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Michael Bildhauer and Martin Fuchs

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Michael Bildhauer

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
bibi@math.uni-sb.de

Martin Fuchs

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

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Michael Bildhauer Martin Fuchs

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Abstract

We discuss different variants of the so-called total variation image inpainting method collecting existence and regularity results related to the proposed techniques.

We start with a description of the problem under consideration: suppose that Ω and D are bounded domains in $\mathbb{R}^2$ having Lipschitz continuous boundaries. Let the closure $\overline{D}$ of D be compactly contained in Ω and suppose that we are given a $\mathcal{L}^2$-measurable function $f\colon \Omega - D \to [0,1]$, where $\mathcal{L}^2$ is Lebesgue's measure in the plane.

In our context $f(x)$ measures the intensity of the grey level at a point $x \in \Omega - D$ of a black and white image in which the region D is missing or damaged in the sense that no data are available.

Our goal is to restore this missing part, which means to find a function $u\colon \Omega \to [0,1]$ representing the undestroyed picture in a sense to be made precise with the help of the given data f, thus we are confronted with an image inpainting problem.

There is a variety of image inpainting techniques established by many prominent authors, without being complete we mention the papers [3-5, 7,8, 12-15, 17, 19, 20] and the references quoted therein.

Here we concentrate on the variational approach involving variational integrals with densities of the form $\Psi(|\nabla u|)$ for a function $\Psi\colon [0,\infty) \to [0,\infty)$ being under our disposal. Popular choices are

$$\Psi_p(|\nabla u|) = |\nabla u|^p \quad \text{with } p \in (1,\infty) \tag{1}$$

and the TV-density

$$\Psi_1(|\nabla u|) = |\nabla u| \ . \tag{2}$$

In the case of (1) one works in the Sobolev space W_p^1 (compare [1] for a definition), and by strict convexity one obtains a unique solution, which turns out to be smooth, i.e. of class C^1.

If (2) is considered, then a reasonable formulation is only possible in the space BV of functions having finite total variation (see, e.g. [18]), we loose uniqueness and in general solutions are rather irregular.

As a compromise between (1) and (2) we propose to study the following family of densities with parameter $\mu \in (1, \infty)$. Let $\Psi(|\nabla u|) = \Phi_\mu(|\nabla u|)$, where

$$\Phi_\mu(t) := \int_0^t \int_0^s (1+r)^{-\mu} \, dr \, ds \, , \qquad t \geq 0 \, . \tag{3}$$

Formula (3) can be replaced by the explicit representation

$$\begin{aligned} \Phi_\mu(t) &= \frac{t}{\mu-1} + \frac{1}{\mu-1}\frac{1}{\mu-2}(t+1)^{-\mu+2} - \frac{1}{\mu-1}\frac{1}{\mu-2}, \ \mu \neq 2 \, , \\ \Phi_2(t) &= t - \ln(1+t) \, , \end{aligned} \tag{4}$$

and from (4) we see that Φ_μ approximates the TV-density in the sense that

$$\lim_{\mu \to \infty} (\mu - 1)\Phi_\mu(t) = t \, , \qquad t \geq 0 \, .$$

Moreover, Φ_μ is of linear growth and the integrand $F_\mu(\xi) := \Phi_\mu(|\xi|), \xi \in \mathbb{R}^2$, is strictly convex, which follows from the condition of μ-ellipticity

$$\nu_0 (1+|\xi|)^{-\mu} |\eta|^2 \leq D^2 F_\mu(\xi)(\eta, \eta) \leq \nu_1 (1+|\xi|)^{-1} |\eta|^2 \, , \qquad \xi, \ \eta \in \mathbb{R}^2 \, , \tag{5}$$

satisfied by F_μ with suitable positive constants ν_0, ν_1.

If we formally let $\mu = 1$ in (3), then we obtain

$$\Phi_1(t) = t \ln(1+t) + \ln(1+t) - t \, ,$$

and our subsequent variational problems have to be formulated in the Orlicz-Sobolev space W_h^1 generated by the function $h(t) := t \ln(1+t), t \geq 0$. As it is shown in [9, 11], this nearly linear growth case is more close to the power growth model (1) with exponent $p > 1$ in the sense that nearly linear growth always leads to smooth solutions.

