Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 339

On a class of variational problems with linear growth related to image inpainting

Michael Bildhauer, Martin Fuchs and Christian Tietz

Saarbrücken 2013

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 339
submitted: December 2, 2013

On a class of variational problems with linear growth related to image inpainting

Michael Bildhauer
Saarland University
Department of Mathematics
P.O. Box 15 11 50
D–66041 Saarbrücken
Germany
bibi@math.uni-sb.de

Martin Fuchs
Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Christian Tietz
Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
tietz@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax:	+ 49 681 302 4443
e-Mail:	preprint@math.uni-sb.de
WWW:	http://www.math.uni-sb.de/

AMS Subject Classification: 49 N 60, 49 Q 20

Keywords: image inpainting, variational method, TV-regularization

Abstract

In this article, we investigate a modification of the total variation image inpainting method and improve the partial regularity results previously established in [8] to $C^{1,\alpha}$ interior differentiability of solutions of this new variational problem using De Giorgi type arguments.

1 Introduction

Suppose we are given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, e.g. a rectangle, a subset D of Ω which is assumed to be measurable with ($\mathcal{L}^2$ denoting Lebesgue's measure on $\mathbb{R}^2$)

$$0 < \mathcal{L}^2(D) < \mathcal{L}^2(\Omega) \tag{1.1}$$

and an observed black and white image described through a measurable function $f : \Omega - D \to [0,1]$, where $f(x)$ is the intensity of the grey level at $x \in \Omega - D$.

Roughly speaking, the region D which is also called “inpainting domain“ (see [10]), represents a certain part of this image for which image data are missing or inaccessible. Our goal is to restore this missing part from the part which is known. In the image processing community, this kind of image interpolation is called “inpainting“ respectively “image inpainting“ (compare [10, 19, 20]).

At this point, we like to add some general comments concerning inpainting: we are concerned with the attempt to recover the original image in terms of a function $u : \Omega \to [0,1]$ which measures the intensity of the grey level at $x \in \Omega$ on the whole domain Ω based on the partial observation $f : \Omega - D \to \mathbb{R}$ which is usually corrupted by noise stemming from transmission or measuring errors. Due to [19], there are essentially four different methods to handle the inpainting problem, depending on being variational or non-variational and local or non-local. Local inpainting methods take the information that is needed to fill in the inpainting domain D only from neighboring points of the boundary of D (compare [19]). In the case that the inpainting domain is quite small, these methods seem to be more desirable (compare [6, 12, 14, 13, 15, 19]).
Using non-local inpainting methods means that all the information of the known

part of the image is taken into account and the information is weighted by its distance to the point that is to be filled in (compare [19]). Although these methods are suitable to fill in structures and textures, there are also several disadvantages as for instance the high computational costs arising in their numerical solution (compare [2, 19]).

We here concentrate on a TV-like variational approach being of non-local type which leads to the minimization of a functional of the type

$$I[u] := \int_\Omega \psi(|\nabla u|)dx + \frac{\lambda}{2} \int_{\Omega - D} (u - f)^2 dx. \tag{1.2}$$

Here, λ is a positive regularization parameter and ψ is supposed to be a convex and increasing function with non-negative values.
The second term on the right-hand side of (1.2) measures the quality of data fitting, i.e. the deviation of the original image u from the given data on $\Omega - D$ while the first term produces a kind of mollification and allows to incorporate some kind of a priori information of the generated image on the entire domain Ω into the minimization process.

In this setting, a common choice of ψ is $\psi(|\nabla u|) := |\nabla u|$. This leads to the total variation inpainting model (compare [3, 19]). To discuss this variational problem, one has to work with functions $\Omega \to \mathbb{R}$ of bounded variation, i.e. in the space $BV(\Omega)$. In this situation, ∇u denotes the distributional gradient which is respresented by a vector valued Radon measure on Ω with finite total variation $\int_\Omega |\nabla u|$ (for details, we refer to [16]).

In the papers [8],[9] and also in the related work [7] the basic idea was to replace the TV-density $\psi(|\nabla u|) = |\nabla u|$ through a family of densities being still of linear growth but with better ellipticity properties leading to appropriate regularity results for the corresponding minimizers.

