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Michael Bildhauer and Martin Fuchs

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On some perturbations of the total variation image inpainting method. Part III: minimization among sets with finite perimeter

Michael Bildhauer

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
bibi@math.uni-sb.de

Martin Fuchs

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

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Abstract

We propose a model for the restoration of images consisting only of completely black or completely white regions with the use of Caccioppoli sets.

The purpose of this short note is to present a technique which might be useful for the restoration of images consisting only of purely black or purely white regions. To be precise, we consider a function $u\colon\ \Omega \to \mathbb{R}$ defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ taking just values in $\{0, 1\}$, which can be seen as a model for the kind of images we have in mind.

Assume that a certain part D of the image is damaged, which means that the observed image is represented through a $\mathcal{L}^2$-measurable function $f\colon\ \Omega - D \to [0, 1]$. Here $\mathcal{L}^2$ stands for Lebesgue's measure in the plane, and D denotes a $\mathcal{L}^2$-measurable subset of Ω with non-empty interior $\mathrm{Int}(D)$ and the property

$$\mathcal{L}^2(D) < \mathcal{L}^2(\Omega)\,. \tag{1}$$

For points $x \in \Omega - D$ the number $f(x)$ is a measure for the intensity of the grey level in the observed image at the point x, and our goal is to restore the missing part $D \to [0, 1]$ of the image from the observed intensity f, where the quality of data fitting is measured through the quantity $\int_{\Omega - D} (u - f)^2 \,\mathrm{d}x$.

Of course our problem is located in the general framework of "image inpainting" discussed under various aspects for example in the papers [ACS], [BHS], [BCMS], [CKS], [CS], [PSS], [Sh] and in the references quoted therein, but one new feature of our analysis might be the requirement

$$u(x) \in \{0, 1\} \ \text{ a.e. on } \Omega\,, \tag{2}$$

which in contrast to our previous investigations (see [BF1], [BF2]) we now impose on the restored image $u\colon\ \Omega \to \mathbb{R}$.

As a suitable method to reconstruct the image we propose to study a TV-like regularization, i.e. we minimize the functional

$$J[u] := \int_\Omega |\nabla u| + \frac{\lambda}{2} \int_{\Omega - D} (u - f)^2 \,\mathrm{d}x \tag{3}$$

on a certain subclass of the space $\mathrm{BV}(\Omega)$ taking care of the constraint (2). In equation (3), $\lambda > 0$ is a free parameter and $\int_\Omega |\nabla u|$ is the total variation of the vector-valued Radon measure ∇u. For a definition of the spave $\mathrm{BV}(\Omega)$ of functions having finite total variation we refer to [AFP] or [Gi].

Clearly our requirement (2) suggests to minimize J among characteristic functions, hence we replace J from (3) through the functional

$$\mathcal{F}[E] := \int_\Omega |\nabla \chi_E| + \frac{\lambda}{2} \int_{\Omega - D} (\chi_E - f)^2 \, \mathrm{d}x \,, \tag{4}$$

E denoting a set of finite perimeter (= Caccioppoli set) in Ω and χ_E its characteristic function (see, e.g., [AFP] or [Gi]).

We recall (compare [Gi], Proposition 3.1) that for a Borel set E there exists a Borel set $\tilde{E}$ equivalent to E, that is, $\tilde{E}$ differs from E only by a set of $\mathcal{L}^2$-measure zero, moreover $\tilde{E}$ has the property

$$0 < \mathcal{L}^2(\tilde{E} \cap B_r(x)) < \pi r^2 \tag{5}$$

for all $x \in \partial \tilde{E}$ and all $r > 0$.

When considering BV-functions, one actually considers equivalence classes of functions being different only on a set of measure zero. In the same spirit, when discussing Caccioppoli sets E, the perimeter and other properties remain unchanged, if we modify E on a set with $\mathcal{L}^2$-measure zero, which means that again we are concerned with equivalence classes and we may therefore assume that (5) holds for any set we consider.