In what follows we like to discuss image inpainting using variational integrals involving the densities $\Phi_\mu(|\nabla u|)$ with parameter $\mu > 1$. To this purpose we introduce some notation: let $G \subset \mathbb{R}^2$ denote a bounded Lipschitz domain. For functions $w \in \mathrm{BV}(G)$ we let

$$K_\mu[w, G] := \int_G \Phi_\mu(|\nabla w|) := \int_G \Phi_\mu(|\nabla^a w|) \, dx + \frac{1}{\mu-1} |\nabla^s w|(G) \, , \tag{6}$$

where $\nabla w = \nabla^a w \llcorner \mathcal{L}^2 + \nabla^s w$ is the decomposition of the vector measure ∇w in its regular and singular part w.r.t. Lebesgue's measure. The reader should note that in accordance with e.g. [16] this definition is a natural extension of the energy $\int_G \Phi_\mu(|\nabla w|)\, dx$ from the space $W^1_1(G)$ to the class $\mathrm{BV}(G)$.

Let us look at

Approach I. *Inpainting with simultaneous denoising.*

For a parameter $\lambda > 0$ we introduce the variational problem

$$J_\mu[u] := K_\mu[u, \Omega] + \frac{\lambda}{2} \int_{\Omega - D} (u - f)^2\, dx \to \min \ \text{ in } \ \mathrm{BV}(\Omega) \tag{7}$$

with K_μ from (6), which means that we *jointly minimize* the quadratic fidelity term calculated over the complement of the inpainting region D and a "suitable" energy measured on the whole region Ω.

In [9, 10], we showed

Theorem 1. *i) Problem (7) admits at least one solution* $u \in \mathrm{BV}(\Omega)$ *and each solution satisfies* $0 \le u(x) \le 1$ *a.e. on* Ω.

ii) If u *and* $\tilde{u}$ *are* J_μ*-minimizing in* $\mathrm{BV}(\Omega)$*, then* $u = \tilde{u}$ *a.e. on* $\Omega - D$*,* $\nabla^a u = \nabla^a \tilde{u}$ *on* Ω *and* $|\nabla^s u|(\Omega) = |\nabla^s \tilde{u}|(\Omega)$.

iii) It holds $\inf_{W^1_1(\Omega)} J_\mu = \inf_{BV(\Omega)} J_\mu$.

iv) Let $\mathcal{M}$ *denote the set of all* L^1*-cluster points of* J_μ*-minimizing sequences from* $W^1_1(\Omega)$*. Then* $\mathcal{M}$ *coincides with the set of all* $\mathrm{BV}(\Omega)$*-solutions of (7).*

v) For any $u \in \mathcal{M}$ *there is an open set* $D_u \subset D$ *such that* $\mathcal{L}^2(D - D_u) = 0$ *and* $u \in C^{1,\alpha}(D_u)$.

vi) Let $1 < \mu < 2$*. Then (7) admits exactly one minimizer* u *being in addition of class* $W^1_1(\Omega) \cap C^{1,\alpha}(\Omega)$.

In general we can not expect an uniqueness result as stated in vi) above, however we have:

Theorem 2. *i) With the notation from Theorem 1 suppose that there exists* $u \in \mathcal{M}$ *such that* $u \in W^1_1(\Omega)$*. Then it follows that* $\mathcal{M} = \{u\}$.

ii) For u*,* $v \in \mathcal{M}$ *we have the estimate*

$$\|u - v\|_{L^2(\Omega)} = \|u - v\|_{L^2(D)} \le \frac{1}{2\sqrt{\pi}} |\nabla^s(u - v)|(\overline{D})\ .$$

