(1+t)$. Therefore and with respect to (1.9) we can interprete the family of densities $\Phi_\mu$ as a smooth curve in the space of integrands deforming the logarithmic density into the density $|\nabla u|$ occurring in the TV-regularization model.*

*iv) A slight modification of the functions $\Phi_\mu$ from (1.6) is given by*

$$\widetilde{\Phi}_\mu(t) := \int_0^t \int_0^s (1+r^2)^{-\mu/2}\,\mathrm{d}r\,\mathrm{d}s,\ t \geq 1\,,$$

*where as before $\mu > 1$. It is easy to check that(1.4) and (1.5) hold for the corresponding integrands $\widetilde{F}_\mu(Z) := \widetilde{\Phi}_\mu(|Z|)$, $Z \in \mathbb{R}^2$. For $\mu = 3$ we obtain the minimal surface density, whereas*

$$\widetilde{F}_2(Z) = |Z|\arctan|Z| - \frac{1}{2}\ln\left(1+|Z|^2\right)\,.$$

After these preparations we formulate our existence and regularity results for the case of $\mu$-elliptic energies.

**Theorem 1.3.** *Let (1.1) hold and define the energy $I$ according to (1.3) with $F$ satisfying (1.4) and (1.5) for some $\mu \in (1,2)$. Then we have:*

*i) the problem $I \to \min$ admits a unique solution $u$ in the space $W^1_1(\Omega)$;*

*ii) it holds $0 \leq u \leq 1$ almost everywhere on $\Omega$.*

**Theorem 1.4.** *i) Under the assumptions and with the notation from Theorem 1.3 it holds $u \in W^1_{p,\mathrm{loc}}(\Omega)$ for any finite $p$, hence $u$ is Hölder continuous in the interior of $\Omega$ for any exponent $< 1$.*

*ii) There is an open subset $\Omega_0$ of $\Omega$ such that $\dim_{\mathcal{H}}(\Omega - \Omega_0) = 0$ and $u \in C^{1,\beta}(\Omega_0)$ for any $\beta < 1$.*

*iii) If $D$ is an open set, then $D \subset \Omega_0$, i.e. $u \in C^{1,\alpha}(D)$ for any $\alpha \in (0,1)$. For arbitrary sets $D$ we have $\mathrm{Int}(D) \subset \Omega_0$, where $\mathrm{Int}(D)$ is the set of interior points of $D$.*

**Remark 1.5.** $\dim_{\mathcal{H}}(\Omega - \Omega_0) = 0$ *by definition means $\mathcal{H}^\varepsilon(\Omega - \Omega_0) = 0$ for any $\varepsilon > 0$, $\mathcal{H}^\varepsilon$ denoting the Hausdorff-measure of dimension $\varepsilon$. Thus the singular set is in some sense very small.*

**Remark 1.6.** *For arbitrary values of* $\mu$ *we do not expect the solvability of the problem* $I \to \min$ *in the Sobolev space* $W^1_1(\Omega)$*: as outlined in [Bi], Theorem 4.39, p. 133, the choice* $\mu > 3$ *seems to be critical, which means that for large values of* $\mu$ *we have to consider suitable relaxed variants of the original problem. We refer the reader to Part II.*

**Remark 1.7.** *Actually we prove (compare (3.13) and pass to the limit* $\delta \to 0$*) that the solution* $u$ *from Theorem 1.3 is in any space* $W^2_{s,\mathrm{loc}}(\Omega)$ *for exponents* $s \in (1,2)$*, which by Sobolev's theorem implies* $u \in W^1_{p,\mathrm{loc}}(\Omega)$ *for all finite* $p$*.*

**Remark 1.8.** *With minor adjustments the results of Theorem 1.1 - 1.4 can be extended to the case that an additional boundary condition like* $u = u_0$ *on* $\partial\Omega$ *is imposed with a sufficiently regular function* $u_0$ *satisfying* $0 \le u_0 \le 1$*. For details we refer to [BF1].*

## 2 Logarithmic inpainting

We start with the

**Proof of Theorem 1.1.** Let

$$\Psi(t) := t\ln(1+t)\,, \quad t \ge 0\,.$$

Then the functional $J$ from (1.2) is well defined on the Orlicz-Sobolev space $W^1_\Psi(\Omega)$ generated by $\Psi$. We note that $W^1_\Psi(\Omega)$ is a proper subspace of the Sobolev class $W^1_1(\Omega)$, moreover, since we consider the 2D-case, the embedding $W^1_\Psi(\Omega) \hookrightarrow L^2(\Omega)$ is compact. We refer the reader to e.g. [Ad] for more details concerning function spaces. Let

$$a := \inf\left\{J[v] : v \in W^1_\Psi(\Omega)\right\} \in [0,\infty)$$

and consider a $J$-minimizing sequence $(u_n)$ in $W^1_\Psi(\Omega)$, i.e.

$$J[u_n] \to a \quad \text{as } n \to \infty\,.$$

We may assume w.l.o.g. that

$$0 \le u_n \le 1 \tag{2.1}$$

holds a.e. on $\Omega$. Otherwise we first replace $u_n$ by the sequence $v_n := \max\{0, u_n\} \in W^1_\Psi(\Omega)$ and observe

$$\int_\Omega \Psi\left(|\nabla v_n|\right)\,\mathrm{d}x \le \int_\Omega \Psi\left(|\nabla u_n|\right)\,\mathrm{d}x\,. \tag{2.2}$$

On the set $[u_n \ge 0]$ is holds $u_n - f = v_n - f$, whereas on $[u_n \le 0]$ we deduce from $f \ge 0$

$$|v_n - f| = f \le |u_n - f|\;,$$

hence

$$\int_{\Omega - D} (v_n - f)^2\,\mathrm{d}x \le \int_{\Omega - D} (u_n - f)^2\,\mathrm{d}x\,, \tag{2.3}$$

and from (2.2) and (2.3) it follows that $v_n$ is also a $J$-minimizing sequence in $W^1_\Psi(\Omega)$. Thus we can directly require the validity of $u_n \geq 0$.