As in the papers [8] and [9], we introduce the energy

$$I[w] := \int_\Omega F(\nabla w)dx + \frac{\lambda}{2} \int_{\Omega - D} (w - f)^2 dx. \tag{1.3}$$

for functions w from the Sobolev space $W^{1,1}(\Omega)$ (for details concerning this space, we refer to [1]) where $F : \mathbb{R}^2 \to [0, \infty)$ is a μ-elliptic density of linear growth satisfying $F \in C^2$, $F(0) = 0$ and $DF(0) = 0$. More precisely, we impose the

following conditions on F:
there exist positive constants ν_1, ν_2, ν_3 and a real number $\mu > 1$ such that for any $Y, Z \in \mathbb{R}^2$ we have

$$|DF(Z)| \leq \nu_1 \tag{1.4}$$

and

$$\nu_2 \frac{1}{(1+|Z|)^\mu}|Y|^2 \leq D^2F(Z)(Y,Y) \leq \nu_3 \frac{1}{1+|Z|}|Y|^2. \tag{1.5}$$

Based on these hypotheses, we can state some useful properties of F which have been established on p. 97/98 in [11] for instance.

Lemma 1.1
Suppose that F satisfies (1.4) and (1.5) for some number $\mu > 1$. Then F is strictly convex on $\mathbb{R}^2$ and it holds

(i) There are real constants $\nu_4 > 0, \nu_5 \in \mathbb{R}$ such that for all $Z \in \mathbb{R}^2$ we have

$$DF(Z) : Z \geq \nu_4|Z| - \nu_5.$$

(ii) F is of linear growth in the sense that for real numbers $\nu_6, \nu_7 > 0, \nu_8, \nu_9 \in \mathbb{R}$ and for all $Z \in \mathbb{R}^2$ it holds

$$\nu_6|Z| - \nu_8 \leq F(Z) \leq \nu_7|Z| + \nu_9. \tag{1.6}$$

(iii) The integrand automatically satisfies a balancing condition: there exists a real constant $\nu_{10} > 0$ such that

$$|D^2F(Z)||Z|^2 \leq \nu_{10}(1 + F(Z))$$

for all $Z \in \mathbb{R}^2$.

In this context, the minimal surface integrand given by $F(Z) := \sqrt{1+|Z|^2}$ serves as the most prominent example for which we have (1.4) and (1.5). It remains to be said that (1.5) holds here for the optimal choice $\mu = 3$.

Furthermore as outlined in [7], another explicit example of a density F satisfying the above hypothesis exactly with a given value $\mu > 1$ is generated by the function

$$\Phi_\mu(t) := \int_0^t \int_0^s (1+r)^{-\mu} dr\, ds,\ t \geq 0,$$

if we define

$$F_\mu(Z) := \Phi_\mu(|Z|),\ Z \in \mathbb{R}^2.$$

Note that $(\mu-1)F_\mu(Z) \to |Z|$ as $\mu \to \infty$, which follows from the formula

$$\Phi_\mu(t) = \frac{t}{\mu-1} + \frac{1}{(\mu-1)(\mu-2)}(t+1)^{-\mu+2} - \frac{1}{(\mu-1)(\mu-2)}, \quad \mu \neq 2 \qquad (1.7)$$

whereas

$$\Phi_2(t) = t - \ln(1+t). \qquad (1.8)$$

With respect to the explicit formulas (1.7) and (1.8), the density $F_\mu(\nabla u)$ serves as a good candidate for an approximation of $|\nabla u|$ by more regular integrands of linear growth.