An interesting feature of problem (7) is the unique solvability of the associated dual problem

$$R_\mu[\tau] \to \max \ \text{ in } \ L^\infty(\Omega, \mathbb{R}^2) \ , \tag{8}$$

where

$$R_\mu[\tau] := \inf_{v \in W_1^1(\Omega)} l_\mu(v, \tau) \ , \qquad \tau \in L^\infty(\Omega, \mathbb{R}^2) \ ,$$

with Lagrangian

$$l_\mu(v, \tau) := \int_\Omega \left[\tau : \nabla v - \Phi_\mu^*\left(|\tau|\right)\right] \, dx + \frac{\lambda}{2} \int_{\Omega - D} (v - f)^2 \, dx \ ,$$

where

$$(v, \tau) \in W_1^1(\Omega) \times L^\infty(\Omega, \mathbb{R}^2)$$

and where Φ_μ^* denotes the conjugate function of Φ_μ.

In [10] we showed

Theorem 3. *i) Problem (8) admits a unique solution σ. It holds $\sigma \in W^1_{2,\mathrm{loc}}(D, \mathbb{R}^2)$ as well as $\sigma = DF_\mu(\nabla^a u)$ a.e. on D. Here $F_\mu(\xi) = \Phi_\mu(|\xi|)$ and u is any solution of (7).*

ii) We have the inf-sup *relation*

$$\inf_{W_1^1(\Omega)} J_\mu = \sup_{L^\infty(\Omega, \mathbb{R}^2)} R_\mu \ .$$

A slight modification of Approach I arises if we incorporate a weight function ρ: $\Omega - D \to [0, \infty)$ in the fidelity term, i.e. if we replace (7) by

$$K_\mu[u, \Omega] + \frac{\lambda}{2} \int_{\Omega - D} \rho (u - f)^2 \, dx \to \min \ \text{ in } BV(\Omega) \ . \tag{7*}$$

Depending on the choice of ρ we can hope for results in the spirit of Theorem 1 - Theorem 3. For example, it might be reasonable to concentrate $\rho(x)$ near points x close to ∂D with small values for $\rho(x)$, if we are near to $\partial\Omega$.

Approach II. *We suggest to proceed in two steps*, i.e.

1^{st} step: denoising on $\Omega - D$,
2^{nd} step: inpainting with natural boundary data.

In step 1 we look at the problem

$$K_\mu\left[w, \Omega - \overline{D}\right] + \frac{\lambda}{2} \int_{\Omega - D} (f - w)^2 \, dx \to \min \ \text{ in } \ \mathrm{BV}(\Omega - \overline{D}) \tag{9}$$

and recall (compare [11])

Theorem 4. *Problem (9) admits a unique solution* $u_0 \in \mathrm{BV}(\Omega - \overline{D})$ *satisfying in addition* $0 \le u_0 \le 1$.

In step 2 we then use the solution u_0 as boundary datum in the sense that we introduce the space

$$\mathrm{BV}(\Omega)_{u_0} := \left\{ w \in \mathrm{BV}(\Omega) : \ w = u_0 \ \text{ on } \ \Omega - \overline{D} \right\} .$$

Next we choose a number $\nu \in (1, \infty)$ not necessarily equal to μ and consider the problem

$$K_\nu[w, \Omega] \to \min \ \text{ in } BV(\Omega)_{u_0} . \tag{10}$$

We have

Theorem 5. *Problem (10) has at least one solution in the space* $BV(\Omega)_{u_0}$. *Any solution* u *satisfies* $0 \le u \le 1$. *If the case* $\nu < 3$ *is considered, then we have* $|\nabla u| \in L^\infty_{\mathrm{loc}}(D)$, *i.e.* u *is locally Lipschitz on the inpainting region* D.

Note that the last statement of Theorem 5 follows from Theorem 2.1 in [2], since obviously u is a local minimizer of the energy $K_\nu[\cdot, D]$.

Approach III. *Inpainting via a limit procedure.*

We like to reconstruct our image by letting u: $\Omega \to [0, 1]$ with

$$u = \begin{cases} f & \text{on} \quad \Omega - D \\ v & \text{on} \quad D \end{cases}$$

for a reasonable function v: $D \to [0, 1]$.