In order to justify the second part of our hypothesis (2.1), we then replace $u_n$ by $w_n := \min\{1, u_n\}$ observing that $f \leq 1$ implies $J[w_n] \leq J[u_n]$.

From (2.1) together with $\lim_{n\to\infty} J[u_n] = a < \infty$ it follows that $(u_n)$ is uniformly bounded in the space $W^1_\Psi(\Omega)$, thus we may extract a subsequence such that (compare [GMS], Theorem 2 on p. 50, for the second statement in (2.4))

$$\begin{aligned} &u_n \to u \text{ in } L^2(\Omega) \text{ and a.e.}\,, \\ &\nabla u_n \rightharpoondown \nabla u \text{ weakly in } L^1(\Omega, \mathbb{R}^2) \end{aligned} \tag{2.4}$$

for a function $u \in W^1_\Psi(\Omega)$ which satisfies (cf. (2.1))

$$0 \leq u \leq 1 \quad \text{a.e. on } \Omega\,. \tag{2.5}$$

By (2.4) we may quote De Giorgi's theorem on lower semicontinuity (see e.g. [Gia], Theorem 2.3, p. 18) to see

$$J[u] \leq \liminf_{n\to\infty} J[u_n]\,,$$

hence $u$ is $J$-minimizing in $W^1_\Psi(\Omega)$ sharing the inequality (2.5).

If $\widetilde{u}$ denotes a second function from $W^1_\Psi(\Omega)$ such that $J[\widetilde{u}] = a$, then (by strict convexity) we must have $\nabla u = \nabla\widetilde{u}$ a.e. on $\Omega$ together with $u = \widetilde{u}$ on $\Omega - D$, hence $u = \widetilde{u} + c$ on $\Omega$ for a suitable constant $c$, and (1.1) immediately implies $c = 0$. This completes the proof of Theorem 1.1. □

In order to proceed we need

**Lemma 2.1.** *Consider the variational problem* ($0 < \delta < 1$)

$$J_\delta[w] := \frac{\delta}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x + J[w] \to \min \quad \text{in } W^1_2(\Omega) \tag{2.6}$$

*with $J$ from (1.2) and with $\Psi(t) := t\ln(1+t)$, $t \geq 0$. Then it holds:*

*i) problem (2.6) has a unique solution $u_\delta$ which satisfies $0 \leq u_\delta \leq 1$;*

*ii) the functions $u_\delta$ belong to the space $W^2_{2,\mathrm{loc}}(\Omega)$;*

*iii) $u_\delta \rightharpoondown u$ in $W^1_1(\Omega)$, $\delta\int_\Omega |\nabla u_\delta|^2\,\mathrm{d}x \to 0$ and $J_\delta[u_\delta] \to J[u]$ as $\delta \to 0$, $u$ denoting the minimizer from Theorem 1.1.*

**Proof of Lemma 2.1.** *i*) follows along the same lines as the proof of Theorem 1.1.

*ii*) is easily established by passing to the Euler equation and applying the difference quotient technique.

*iii*) We have $J_\delta[u_\delta] \le J_\delta[0] = J[0]$, thus

$$\sup_\delta \int_\Omega \Psi\left(|\nabla u_\delta|\right) \mathrm{d}x < \infty \,. \tag{2.7}$$

Combining (2.7) with $0 \le u_\delta \le 1$, we find a sequence $\delta_m \to 0$ and a function $\overline{u} \in W^1_\Psi(\Omega)$ such that ($u_m := u_{\delta_m}$)

$$u_m \to \overline{u} \;\text{ in } L^2(\Omega) \text{ and a.e.,} \quad \nabla u_m \rightharpoondown \nabla \overline{u} \;\text{ weakly in } L^1(\Omega, \mathbb{R}^2)\,. \tag{2.8}$$

As in the proof of Theorem 1.1 we get from (2.8)

$$J[\overline{u}] \le \liminf_{m\to\infty} J[u_m]\,,$$

thus ($J_m := J_{\delta_m}$)

$$J[\overline{u}] \le \liminf_{m\to\infty} J_m[u_m]\,. \tag{2.9}$$

Consider $w \in W^1_2(\Omega)$. The $J_m$-minimality of $u_m$ in the space $W^1_2(\Omega)$ implies

$$J_m[u_m] \le J_m[w] \to J[w] \quad \text{as } m \to \infty\,, \tag{2.10}$$

and in conclusion $J[\overline{u}] \le J[w]$.

Now, if $v \in W^1_\Psi(\Omega)$ is given, we may choose a sequence $(w_k)$ in $W^1_2(\Omega)$ such that $w_k \to v$ in the norm of the space $W^1_\Psi(\Omega)$. This yields $J[\overline{u}] \le J[v]$, hence $\overline{u}$ is $J$-minimal in $W^1_\Psi(\Omega)$. Theorem 1.1 then shows $\overline{u} = u$.

Note that the above arguments can be repeated starting with any subsequence of $(u_\delta)$, thus (2.8) is true for any choice $\delta_m \to 0$.

Finally recall that for $w \in W^1_2(\Omega)$ we have (compare (2.9) and (2.10))

$$J[u] \le \liminf_{\delta\to 0} J_\delta[u_\delta] =: \tau_1 \le \tau_2 := \limsup_{\delta\to 0} J_\delta[u_\delta] \le J[w]\,,$$

and if we consider $W^1_2(\Omega) \ni w_k \to u$ in $W^1_\Psi(\Omega)$, we end up with

$$J[u] = \tau_1 = \tau_2\,,$$

which immediately leads to the remaining claims in *iii*). $\square$

Next we pass to the

**Proof of Theorem 1.2.** In what follows we make use of the functions $u_\delta$ introduced in Lemma 2.1. Letting

$$H_\delta(\xi) := \Psi_\delta\left(|\xi|\right)\,, \quad \Psi_\delta(t) := \frac{\delta}{2}t^2 + \Psi(t)\,,$$