We have the following result:

Theorem 1. *Suppose that D satisfies (1) and consider a $\mathcal{L}^2$-measurable function $f\colon \Omega - D \to [0,1]$.*

i) Then there exists a set E of finite perimeter in Ω such that

$$\mathcal{F}[E] \le \mathcal{F}[G]$$

for any Caccioppoli set $G \subset \Omega$, where $\mathcal{F}$ is defined in (4).

ii) The boundary part $\partial F \cap \Omega$ of any $\mathcal{F}$-minimizer F is a C^1-curve.

iii) If E is a $\mathcal{F}$-minimizing set and if $x \in \partial E$ belongs to the interior of D, then for a suitable disk $B_r(x) \subset \mathrm{Int}(D)$ the intersection $\partial E \cap B_r(x)$ is contained in a straight line.

Remark 1. *From the analytical point of view the smoothness of $\partial E \cap \Omega$ seems to be a nice result but it might be unnatural or too restrictive in the framework of image analysis. We also do not know if Theorem 1 iii) contaims a realistic statement.*

Proof of Theorem 1. Let (E_n) denote a $\mathcal{F}$-minimizing sequence of Caccioppoli sets. Then we have ($u_n := \chi_{E_n}$)

$$\sup_n \left[\int_\Omega |\nabla u_n| + \int_\Omega |u_n| \, \mathrm{d}x \right] < \infty \,, \tag{6}$$

and by BV-compactness ([Gi], Theorem 1.19) inequality (6) implies the existence of $u \in L^1(\Omega)$ such that

$$\tilde{u}_n \to u \quad \text{in } L^1(\Omega) \text{ and a.e. on } \Omega \tag{7}$$

for a subsequence $(\tilde{u}_n)$ of (u_n). From (7) we infer (compare [Gi], Theorem 1.9)

$$\int_\Omega |\nabla u| \leq \liminf_{n\to\infty} \int_\Omega |\nabla \tilde{u}_n| \,, \tag{8}$$

thus $u \in \mathrm{BV}(\Omega)$, and from $\tilde{u}_n \to u$ a.e. it follows $u(x) \in \{0,1\}$ as well as

$$\int_{\Omega - D} (\tilde{u}_n - f)^2 \,\mathrm{d}x \to \int_{\Omega - D} (u - f)^2 \,\mathrm{d}x \,. \tag{9}$$

If we let

$$E := \{x \in \Omega : \ u(x) = 1\} \,,$$

then $u = \chi_E$ and (8), (9) imply the $\mathcal{F}$-minimality of the set E. This proves part $i)$ of Theorem 1.

In order to verify $ii)$ we show that any $\mathcal{F}$-minimizer F is almost minimal in the sense of [Ta]: consider a Caccioppoli set $\tilde{F}$ such that

$$F \Delta \tilde{F} := (F - \tilde{F}) \cup (\tilde{F} - F) \Subset B_r(x)$$

for a disk $B_r(x) \Subset \Omega$. The $\mathcal{F}$-minimality of F then yields (recall $0 \leq f \leq 1$ a.e. on Ω)

$$\begin{aligned} \int_{B_r(x)} |\nabla \chi_F| &\leq \int_{B_r(x)} |\nabla \chi_{\tilde{F}}| + \frac{\lambda}{2} \int_{B_r(x) \cap (\Omega - D)} \Big[(\chi_{\tilde{F}} - f)^2 - (\chi_F - f)^2 \Big] \,\mathrm{d}y \\ &\leq \int_{B_r(x)} |\nabla \chi_{\tilde{F}}| + \frac{\lambda}{2} \mathcal{L}^2(B_r(x)) \\ &= \int_{B_r(x)} |\nabla \chi_F| + \frac{\lambda}{2} \pi r^2 \,, \end{aligned}$$

and we can quote [Ta], Section 1.9, to see that $\partial F \cap \Omega$ is a C^1-curve.