If f has a trace on ∂D, then v might be obtained by solving a suitable boundary value or minimization problem on D. However, for an observed image (with noise) we just may assume $f \in L^\infty(\Omega - D)$ and therefore we suggest to proceed in the following way.

For $\varepsilon > 0$ sufficiently small let

$$D_\varepsilon := \left\{ x \in \Omega : \ \mathrm{dist}\left(x, \overline{D}\right) < \varepsilon \right\}$$

and consider the variational problem similar to (7) (replace Ω by D_ε in (7) and choose $\lambda = \lambda_\varepsilon$)

$$K_\mu\left[w, D_\varepsilon\right] + \lambda_\varepsilon \int_{D_\varepsilon - D} (f - w)^2 \, dx \to \min \ \text{ in } \ \mathrm{BV}(D_\varepsilon) \, , \tag{11}$$

where $\lambda_\varepsilon := \mathcal{L}^2\left(D_\varepsilon - D\right)^{-1}$.

From Theorem 1 we deduce the existence of a solution $u_\varepsilon \in BV(D_\varepsilon)$ to problem (11) which in addition satisfies

$$0 \le u_\varepsilon \le 1 \text{ on } D_\varepsilon\,, \qquad \sup_\varepsilon |\nabla u_\varepsilon|(D_\varepsilon) < \infty\,,$$

thus we find $v \in \mathrm{BV}(D)$ such that $0 \le v \le 1$ and $u_\varepsilon \to v$ in $L^1(D)$.

Now v seems to be a reasonable candidate in the previous definition of u. We note that clearly $v = a$ in D in case that $f = a$ on $D_{\varepsilon_0} - D$ for some $\varepsilon_0 > 0$ and a number $a \in [0,1]$, since then $u_\varepsilon \equiv a$ on D_ε for all $\varepsilon \le \varepsilon_0$. In general it holds

Theorem 6. *Any function $v \in \mathrm{BV}(D)$ obtained by the above limit procedure is a local $K_\mu[\cdot, D]$-minimizer in $\mathrm{BV}(D)$ and thereby locally Lipschitz, if the case $\mu \in (1,3)$ is considered.*

Proof of Theorem 6. The second claim is a consequence of Theorem 2.1 in [2].

In order to establish the first statement consider $v \in \mathrm{BV}(D)$ such that $u_n \to v$ in $L^1(D)$ for a sequence $u_n := u_{\varepsilon_n}$ of solutions to problem (11) with parameter $\varepsilon_n \to 0$.

Given $w \in \mathrm{BV}(D)$ such that $C := \mathrm{spt}(v - w)$ is a compact subset of D we have to show that

$$K_\mu[v, D] \le K_\mu[w, D] \tag{12}$$

is true. Let us choose a smooth region G such that $C \subset G \Subset D$ and with the additional property

$$|\nabla u_n|\,(\partial G) = 0\,, \qquad |\nabla v|\,(\partial G) = 0 \tag{13}$$

for any $n \in \mathbb{N}$. In order to construct such a region G we may choose a sufficiently regular function η: $\mathbb{R}^2 \to \mathbb{R}$ such that for $t \in [0, \delta]$ the sets $G_t := \{x \in \mathbb{R}^2 : \eta(x) < t\}$ are smooth domains (with boundaries $\partial G_t = \{x \in \mathbb{R}^2 : \eta(x) = t\}$) such that $C \subset G_t \Subset D$. Let $M := \{(x,t) \in \mathbb{R}^2 \times [0,\delta] : \eta(x) = t\}$ with sections M_x and M_t, respectively. From Fubini's theorem it follows for any Radon measure ρ on $\mathbb{R}^2$

$$\begin{aligned}\int_0^\delta \rho\,(\partial G_t)\, d\mathcal{L}^1(t) &= \int_0^\delta \rho\,(M_t)\, d\mathcal{L}^1(t)\\ &= \int_{\mathbb{R}^2} \mathcal{L}^1\,(M_x)\, d\rho(x)\\ &= \int_{\mathbb{R}^2} \mathcal{L}^1\,(\{\eta(x)\})\, d\rho(x) = 0\,,\end{aligned}$$

hence $\rho\,(\partial G_t) = 0$ for $\mathcal{L}^1$-almost all $t \in [0,\delta]$. Applying this result to the measures $\rho = |\nabla u_n|$, $|\nabla v|$, $n \in \mathbb{N}$, we see that (13) holds for $G := G_t$ and almost all $t \in [0,\delta]$.