we drop the index $\delta$ and obtain from the minimizing property

$$0 = \int_\Omega DH\left(\nabla u\right)\cdot\nabla\varphi\,\mathrm{d}x + \lambda\int_{\Omega - D}(u-f)\varphi\,\mathrm{d}x \tag{2.11}$$

valid for $\varphi \in C_0^\infty(\Omega)$. Replacing $\varphi$ by $\partial_\alpha\varphi$, $\alpha = 1, 2$, equation (2.11) implies after an integration by parts

$$\int_\Omega D^2H(\nabla u)\left(\partial_\alpha\nabla u, \nabla\varphi\right)\,\mathrm{d}x = \lambda\int_\Omega \mathbf{1}_{\Omega - D}\,(u-f)\,\partial_\alpha\varphi\,\mathrm{d}x\,, \tag{2.12}$$

and (2.12) remains valid for functions $\varphi \in W_2^1(\Omega)$ having compact support. Fix a disk $B := B_R(x_0)$ such that $2B := B_{2R}(x_0) \Subset \Omega$ and let $\eta \in C_0^1(2B)$ with the properties $\eta = 1$ on $B$, $0 \le \eta \le 1$ and $|\nabla\eta| \le c/R$. In (2.12) we then use $\varphi := \eta^2\partial_\alpha u$ and get (taking the sum w.r.t. $\alpha$)

$$\begin{aligned}
&\int_{2B} D^2H\left(\nabla u\right)\left(\partial_\alpha\nabla u, \partial_\alpha\nabla u\right)\eta^2\,\mathrm{d}x \\
&= \; -2\int_{2B} D^2H\left(\nabla u\right)\left(\partial_\alpha\nabla u, \nabla\eta\right)\partial_\alpha u\,\eta\,\mathrm{d}x \\
&\quad +\lambda\int_{2B}\mathbf{1}_{\Omega-D}\,(u-f)\,\partial_\alpha\left(\eta^2\partial_\alpha u\right)\,\mathrm{d}x =: T_1 + T_2\,.
\end{aligned} \tag{2.13}$$

Equation (2.13) corresponds to the identity (2.5) in [BF1], and as outlined there (compare (2.8) in [BF1]) we deduce from (2.13) (with constants $c$ independent of $B$ and $\delta$)

$$\int_{2B}\eta^2\frac{\left|\nabla^2 u\right|^2}{1+|\nabla u|}\,\mathrm{d}x \le c\left[R^{-2}\int_{2B}\Psi\left(|\nabla u|\right)\,\mathrm{d}x + |T_2|\right]\,. \tag{2.14}$$

Recalling $0 \le u,\ f \le 1$ we estimate

$$|T_2| \le c\left\{\int_{2B}|\nabla\eta|\,|\nabla u|\,\mathrm{d}x + \int_{2B}\eta^2\left|\nabla^2 u\right|\,\mathrm{d}x\right\}\,,$$

the first integral being uniformly bounded w.r.t. the hidden parameter $\delta$ (recall (2.7)). Observing

$$\int_{2B}\eta^2\left|\nabla^2 u\right|\,\mathrm{d}x \le \varepsilon\int_{2B}\eta^2\frac{\left|\nabla^2 u\right|^2}{1+|\nabla u|}\,\mathrm{d}x + c(\varepsilon)\int_{2B}\eta^2\left(1+|\nabla u|\right)\,\mathrm{d}x$$

and choosing $\varepsilon > 0$ small enough, we infer from (2.14)

$$\sup_{0<\delta<1} \int_B \frac{|\nabla^2 u_\delta|^2}{1 + |\nabla u_\delta|} \, \mathrm{d}x < \infty \, . \tag{2.15}$$

Now let $\psi_\delta := \sqrt{1 + |\nabla u_\delta|}$ . Combining (2.7) with (2.15) we see $\psi_\delta \in W^1_{2,\mathrm{loc}}(\Omega)$ uniformly w.r.t. $\delta$, hence $\psi_\delta \in L^p_{\mathrm{loc}}(\Omega)$ for any finite $p$ uniformly w.r.t. $\delta$ and in conclusion

$$\int_{\Omega'} |\nabla u_\delta|^s \, \mathrm{d}x \le c(s, \Omega') < \infty \tag{2.16}$$

for any $s < \infty$ and any subdomain $\Omega' \Subset \Omega$.

Again referring to the inequality $0 \le u,\ f \le 1$ we clearly obtain (2.14) from the paper [BF1] by using the modified test function $\varphi = \eta^2 (\partial_\alpha u - \xi_\alpha)$, $\xi := \fint_{2B-\overline{B}} \nabla u \, \mathrm{d}x$ in equation (2.12). But then we may exactly follow the calculations carried out after (2.14) in [BF1] quoting also the above estimate (2.16) at appropriate places to finish the proof of interior $C^{1,\alpha}$-regularity of our solution $u$. □

# 3 Inpainting with $\mu$-elliptic energies

Consider the energy $I$ defined according to (1.3) with $F$: $\mathbb{R}^2 \to [0, \infty)$ satisfying (1.4) and (1.5) for some $\mu \in (1, 2)$. As in the previous section we let for $\delta > 0$

$$\begin{aligned} I_\delta[w] &:= \int_\Omega F_\delta(\nabla w) \, \mathrm{d}x + \frac{\lambda}{2} \int_{\Omega - D} (w - f)^2 \, \mathrm{d}x \, , \\ F_\delta(\xi) &:= \frac{\delta}{2} |\xi|^2 + F(\xi) \, , \quad \xi \in \mathbb{R}^2 \, , \end{aligned}$$

and denote by $u_\delta \in W^1_2(\Omega)$ the unique solution of the problem $I_\delta[u_\delta] \to \min$ in $W^1_2(\Omega)$. As before we have

$$u_\delta(x) \in [0, 1] \ \text{ a.e.} \, , \quad u_\delta \in W^2_{2,\mathrm{loc}}(\Omega) \, , \tag{3.1}$$

and from $I_\delta[u_\delta] \le I[0]$ we immediately get

$$\int_\Omega |\nabla u_\delta| \, \mathrm{d}x \le \ \text{const} \ < \infty \, . \tag{3.2}$$

**Lemma 3.1.** *We have $u_\delta \in W^1_{2,\mathrm{loc}}(\Omega)$ uniformly w.r.t. the parameter $\delta$.*

Given Lemma 3.1, we first present the

**Proof of Theorem 1.3.** From (3.1) and (3.2) it follows (by BV-compactness) that there exists $\overline{u} \in BV(\Omega)$ such that $u_\delta \to \overline{u}$ in $L^1(\Omega)$ (and a.e. on $\Omega$) at least for a subsequence.