In this note we like to improve the regularity results concerning interior differentiability of minimizers of the functional I with μ-elliptic density F. Up to this point, the following theorem has been proven in [8]:

Theorem 1.2
Let (1.1) *hold and define the energy* I *according to* (1.3) *with* F *satisfying* (1.4) *and* (1.5) *for some* $\mu \in (1,2)$. *Then we have:*

(i) the problem $I \to$ *min admits an unique solution* u *in the space* $W^{1,1}(\Omega)$*;*

(ii) the solution satisfies $0 \le u \le 1$ *a.e. on* Ω*;*

(iii) it holds $u \in W^{1,p}_{loc}(\Omega)$ *for any finite* p*, hence* u *is Hölder continuous in the interior of* Ω *for any exponent* < 1*;*

(iv) there is an open subset Ω_0 *of* Ω *such that* $dim_{\mathcal{H}}(\Omega - \Omega_0) = 0$ *and* $u \in C^{1,\beta}(\Omega_0)$ *for any* $\beta < 1$*;*

(v) if D *is an open set, then* $D \subset \Omega_0$*, i.e.* $u \in C^{1,\alpha}(D)$ *for any* $\alpha \in (0,1)$*. For arbitrary sets* D *we have* $Int(D) \subset \Omega_0$*, where* $Int(D)$ *is the set of interior points of* D*.*

Remark 1.3
Recall that $dim_{\mathcal{H}}(\Omega - \Omega_0) = 0$ *by definiton means* $\mathcal{H}^\varepsilon(\Omega - \Omega_0) = 0$ *for any* $\varepsilon > 0$ *(*$\mathcal{H}^\varepsilon$ *denoting the Hausdorff-measure of dimension* ε*).*

Now we want to show that actually interior singularities can be excluded, more precisely we have the following substantial improvement of Theorem 1.2 (iv):

Theorem 1.4
Suppose that (1.1), (1.4) *and* (1.5) *hold together with* $\mu \in (1,2)$. *Then we have* $u \in C^{1,\alpha}(\Omega)$ *for any* $0 < \alpha < 1$ *where* u *is the solution from Theorem 1.2.*

Remark 1.5
We emphasize that it is easy to check that Theorem 1.2 and Theorem 1.4 also hold in the case $D = \emptyset$ *("pure denoising of* f*").*

The rest of the paper is organized as follows:
In Section 2, we are going to prove Theorem 1.4 using De Giorgi type arguments. As a technical tool we will make use of the approximation lemma 2.1 (proven in [8]) where the original variational problem is replaced by a sequence of more regular problems with smooth solutions u_δ. By means of this lemma we are going to prove in Lemma 2.4 that ∇u_δ is locally uniformly bounded w.r.t. δ, thus u_δ is locally Lipschitz continuous uniformly in δ implying the local Lipschitz regularity of u. From this fact we deduce (in the case $\mu \in (1,2)$) that u has locally Hölder continuous first partial derivatives in Ω using standard results about elliptic partial differential equations of second order. In the Appendix we sketch the proof of a technical lemma needed in Section 2.

2 Proof of Theorem 1.4

We start with the following approximation lemma:

Lemma 2.1
Suppose that we have (1.1), (1.4) *and* (1.5) *for some* $\mu > 1$. *For* $\delta > 0$ *we let*

$$I_\delta[w] := \int\limits_\Omega F_\delta(\nabla w)dx + \frac{\lambda}{2}\int\limits_{\Omega - D}(w-f)^2 dx,$$

$$F_\delta(Z) := \frac{\delta}{2}|Z|^2 + F(Z), Z \in \mathbb{R}^2.$$

Then it holds:

(i) The problem $I_\delta \to \min$ *in* $W^{1,2}(\Omega)$ *admits an unique solution* u_δ *and we have* $0 \le u_\delta \le 1$ *a.e. in* Ω *as well as* $u_\delta \in W^{2,2}_{loc}(\Omega)$.

In particular, for $\mu \in (1,2)$ we have

(ii) $I_\delta[u_\delta] \to I[u]$, $\delta \int\limits_\Omega |\nabla u_\delta|^2 dx \to 0$, $u_\delta \to u$ *in* $L^1_{loc}(\Omega)$ *as* $\delta \to 0$.