The reader should note that according to [18], 2.13 Remark, G has been selected in such a way that the traces of each u_n and also of v from inside and from outside coincide on ∂G.

Next we let

$$w_n := \left\{ \begin{array}{ccc} w & \text{on} & G \\ u_n & \text{on} & D_{\varepsilon_n} - G \end{array} \right\} \in \mathrm{BV}(D_{\varepsilon_n})$$

and obtain from the minimizing property of u_n

$$\int_{D_{\varepsilon_n}} \Phi_\mu \left(|\nabla u_n|\right) \leq \int_{D_{\varepsilon_n}} \Phi_\mu \left(|\nabla w_n|\right) . \tag{14}$$

On the open set $D_{\varepsilon_n} - \overline{G}$ it holds $\nabla w_n = \nabla u_n$ (as measures), thus (14) implies

$$\int_{\overline{G}} \Phi_\mu \left(|\nabla u_n|\right) \leq \int_{\overline{G}} \Phi_\mu \left(|\nabla w_n|\right) . \tag{15}$$

To proceed we recall the $L^1(D)$-convergence $u_n \to v$ which implies $L^1(\partial G_t)$-convergence of the traces (at least for a subsequence) for $\mathcal{L}^1$-almost all $t \in [0, \delta]$.

We assume that this condition is satisfied for our choice $G = G_t$. We claim the validity of

$$\lim_{n\to\infty} \int_{\partial G} |\nabla w_n| = 0 . \tag{16}$$

In order to justify (16) we let

$$\tilde{w} := \left\{ \begin{array}{ccc} w & \text{on} & G \\ 0 & \text{on} & D - G \end{array} \right\} , \quad \tilde{u}_n := \left\{ \begin{array}{ccc} 0 & \text{on} & G \\ u_n & \text{on} & D - G \end{array} \right\} \in \mathrm{BV}(D)$$

and quote [6], Corollary 3.89, p. 183: according to this reference we have the formula

$$\nabla w_n = \nabla \tilde{w} + \nabla \tilde{u}_n + (w_{|\partial G} - u_{n|\partial G}) \nu_{\partial G} \mathcal{H}^1 \llcorner \partial G$$

for the total variation measure ∇w_n on the domain D, where $w_{|\partial G}$ $(= v_{|\partial G})$ and $u_{n|\partial G}$ denote the traces of the corresponding functions (recall the choice of t). From (13) and the above representation we get as $n \to \infty$ (recalling also $|\nabla w|(\partial G) = |\nabla v|(\partial G)$)

$$\int_{\partial G} |\nabla w_n| \leq \int_{\partial G} \left| w_{|\partial G} - u_{n|\partial G} \right| d\mathcal{H}^1 = \int_{\partial G} \left| v_{|\partial G} - u_{n|\partial G} \right| d\mathcal{H}^1 \to 0 .$$

This implies (16) and thereby

$$\lim_{n\to\infty} \int_{\partial G} \Phi_\mu(|\nabla w_n|) = 0 . \tag{17}$$

We have quoting (13)

$$\int_{\overline{G}} \Phi_\mu(|\nabla u_n|) = \int_G \Phi_\mu(|\nabla u_n|) \tag{18}$$

and by lower-semicontinuity it holds

$$\int_G \Phi_\mu(|\nabla v|) \le \liminf_{n\to\infty} \int_G \Phi_\mu(|\nabla u_n|) \ . \tag{19}$$

If we write

$$\int_{\overline{G}} \Phi_\mu(|\nabla w_n|) = \int_G \Phi_\mu(|\nabla w|) + \int_{\partial G} \Phi(|\nabla w_n|) \ ,$$

then we deduce from (15) and (17)-(19) the inequality $K_\mu[v,G] \le K_\mu[w,G]$ which gives (12) by the choice of G. $\square$

Remark 1. *We strongly suggest to compare our proposals I - III for concrete images and for different choices of the parameter μ, e.g. for μ close to 1 and for μ being very large.*

Remark 2. *As a matter of fact our results extend to any μ-elliptic linear growth integrand $F = F(\nabla u)$, where the notion of μ-ellipticity is defined according to (5).*

References

[1] R.A. Adams, *Sobolev spaces*, Academic Press, New York-London, 1975, Pure and Applied Mathematics, Vol. 65.