From Lemma 3.1 we infer $\overline{u} \in W^1_{2,\mathrm{loc}}(\Omega)$. Since $BV(\Omega) \cap W^1_{2,\mathrm{loc}}(\Omega)$ is a subspace of $W^1_1(\Omega)$, we arrive at $\overline{u} \in W^1_1(\Omega)$ and thereby $I[\overline{u}]$ is well defined.

By lower semicontinuity we have

$$I[\overline{u}] \leq \liminf_{\delta\to 0} I[u_\delta] \tag{3.3}$$

(see below), and (3.3) implies for $v \in W^1_2(\Omega)$ (quoting the minimality of $u_\delta$ in $W^1_2(\Omega)$ for $I_\delta$)

$$I[\overline{u}] \leq \liminf_{\delta\to 0} I_\delta[u_\delta] \leq I[v]\,,$$

and from this we obtain $I[\overline{u}] \leq I[w]$ for any $w \in W^1_1(\Omega)$ by approximating $w$ through a sequence $(v_k) \subset W^1_2(\Omega)$ in the topology of $W^1_1(\Omega)$.

In other words: $\overline{u}$ is an $I$-minimizer in the class $W^1_1(\Omega)$ satisfying (see (3.1)) $0 \leq \overline{u} \leq 1$ a.e., and by construction we additionally have $\overline{u} \in W^1_{2,\mathrm{loc}}(\Omega)$.

As in the proof of Theorem 1.1 it follows that $\overline{u}$ is the only $I$- minimizer in $W^1_1(\Omega)$. This completes the proof of Theorem 1.3 modulo (3.3): as stated in the beginning of the proof we know $u_\delta \rightharpoonup \overline{u}$ in $W^1_{2,\mathrm{loc}}(\Omega)$ for a subsequence, thus $u_\delta \to \overline{u}$ in $L^2_{\mathrm{loc}}(\Omega)$ and thereby

$$\int_{\Omega^*\cap(\Omega-D)} (u_\delta - f)^2 \,\mathrm{d}x \to \int_{\Omega^*\cap(\Omega-D)} (\overline{u} - f)^2 \,\mathrm{d}x$$

as $\delta \to 0$ for compact subregions $\Omega^*$ of $\Omega$. At the same time it follows from De Giorgi's theorem (using $\nabla u_\delta \rightharpoonup \nabla\overline{u}$ in $L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$)

$$\int_{\Omega^*} F(\nabla\overline{u}) \,\mathrm{d}x \leq \liminf_{\delta\to 0} \int_{\Omega^*} F(\nabla u_\delta) \,\mathrm{d}x\,,$$

hence

$$\int_{\Omega^*} F(\nabla\overline{u}) \,\mathrm{d}x + \frac{\lambda}{2} \int_{\Omega^*\cap(\Omega-D)} (\overline{u} - f)^2 \,\mathrm{d}x \leq \liminf_{\delta\to 0} I[u_\delta]\,,$$

and if we consider $\Omega^* \nearrow \Omega$, we obtain (3.3). □

**Remark 3.1.** *From (3.3) we get along the lines of the proof of Lemma 2.1 iii) that*

$$\begin{aligned} I_\delta[u_\delta] &\to I[\overline{u}]\,, \\ \delta \int_\Omega |\nabla u_\delta|^2 \,\mathrm{d}x &\to 0\,, \\ u_\delta &\to \overline{u} \quad \text{in } L^1(\Omega)\,, \\ u_\delta &\rightharpoonup \overline{u} \quad \text{in } W^1_{2,\mathrm{loc}}(\Omega) \end{aligned} \tag{3.4}$$

*as $\delta \to 0$ not only for a subsequence.*

Next we present the

**Proof of Lemma 3.1.** As usual we drop the index $\delta$. We then obtain (2.13) now with $F(=F_\delta)$ in place of the energy density $H(=H_\delta)$, and the condition (1.5) of $\mu$-ellipticity implies (for a suitable positive constant $c$ independent of $\delta$ and $B$)

$$c\int_{2B}\eta^2\frac{|\nabla^2 u|^2}{(1+|\nabla u|)^\mu}\,\mathrm{d}x \le c(R)+T_2 \tag{3.5}$$

with $T_2$, $\eta$ and $B$ as in (2.13) (compare the derivation of (2.14)).

We have

$$\begin{aligned} T_2 &= \lambda\int_{2B}(u-f)\partial_\alpha(\eta^2\partial_\alpha u)\,\mathrm{d}x - \lambda\int_{2B\cap D}(u-f)\partial_\alpha(\eta^2\partial_\alpha u)\,\mathrm{d}x \\ &= -\lambda\int_{2B}\eta^2|\nabla u|^2\,\mathrm{d}x - \lambda\int_{2B} f\partial_\alpha(\eta^2\partial_\alpha u)\,\mathrm{d}x - \lambda\int_{2B\cap D}(u-f)\partial_\alpha(\eta^2\partial_\alpha u)\,\mathrm{d}x\,, \end{aligned}$$

hence we get from (3.5) by recalling $0\le u$, $f\le 1$ as well as (3.2)

$$c\int_{2B}\eta^2\frac{|\nabla^2 u|^2}{(1+|\nabla u|)^\mu}\,\mathrm{d}x + \lambda\int_{2B}\eta^2|\nabla u|^2\,\mathrm{d}x \le c(R) + c\int_{2B}\eta^2|\nabla^2 u|\,\mathrm{d}x\,.$$