(iiii) $\nabla u_\delta \in L^p_{loc}(\Omega, \mathbb{R}^2)$ *uniformly in δ for any finite p, i.e.*

$$\sup_\delta \int\limits_{\Omega'} |\nabla u_\delta|^p dx = c(p, \Omega') < \infty \tag{2.1}$$

where $\Omega' \Subset \Omega$. As a consequence we also have $\nabla u_\delta \rightharpoonup \nabla u$ in $L^p_{loc}(\Omega, \mathbb{R}^2)$ as $\delta \to 0$ for all $p < \infty$.

Proof. The lemma has been established in [8], Section 3. □

Remark 2.2
We emphasize that the condition $\mu < 2$ is needed to get the local uniform p-integrability of ∇u_δ for any finite exponent p.

First, we like to introduce some notation. We fix a point $x_0 \in \Omega$ and consider radii $0 < r < R < R_0$ with $B_{R_0}(x_0) \Subset \Omega$.
Moreover, we let

$$\Gamma_\delta := 1 + |\nabla u_\delta|^2, \; A_\delta(k, R) := \{x \in B_R(x_0) : \Gamma_\delta > k\}, \; k > 0,$$

and consider $\eta \in C_0^\infty(B_R(x_0))$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(x_0)$ and $|\nabla \eta| \le \frac{c}{R-r}$.
Finally, for functions $v : \Omega \to \mathbb{R}$ we denote $\max\{v, 0\}$ by v^+.

At first we are going to establish a Caccioppoli-type inequality. Note that this inequality is not depending on $\mu > 1$.

Lemma 2.3
With the previous notation and under the assumption of Lemma 2.1 (i), in par-

ticular for any $\mu > 1$, we have the following variant of Caccioppoli's inequality

$$\begin{aligned}
\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{-\mu}{2}} |\nabla \Gamma_\delta|^2 \eta^2 dx \leq c & \int\limits_{A_\delta(k,R)} |D^2 F_\delta(\nabla u_\delta)||\nabla\eta|^2 (\Gamma_\delta - k)^2 dx \\
& + c \int\limits_{A_\delta(k,R)} \eta^2 |\nabla u_\delta|^{2+\mu} dx + \int\limits_{A_\delta(k,R)} \eta |\nabla \eta||\nabla u_\delta|^3 dx \\
& \leq \frac{c}{(R-r)^2} \int\limits_{A_\delta(k,R)} |D^2 F_\delta(\nabla u_\delta)|(\Gamma_\delta - k)^2 dx \\
& + \frac{c}{R-r} \int\limits_{A_\delta(k,R)} \eta |\nabla u_\delta|^{2+\mu} dx
\end{aligned} \tag{2.2}$$

for a suitable positive constant c independent of δ, r and R.

Proof. We note that the second inequality follows from the first since w.l.o.g. we may assume $R_0 \leq 1$ and $k \geq 2$. To prove (2.2) we observe that u_δ is solution of the Euler equation

$$\int\limits_\Omega DF_\delta(\nabla u_\delta)\nabla\varphi dx = - \int\limits_{\Omega - D} \lambda(u_\delta - f)\varphi dx$$

for all $\varphi \in C_0^\infty(\Omega)$.
Setting $\varphi = \partial_\alpha \psi$, $\alpha \in \{1,2\}$, with $\psi \in C_0^\infty(\Omega)$ we have

$$\int\limits_\Omega DF_\delta(\nabla u_\delta)\partial_\alpha \nabla\psi dx = - \int\limits_{\Omega - D} \lambda(u_\delta - f)\partial_\alpha\psi dx$$

for all $\psi \in C_0^\infty(\Omega)$.
Since $u_\delta \in W^{2,2}_{\mathrm{loc}}(\Omega)$ (compare Lemma 2.1,(i)) and thereby $DF_\delta(\nabla u_\delta) \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$, an integration by parts leads to

$$\int\limits_{B_R(x_0)} D^2 F_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \nabla\psi) dx = \int\limits_{B_R(x_0) - D} \lambda(u_\delta - f)\partial_\alpha \psi dx.$$

for all $\psi \in W_0^{1,2}(\Omega)$ with compact support in Ω.
Observing that $\psi = \eta^2 \partial_\alpha u_\delta (\Gamma_\delta - k)^+$ is admissible we get (from now on summation