[2] D. Apushkinskaya, M. Bildhauer and M. Fuchs, *On local generalized minimizers and local stress tensors for variational problems with linear growth*, J. of Math. Sciences, Vol. 165, No.1 (2010), 39–54.

[3] P. Arias, V. Caselles, G. Facciolo, V. Lazcano and R. Sadek, *Nonlocal variational models for inpainting and interpolation* Math. Models Methods Appl. Sci. 22 (2012), no. suppl. 2.

[4] P. Arias, V. Casseles and G. Sapiro, *A variational framework for non-local image inpainting*. IMA Preprint Series No. 2265 (2009).

[5] P. Arias, G. Facciolo, V. Casseles and G. Sapiro, *A variational framework for exemplar-based image inpainting*, Int. J. Comput. Vis. 93 (2011), no. 3, 319–347.

[6] Ambrosio, L., Fusco, N., Pallara, D., Functions of bounded variation and free discontinuity problems. Oxford Science Publications, Clarendon Press, Oxford (2000).

[7] G.Aubert and P. Kornprobst, *Mathematical problems in image processing*, Applied Mathematical Sciences, Vol. 147, Springer-Verlag, New York, 2002.

[8] M. Bertalmio, G. Sapiro, V, Caselles and C. Ballester, *Image inpainting*, Proceedings of the 27th annual conference on Computer graphics and interactive techniques ACM press/Addison-Wesley Publishing Co. (2000), 417–424.

[9] M. Bildhauer and M. Fuchs, *On some perturbations of the total variation image inpainting method. part 1: Regularity theory*, Preprint Nr. 328 Saarland University.

[10] M. Bildhauer and M. Fuchs, *On some perturbations of the total variation image inpainting method. part 2: Relaxation and dual variational formulation*, Preprint Nr. 332 Saarland University.

[11] M. Bildhauer and M. Fuchs, *A variational approach to the denoising of images based on different variants of the TV-regularization*, Appl. Math. Optim. 66 (2012), no. 3, 331–361.

[12] M. Burger, L. He and C.-B. Schönlieb, *Cahn-Hilliard inpainting and a generalization for grayvalue images* SIAM J. Imaging Sci 2 (2009), no. 4, 1129–1167.

[13] T.F. Chan, S.H. Kang and J. Shen, *Euler's elastica and curvature based inpaintings*, SIAM J. Appl. Math. 63 (2002), no. 2, 564–592.

[14] T.F. Chan and J. Shen, *Nontexture inpainting by curvature-driven diffusions*, Journal of Visual Communication and Image Representation 12 (2001), No. 4, 436–449.

[15] T.F. Chan and J. Shen, *Mathematical models for local nontexture inpaintings*, SIAM J. Appl. Math. 62 (2001/02), no. 3, 1019–1043.

[16] Demengel, F., Temam, R., Convex functions of a measure and applications. Ind. Univ. Math. J. 33, 673–709 (1984).

[17] S. Esedoglu and J. Shen, *Digital inpainting based on the Mumford-Shah-Euler image model*, European Journal of Applied Mathematics 13 (2002), no. 4, 353–370.

[18] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, Vol. 80, Birkhäuser Verlag, Basel, 1984.

[19] K. Papafitsoros, B. Sengul and C.-B. Schönlieb, *Combined first and second order total variation impainting using split Bregman*, IPOL Preprint (2012).

[20] J. Shen, J. *Inpainting and the fundamental problem of image processing*, SIAM News 36, No. 5 (2003), 1–4.