Young's inequality gives

$$\int_{2B}\eta^2|\nabla^2 u|\,\mathrm{d}x \le \varepsilon\int_{2B}\eta^2\frac{|\nabla^2 u|^2}{(1+|\nabla u|)^\mu}\,\mathrm{d}x + c(\varepsilon)\int_{2B}\eta^2(1+|\nabla u|)^\mu\,\mathrm{d}x\,,$$

and for $\varepsilon$ sufficiently small we obtain

$$\int_{2B}\eta^2\frac{|\nabla^2 u|^2}{(1+|\nabla u|)^\mu}\,\mathrm{d}x + \int_{2B}\eta^2|\nabla u|^2\,\mathrm{d}x \le c(R) + c\int_{2B}\eta^2(1+|\nabla u|)^\mu\,\mathrm{d}x\,. \tag{3.6}$$

Clearly (3.6) implies the claim of Lemma 3.1: since we assume $\mu<2$, we can apply Young's inequality one more time to get

$$\int_B\frac{|\nabla^2 u|^2}{(1+|\nabla u|)^\mu} + \int_B|\nabla u|^2\,\mathrm{d}x \le c(R)\,. \tag{3.7}$$

□

Next we let $\varphi := (1+|\nabla u|)^{1-\mu/2}$. From (3.7) it follows

$$\int_B|\nabla\varphi|^2\,\mathrm{d}x \le c(R)\,, \tag{3.8}$$

and (3.8) holds uniformly in $\delta$. From (3.8) we infer

$$\varphi=\varphi_\delta\in W^1_{2,\mathrm{loc}}(\Omega)\quad\text{uniformly in }\delta \tag{3.9}$$

and (3.9) together with Sobolev's embedding theorem gives $\varphi = \varphi_\delta \in L^p_{\mathrm{loc}}(\Omega)$ uniformly, thus

$$\nabla u_\delta \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \quad \text{uniformly in } \delta \text{ and for all } p < \infty \, . \tag{3.10}$$

Finally we give the

**Proof of Theorem 1.4.** The first claim is already established (compare (3.10)). With the notation introduced at the beginning of this section we claim the validity of the following statements uniformly w.r.t. the parameter $\delta$

$$\Psi_\delta := (1 + |\nabla u_\delta|)^{\frac{\mu}{2}} \in W^1_{2,\mathrm{loc}}(\Omega) \, , \tag{3.11}$$

$$\omega_\delta := D^2 F_\delta \left(\nabla u_\delta\right) \left(\partial_\alpha \nabla u_\delta, \partial_\alpha \nabla u_\delta\right)^{\frac{1}{2}} \in L^2_{\mathrm{loc}}(\Omega) \, , \tag{3.12}$$

$$u_\delta \in W^2_{s,\mathrm{loc}}(\Omega) \quad \text{for any } 1 < s < 2 \, . \tag{3.13}$$

We postpone the proofs and observe that in place of (2.12) we have ($\alpha = 1$, 2) for any $\varphi \in C^1_0(\Omega)$

$$\int_\Omega D^2 F_\delta \left(\nabla u_\delta\right) \left(\partial_\alpha \nabla u_\delta, \nabla \varphi\right) \mathrm{d}x = \int_\Omega g_\delta \partial_\alpha \varphi \, \mathrm{d}x \, , \tag{3.14}$$

$g_\delta := \lambda \mathbf{1}_{\Omega - D}(u_\delta - f)$ being uniformly bounded.

As sketched at the end of Section 2 we let $\varphi = \eta^2 \left(\partial_\alpha u_\delta - (\xi_\delta)_\alpha\right)$, $\xi_\delta := \fint_T \nabla u_\delta \, \mathrm{d}x$, where $B$ and $\eta$ are chosen as done after (2.12) and where $T := 2B - \overline{B}$. (3.14) easily gives

$$\begin{aligned} \int_{2B} \eta^2 \omega_\delta^2 \, \mathrm{d}x \;\; \leq \;\; & c \left[ \frac{1}{R} \int_T \left| D^2 F_\delta \left(\nabla u_\delta\right) \right| \left| \nabla^2 u_\delta \right| \left| \nabla u_\delta - \xi_\delta \right| \mathrm{d}x \right] \\ & + \int_T \left| \partial_\alpha \left( \eta^2 \left[ \partial_\alpha u_\delta - (\xi_\delta)_\alpha \right] \right) \right| \mathrm{d}x =: S_1 + S_2 \, . \end{aligned} \tag{3.15}$$

From our assumption (1.5) it follows

$$S_1 \leq c R^{-1} \int_T \frac{1}{1 + |\nabla u_\delta|} \left| \nabla^2 u_\delta \right| \left| \nabla u_\delta - \xi_\delta \right| \mathrm{d}x$$

and since (again by (1.5))

$$c \left(1 + |\nabla u_\delta|\right)^{-\frac{\mu}{2}} \left| \nabla^2 u_\delta \right| \leq \omega_\delta \, ,$$

we get (recall $\mu < 2$ and use Hölder's inequality)

$$\begin{aligned} S_1 \;\; & \leq \;\; c R^{-1} \int_T \omega_\delta \left(1 + |\nabla u_\delta|\right)^{-1 + \frac{\mu}{2}} \left| \nabla u_\delta - \xi_\delta \right| \mathrm{d}x \\ & \leq \;\; c R^{-1} \int_T \omega_\delta \left| \nabla u_\delta - \xi_\delta \right| \mathrm{d}x \\ & \leq \;\; c R^{-1} \left[ \int_T \omega_\delta^2 \, \mathrm{d}x \right]^{\frac{1}{2}} \left[ \int_T \left| \nabla u_\delta - \xi_\delta \right|^2 \mathrm{d}x \right]^{\frac{1}{2}} , \end{aligned}$$