w.r.t. $\alpha \in \{1,2\}$)

$$
\begin{aligned}
&\int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha \nabla u_\delta)(\Gamma_\delta - k)\eta^2 dx \\
&+ \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha u \nabla \Gamma_\delta)\eta^2 dx \\
&+ 2 \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha u \nabla \eta)\eta(\Gamma_\delta - k) dx \\
&= \int\limits_{B_R(x_0)-D} \lambda(u_\delta - f)\partial_\alpha[\eta^2 \partial_\alpha u_\delta (\Gamma_\delta - k)^+] dx.
\end{aligned} \tag{2.3}
$$

For the second integral on the l.h.s. it holds

$$
\begin{aligned}
&\int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha u \nabla \Gamma_\delta)\eta^2 dx \\
&= \frac{1}{2} \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \Gamma_\delta, \nabla \Gamma_\delta)\eta^2 dx.
\end{aligned} \tag{2.4}
$$

In accordance with (2.4) we also have for the third integral on the l.h.s. of (2.3)

$$
\begin{aligned}
&\int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha u \nabla \eta)\eta(\Gamma_\delta - k) dx \\
&= \frac{1}{2} \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \eta, \nabla \Gamma_\delta)\eta(\Gamma_\delta - k) dx.
\end{aligned} \tag{2.5}
$$

Summarizing, (2.3)-(2.5) imply with the Cauchy-Schwarz inequality applied to the bilinear form $D^2F_\delta(\nabla u_\delta)$ and with Young's inequality ($\varepsilon > 0$)

$$\begin{aligned}
&\int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\alpha \nabla u_\delta, \partial_\alpha \nabla u_\delta)(\Gamma_\delta - k)\eta^2 dx \\
&+ \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \Gamma_\delta, \nabla \Gamma_\delta)\eta^2 dx \\
&\leq \varepsilon \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \Gamma_\delta, \nabla \Gamma_\delta)\eta^2 dx \\
&+ \frac{1}{\varepsilon} \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \eta, \nabla \eta)(\Gamma_\delta - k)^2 dx \\
&+ \int\limits_{B_R(x_0)-D} \lambda(u_\delta - f)\partial_\alpha[\eta^2 \partial_\alpha u_\delta (\Gamma_\delta - k)^+] dx.
\end{aligned} \tag{2.6}$$

In what follows we concentrate on the last integral on the r.h.s. of (2.6). We set

$$I_1 := \int\limits_{B_R(x_0)-D} \lambda(u_\delta - f)\partial_\alpha[\eta^2 \partial_\alpha u_\delta (\Gamma_\delta - k)^+] dx$$

and get by recalling $0 \leq u_\delta, f \leq 1$ a.e.

$$\begin{aligned}
I_1 \leq c &\int\limits_{A_\delta(k,R)} |\nabla \eta| \eta |\nabla u_\delta| (\Gamma_\delta - k) dx + c \int\limits_{A_\delta(k,R)} \eta^2 |\nabla^2 u_\delta| (\Gamma_\delta - k) dx \\
+ c &\int\limits_{A_\delta(k,R)} \eta^2 |\nabla u_\delta| |\nabla \Gamma_\delta| dx.
\end{aligned} \tag{2.7}$$

Another application of Young's inequality ($\varepsilon > 0$) gives

$$\begin{aligned}
\int\limits_{A_\delta(k,R)} \eta^2 |\nabla^2 u_\delta| (\Gamma_\delta - k) dx \leq \varepsilon &\int\limits_{A_\delta(k,R)} \eta^2 |\nabla^2 u_\delta|^2 (\Gamma_\delta - k) \Gamma_\delta^{\frac{-\mu}{2}} dx \\
+ \frac{1}{\varepsilon} &\int\limits_{A_\delta(k,R)} \eta^2 (\Gamma_\delta - k) \Gamma_\delta^{\frac{\mu}{2}} dx
\end{aligned} \tag{2.8}$$

as well as

$$\begin{aligned}
\int\limits_{A_\delta(k,R)} \eta^2 |\nabla u_\delta| |\nabla \Gamma_\delta| dx \leq \varepsilon &\int\limits_{A_\delta(k,R)} \eta^2 |\nabla \Gamma_\delta|^2 \Gamma_\delta^{\frac{-\mu}{2}} dx \\
+ \frac{1}{\varepsilon} &\int\limits_{A_\delta(k,R)} \eta^2 |\nabla u_\delta|^2 \Gamma_\delta^{\frac{\mu}{2}} dx.
\end{aligned} \tag{2.9}$$