Due to the smoothness of $\partial E \cap \Omega$ for $\mathcal{F}$-minimizing sets E we have ($\mathcal{H}^s$ denoting the Hausdorff measure of dimension s)

$$\int_U |\nabla \chi_E| = \mathcal{H}^1(\partial E \cap U)$$

for any open set $U \Subset \Omega$, and if we choose $U \Subset \mathrm{Int}(D)$, we see that ∂E is a local minimizer of the curve length within the set U, which implies $iii)$ of Theorem 1. $\square$

Extension 1.
We briefly mention another approach towards the restoration of images consisting only

of black and white zones: let $\Phi\colon\ \overline{\Omega}\times\mathbb{R}^2\to[0,\infty)$ denote a parametric integrand, i.e. a continuous function satisfying the homogeneity condition

$$\Phi(x,tp)=t\Phi(x,p)\,,\qquad x\in\overline{\Omega}\,,\ \ p\in\mathbb{R}^2\,,\ \ t\geq 0\,, \tag{10}$$

and being convex in p for each fixed $x\in\overline{\Omega}$. Moreover, we assume the coercivity of Φ in the sense that

$$\Phi(x,p)\geq\nu_1|p| \tag{11}$$

holds for all $x\in\overline{\Omega}$ and $p\in\mathbb{R}^2$ with a suitable constant $\nu_1>0$.

An important example (considered in Theorem 1) is $\Phi(p):=|p|$, alternatively we may look at

$$\Phi(x,p)=\Bigg[\sum_{\alpha,\beta=1}^{2}a_{\alpha\beta}(x)p_\alpha p_\beta\Bigg]^{\frac{1}{2}}$$

with continuous coefficients $a_{\alpha\beta}$ such that

$$\nu_2|p|^2\leq\sum_{\alpha,\beta=1}^{2}a_{\alpha\beta}(x)p_\alpha p_\beta\leq\nu_3|p|^2\,,\qquad x\in\overline{\Omega}\,,\ \ p\in\mathbb{R}^2\,,$$

where ν_2, $\nu_3>0$.

If Φ satisfies (10) and (11), we then replace $\mathcal{F}$ from (4) by

$$\mathcal{G}[E]:=\int_\Omega\Phi(x,\nabla\chi_E)+\frac{\lambda}{2}\int_{\Omega-D}(\chi_E-f)^2\,\mathrm{d}x \tag{12}$$

for Caccioppoli sets $E\subset\Omega$. In (12) we use the following notation (compare [GMS1], [GMS2]): if μ denotes a $\mathbb{R}^2$-valued Radon-measure, we let

$$\int_\Omega\Phi(x,\mu):=\int_\Omega\Phi\Big(x,\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\Big)\,\mathrm{d}|\mu|\,,$$

where $\mathrm{d}\mu/\mathrm{d}|\mu|$ is the Radon-Nikodym derivative of the measure μ w.r.t. the measure $|\mu|$. Note that $\mathrm{d}\mu/\mathrm{d}|\mu|$ is a unit vector $|\mu|$-a.e.

Then we have:

there exists a set of finite perimeter $E\subset\Omega$ such that

$$\mathcal{G}[E]\leq\mathcal{G}[F]$$

for any other set F of finite perimeter.

The proof can be carried out as done in Theorem 1 for the particular case $\Phi(p) = |p|$.

Extension 2.
Suppose that we want to restore an image using a finite number of distinct grey levels, which means that now we consider BV-functions $u \in \mathrm{BV}(\Omega, A)$ taking their values in the set

$$A := \{a_1, \ldots, a_n\}, \quad 0 \le a_1 < a_2 < \cdots < a_n \le 1,$$

with given numbers a_i. We then replace $\mathcal{G}$ from (12) through

$$\overline{\mathcal{G}}[u] := \int_\Omega \Phi(x, \nabla u) + \frac{\lambda}{2} \int_{\Omega - D} (u - f)^2 \, \mathrm{d}x$$

and get:

the problem $\overline{\mathcal{G}} \to \min$ *in* $\mathrm{BV}(\Omega, A)$ *admits at least one solution.*

Extension 3.
It might be of interest to include a "volume constraint" which means to consider the problem $\mathcal{F}[E] \to \min$ among all Caccioppoli sets E in Ω satisfying

$$\mathcal{L}^2(E) = m. \tag{13}$$

Here $\mathcal{F}$ is defined according to (4) and m denotes some fixed number in the interval $(0, \mathcal{L}^2(\Omega))$. The requirement (13) can be understood in the sense that we have to restore the image using a given amount of black color. The existence of a solution is easily established along the lines of the proof of Theorem 1. For a discussion of the analytical and topological properties of minimizing sets subject to the constraint (13) we refer the reader to [Qi]. In particular it follows from Theorem 4.5 of this reference that the boundary of a minimizing set is a smooth curve.

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