hence by the Sobolev-Poincaré inequality

$$\begin{aligned} S_1 &\leq cR^{-1}\left[\int_T \omega_\delta^2 \,\mathrm{d}x\right]^{\frac{1}{2}} \int_T \left|\nabla^2 u_\delta\right| \,\mathrm{d}x \\ &\leq cR^{-1}\left[\int_T \omega_\delta^2 \,\mathrm{d}x\right]^{\frac{1}{2}} \int_T \omega_\delta \Psi_\delta \,\mathrm{d}x \end{aligned} \tag{3.16}$$

with $\Psi_\delta$ from (3.11). For $S_2$ we have

$$\begin{aligned} |S_2| &\leq c\left[R^{-1}\int_T |\nabla u_\delta - \xi_\delta| \,\mathrm{d}x + \int_{2B} \eta^2 \left|\nabla^2 u_\delta\right| \,\mathrm{d}x\right] \\ &\leq c\left[\int_T \left|\nabla^2 u_\delta\right| \,\mathrm{d}x + \int_{2B} \eta^2 \left|\nabla^2 u_\delta\right| \,\mathrm{d}x\right] \\ &\leq c\left[\int_T \omega_\delta \Psi_\delta \,\mathrm{d}x + \int_{2B} \eta^2 \omega_\delta \Psi_\delta \,\mathrm{d}x\right] . \end{aligned}$$

Inserting this estimate together with (3.16) into (3.15) it follows

$$\int_{2B} \eta^2 \omega_\delta^2 \,\mathrm{d}x \leq cR^{-1}\left[\int_T \omega_\delta^2 \,\mathrm{d}x + R^2\right]^{\frac{1}{2}} \int_T \omega_\delta \Psi_\delta \,\mathrm{d}x + c\int_{2B} \eta^2 \omega_\delta \Psi_\delta \,\mathrm{d}x\,. \tag{3.17}$$

To the last integral on the r.h.s. of (3.17) we apply Young's inequality and recall the support properties of $\eta$ with the result

$$\int_B \omega_\delta^2 \,\mathrm{d}x \leq cR^{-1}\left[\int_T \omega_\delta^2 + R^2\right]^{\frac{1}{2}} \int_T \omega_\delta \Psi_\delta \,\mathrm{d}x + c\int_{2B} \Psi_\delta^2 \,\mathrm{d}x\,. \tag{3.18}$$

Note that (3.10) implies for any $\varepsilon > 0$

$$\int_{2B} \Psi_\delta^2 \,\mathrm{d}x \leq c\,(\varepsilon, \Omega^*)\, R^{2-\varepsilon}\,, \tag{3.19}$$

provided we require $2B \subset \Omega^*$ for some subdomain $\Omega^* \Subset \Omega$ which we will assume from now on.

With the exception of the term $\int_{2B} \Psi_\delta^2 \,\mathrm{d}x$ estimate (3.18) corresponds to inequality (4.22) of Lemma 4.1 in [FS]. Moreover, on account of (3.11) and (3.12), the remaining hypothesis of Lemma 4.1 in [FS] are satisfied.

But as outlined in [ABF], p. 295, or in [BF2], p. 1615, the growth estimate (3.19) is sufficient to deduce

$$\int_{B_R(x_0)} \omega_\delta^2 \,\mathrm{d}x \leq c\,\frac{1}{\ln(1/R)} \tag{3.20}$$

at least locally and uniformly w.r.t. the parameter $\delta$. Let

$$\sigma_\delta := DF_\delta\left(\nabla u_\delta\right) .$$

We have by the Cauchy-Schwarz inequality

$$\begin{aligned}\partial_\alpha \sigma_\delta \cdot \partial_\alpha \sigma_\delta &= D^2F_\delta\left(\nabla u_\delta\right)\left(\partial_\alpha \nabla u_\delta, \partial_\alpha \sigma_\delta\right)\\ &\le \left(D^2F_\delta\left(\nabla u_\delta\right)\left(\partial_\alpha \nabla u_\delta, \partial_\alpha \nabla u_\delta\right)\right)^{\frac{1}{2}}\left(D^2F_\delta\left(\nabla u_\delta\right)\left(\partial_\alpha \sigma_\delta, \partial_\alpha \sigma_\delta\right)\right)^{\frac{1}{2}}\ ,\end{aligned}$$

and if we use (1.5) we find

$$|\nabla\sigma_\delta|^2 \le c\,\omega_\delta \frac{1}{\sqrt{1+|\nabla u_\delta|}}\,|\nabla\sigma_\delta|\ , \quad \text{i.e.} \quad |\nabla\sigma_\delta| \le c\,\omega_\delta\,,$$

thus by (3.20)

$$\int_{B_R(x_0)} |\nabla\sigma_\delta|^2\,\mathrm{d}x \le c\,\frac{1}{\ln(1/R)} \tag{3.21}$$

for disks $B_{2R}(x_0) \subset \Omega^*$ and any $0 < \delta < 1$.

Due to a result of Frehse [Fr], (3.21) together with $\sup_{0<\delta<1} \|\sigma_\delta\|_{L^2(\Omega)} < \infty$ implies the continuity of $\sigma_\delta$ with local modulus of continuity being bounded independent of $\delta$. Hence there exists a continuous function $\sigma$ such that $\sigma_\delta \to \sigma$ locally uniformly. We claim that a.e. on $\Omega$

$$\sigma = DF(\nabla\overline{u}) \tag{3.22}$$

is true, $\overline{u}$ denoting the unique $W^1_1$-minimizer the functional $I$ constructed during the proof of Theorem 1.3.

In fact, by (3.13) we may assume that $\nabla u_\delta \to \nabla\overline{u}$ holds a.e. on $\Omega$. Therefore $DF_\delta\left(\nabla u_\delta\right) \to DF\left(\nabla\overline{u}\right)$ a.e. on $\Omega$ and (3.22) is a consequence of the definition of $\sigma_\delta$.

Now let

$$\Omega_0 := \left\{ x_0 \in \Omega : \lim_{\rho\downarrow 0} \fint_{B_\rho(x_0)} \nabla\overline{u}\,\mathrm{d}x \text{ exists in } \mathbb{R}^2 \right\}.$$

From [Gia], p. 100, Theorem 2.1, and from (3.13) (by passing to the limit $\delta \to 0$) it follows that $\Sigma := \Omega - \Omega_0$ satisfies $\dim_{\mathcal{H}}(\Sigma) = 0$, i.e. $\mathcal{H}^\varepsilon(\Sigma) = 0$ for any $\varepsilon > 0$, where $\mathcal{H}^\varepsilon$ is the Hausdorff-measure of dimension $\varepsilon > 0$. (3.22) implies

$$\sigma(x_0) = DF\left(\nabla\overline{u}(x_0)\right),\ x_0 \in \Omega_0\,, \tag{3.23}$$

with an obvious meaning of $\nabla\overline{u}(x_0)$.

Now, by the open mapping theorem, $\mathrm{Im}(DF)$ is an open set and (3.23) shows that $\sigma(x_0) \in \mathrm{Im}(DF)$. By continuity $\sigma(x) \in \mathrm{Im}(DF)$ for all $x \in B_r(x_0)$ with $r$ sufficiently small, hence $(DF)^{-1} \circ \sigma$ is well defined and continuous on $B_r(x_0)$.

By (3.22) we see that $\nabla\overline{u}$ has a continuous representative on $B_r(x_0)$, and therefore $\overline{u} \in C^1\left(B_r(x_0)\right)$. This shows that $\Omega_0$ is open and $\overline{u} \in C^1(\Omega_0)$.

The $C^{1,\beta}$-regularity of $\overline{u}$ on $\Omega_0$ follows by showing that $\partial_k \overline{u}$, $k = 1, 2$, satisfies an elliptic equation with coefficients $a_{ij}(x)$ to which the De Giorgi-Moser-Nash theory (compare [GT]) applies, which means that the coefficients $a_{ij}(x)$ belong to $L^\infty_{\mathrm{loc}}(\Omega_0)$. This however is an immediate consequence of $\nabla\overline{u} \in L^\infty_{\mathrm{loc}}(\Omega_0)$.

Suppose now that $D$ is an open set. Then we exactly follow the arguments in [Bi], proof of Lemma 4.29 and Theorem 4.28, using our sequence $u_\delta$ to get

$$\nabla u_\delta \in L^\infty_{\mathrm{loc}}(D, \mathbb{R}^2) \tag{3.24}$$

uniformly w.r.t. $\delta$. Here we remark that in the functional $I_\delta$ the term $\lambda \int_{\Omega - D} (w-f)^2 \, \mathrm{d}x$ drops out if we linearize with test-functions having support in $D$. From (3.24) we deduce $\nabla u \in L^\infty_{\mathrm{loc}}(D, \mathbb{R}^2)$ and thereby $u \in C^{1,\beta}(D, \mathbb{R}^2)$ for any $0 < \beta < 1$.

In order to finish the proof of Theorem 1.4 we have to justify (3.11) - (3.13).

Ad (3.13). From (3.7) it follows that for $\Omega^* \Subset \Omega$

$$\int_{\Omega^*} (1 + |\nabla u_\delta|)^{-\mu} \left|\nabla^2 u_\delta\right|^2 \, \mathrm{d}x \le c\,(\Omega^*) < \infty \, .$$

(Note that in (3.7) actually $u_\delta$ is considered.) For $1 < s < 2$ we obtain by Hölder's inequality

$$\begin{aligned} \int_{\Omega^*} \left|\nabla^2 u_\delta\right|^s \, \mathrm{d}x &= \int_{\Omega^*} \left[(1 + |\nabla u_\delta|)^{-\mu} \left|\nabla^2 u_\delta\right|^2\right]^{\frac{s}{2}} (1 + |\nabla u_\delta|)^{\mu \frac{s}{2}} \, \mathrm{d}x \\ &\le \left[\int_{\Omega^*} (1 + |\nabla u_\delta|)^{-\mu} \left|\nabla^2 u_\delta\right|^2 \, \mathrm{d}x\right]^{\frac{s}{2}} \left[\int_{\Omega^*} (1 + |\nabla u_\delta|)^t \, \mathrm{d}x\right]^{1-\frac{s}{2}} \end{aligned}$$

for a suitable positive exponent $t$, and (3.10) implies (3.13).

Ad (3.12). We use (3.14) with $\varphi := \eta^2 \partial_\alpha u_\delta$, where $\eta$ is as usual, and get

$$\begin{aligned} &\int_{2B} \eta^2 D^2 F_\delta(\nabla u_\delta)\left(\partial_\alpha \nabla u_\delta, \partial_\alpha \nabla u_\delta\right) \mathrm{d}x \\ &\quad + \; 2 \int_{2B} D^2 F_\delta(\nabla u_\delta)\left(\partial_\alpha \nabla u_\delta \eta, \nabla\eta \partial_\alpha u_\delta\right) \mathrm{d}x \\ &\qquad = \int_{2B} g_\delta \partial_\alpha \left[\eta^2 \partial_\alpha u_\delta\right] \, \mathrm{d}x \, . \end{aligned}$$

To the second integral on the l.h.s. we first apply the Cauchy-Schwarz inequality and then use Young's inequality with the result

$$\begin{aligned}\int_{2B} \eta\omega_\delta^2 \,\mathrm{d}x &\le c\Bigg[\int_{2B} D^2F_\delta(\nabla u_\delta)(\nabla\eta,\nabla\eta)\,|\nabla u_\delta|^2\,\mathrm{d}x \\ &\quad + \int_{2B} |\nabla\eta|\,|\nabla u_\delta|\,\mathrm{d}x + \int_{2B} \eta^2\left|\nabla^2 u_\delta\right|\,\mathrm{d}x\Bigg]\,,\end{aligned}$$

and by (3.10) it remains to discuss the last integral on the r.h.s. But clearly

$$\int_{2B} \eta^2\left|\nabla^2 u_\delta\right|\,\mathrm{d}x \le \varepsilon\int_{2B}\eta^2(1+|\nabla u_\delta|)^{-\mu}\left|\nabla^2 u_\delta\right|^2\mathrm{d}x + c(\varepsilon)\int_{2B}(1+|\nabla u_\delta|)^{\mu}\,\mathrm{d}x\,,$$

and if we choose $\varepsilon$ sufficiently small, recall (1.5) and apply (3.10) in various places, (3.12) follows.

Ad (3.11). Let $\Gamma_\delta := 1+|\nabla u_\delta|^2$. It holds

$$\begin{aligned}\int_B |\nabla\Psi_\delta|^2\,\mathrm{d}x &\le c\int_B \left|\nabla^2 u_\delta\right|^2(1+|\nabla u_\delta|)^{\mu-2}\,\mathrm{d}x \\ &= c\int_B \left(1+|\nabla u_\delta|^2\right)^{-\mu}\left|\nabla^2 u_\delta\right|^2(1+|\nabla u_\delta|)^{2\mu-2}\,\mathrm{d}x \\ &\overset{(1.5)}{\le} c\int_{2B}\eta^2 D^2F_\delta(\nabla u_\delta)(\partial_\alpha\nabla u_\delta,\partial_\alpha\nabla u_\delta)\,\Gamma_\delta^{\mu-1}\,\mathrm{d}x \\ &\overset{(3.14)}{=} c\Bigg[\int_{2B} g_\delta\partial_\alpha\left(\eta^2\partial_\alpha u_\delta\Gamma_\delta^{\mu-1}\right)\mathrm{d}x \\ &\quad -\int_{2B} D^2F_\delta(\nabla u_\delta)\left(\partial_\alpha\nabla u_\delta,\nabla\eta^2\partial_\alpha u_\delta\Gamma_\delta^{\mu-1}\right)\mathrm{d}x \\ &\quad -\int_{2B} D^2F_\delta(\nabla u_\delta)\left(\partial_\alpha\nabla u_\delta,\eta^2\partial_\alpha u_\delta\nabla\Gamma_\delta^{\mu-1}\right)\mathrm{d}x\Bigg] \\ &= c\,[V_1 - V_2 - V_3]\,.\end{aligned}$$

By the Cauchy-Schwarz inequality and Hölder's estimate we have

$$|V_2| \le c\left[\int_{2B}\omega_\delta^2\,\mathrm{d}x\right]^{\frac12}\left[\int_{2B} D^2F_\delta(\nabla u_\delta)(\nabla\eta,\nabla\eta)\,\Gamma_\delta^q\,\mathrm{d}x\right]^{\frac12}$$

for some power $q$, thus we can apply (3.10) and (3.12).

Since

$$V_3 = \frac12\int_{2B} D^2F_\delta(\nabla u_\delta)\left(\nabla|\partial_\alpha u_\delta|^2,\eta^2\nabla\Gamma_\delta^{\mu-1}\right)\mathrm{d}x \ge 0\,,$$

we just neglect this term.

For $V_1$ we use

$$|V_1| \leq c \left[ \int_{2B} |\nabla \eta| \, |\nabla u_\delta| \, \Gamma_\delta^{\mu-1} + \int_{2B} \eta^2 \left| \nabla^2 u_\delta \right| \Gamma_\delta^{\mu-1} \, \mathrm{d}x \right]$$

and observe

$$\int_{2B} \left| \nabla^2 u_\delta \right| \Gamma_\delta^{\mu-1} \, \mathrm{d}x \leq c \left[ \int_{2B} \omega_\delta^2 \, \mathrm{d}x + \int_{2B} \Gamma_\delta^{2\mu-2+\frac{\mu}{2}} \, \mathrm{d}x \right] ,$$

so that we can use (3.10) and (3.12) again. Altogether we have shown

$$\int_{2B} |\nabla \Psi_\delta|^2 \, \mathrm{d}x \leq c(B) < \infty$$

uniform in $\delta$, and since the uniform $L^2$-boundedness of $\Psi_\delta$ is immediate, we arrive at (3.11). This completes the proof of Theorem 1.4. □

# 4 Some related problems

Let $W(\nabla u)$ either denote the logarithmic density $|\nabla u| \ln (1 + |\nabla u|)$ or the $\mu$-elliptic version, i.e. $W(\nabla u) = F(\nabla u)$ with $F$ satisfying (1.4) and (1.5).

i) Inpainting with inner obstacles
Suppose that $D \subset \Omega$ is open satisfying (1.1) and with sufficiently regular boundary. Consider Lipschitz functions $g_1$, $g_2$: $\overline{D} \to \mathbb{R}$ such that $0 \leq g_1(x) < g_2(x) \leq 1$ on $\overline{D}$. For a given measurable function $f$: $\Omega - D \to [0,1]$ and a parameter $\lambda > 0$ we then consider the problem

$$\left. \begin{array}{l} K[u] := \displaystyle\int_\Omega W(\nabla u) \, \mathrm{d}x + \frac{\lambda}{2} \int_{\Omega - D} (u - f)^2 \, \mathrm{d}x \to \min \\ \text{subject to the constraint } g_1 \leq u \leq g_2 \text{ a.e. on } D \, . \end{array} \right\} \tag{4.1}$$

Thus the image is reconstructed under the requirement that on the inpainting region the intensity of the grey level $u(x)$ at $x \in D$ has to satisfy $g_1(x) \leq u(x) \leq g_2(x)$.

ii) Inpainting with global obstacles
Now we are given functions $g_1$, $g_2$: $\Omega \to [0,1]$ such that $g_1(x) < g_2(x)$, and (4.1) is replaced by

$$K[u] \to \min \quad \text{among all } u\colon \Omega \to \mathbb{R} \text{ such that } g_1 \leq u \leq g_2 \text{ a.e. on } \Omega \, . \tag{4.2}$$

We believe that at least for problem (4.2) we have results in the spirit of Theorem 1.1 - 1.4, which in particular means that interior $C^1$-regularity of minimizers probably holds for $g_1$, $g_2$ of class $C^{1,\gamma}(\Omega)$. The details will be the subject of further investigations.

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