Choosing $k \geq 2$ w.l.o.g. we have $|\nabla u_\delta| \geq 1$ on $A_\delta(k,R)$ and therefore $\Gamma_\delta \leq c|\nabla u_\delta|^2$ on $A_\delta(k,R)$. It follows by incorporating (2.8) and (2.9) in (2.7)

$$\begin{aligned} I_1 \leq c \int\limits_{A_\delta(k,R)} |\nabla\eta|\eta|\nabla u_\delta|^3 dx + c\varepsilon \int\limits_{A_\delta(k,R)} \eta^2|\nabla^2 u_\delta|^2(\Gamma_\delta - k)\Gamma_\delta^{\frac{-\mu}{2}}\, dx \\ + c\varepsilon \int\limits_{A_\delta(k,R)} \eta^2|\nabla\Gamma_\delta|^2\Gamma_\delta^{\frac{-\mu}{2}}\, dx + \frac{c}{\varepsilon} \int\limits_{A_\delta(k,R)} \eta^2|\nabla u_\delta|^{2+\mu} dx. \end{aligned} \tag{2.10}$$

Connecting (2.10) with (2.6) we get

$$\begin{aligned} &\int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\alpha\nabla u_\delta, \partial_\alpha\nabla u_\delta)(\Gamma_\delta - k)\eta^2 dx \\ &+ \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla\Gamma_\delta, \nabla\Gamma_\delta)\eta^2 dx \\ &\leq \varepsilon \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla\Gamma_\delta, \nabla\Gamma_\delta)\eta^2 dx \\ &+ \frac{1}{\varepsilon} \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla\eta, \nabla\eta)(\Gamma_\delta - k)^2 dx \\ &+ c \int\limits_{A_\delta(k,R)} |\nabla\eta|\eta|\nabla u_\delta|^3 dx + c\varepsilon \int\limits_{A_\delta(k,R)} \eta^2|\nabla^2 u_\delta|^2(\Gamma_\delta - k)\Gamma_\delta^{\frac{-\mu}{2}}\, dx \\ &+ c\varepsilon \int\limits_{A_\delta(k,R)} \eta^2|\nabla\Gamma_\delta|^2\Gamma_\delta^{\frac{-\mu}{2}}\, dx + \frac{c}{\varepsilon} \int\limits_{A_\delta(k,R)} \eta^2|\nabla u_\delta|^{2+\mu} dx. \end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small and using (1.5) it follows

$$\begin{aligned} &\int\limits_{A_\delta(k,R)} \eta^2|\nabla^2 u_\delta|^2(\Gamma_\delta - k)\Gamma_\delta^{\frac{-\mu}{2}}\, dx \\ &+ \int\limits_{A_\delta(k,R)} \eta^2|\nabla\Gamma_\delta|^2\Gamma_\delta^{\frac{-\mu}{2}}\, dx \\ &\leq c \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla\eta, \nabla\eta)(\Gamma_\delta - k)^2 dx \\ &+ c \int\limits_{A_\delta(k,R)} |\nabla\eta|\eta|\nabla u_\delta|^3 dx + c \int\limits_{A_\delta(k,R)} \eta^2|\nabla u_\delta|^{2+\mu} dx. \end{aligned} \tag{2.11}$$

Since we may assume $R_0 < 1$ and since we may neglect the non-negative first integral on the left hand side of (2.11) it finally holds

$$\int\limits_{A_\delta(k,R)} \eta^2 |\nabla\Gamma_\delta|^2 \Gamma_\delta^{\frac{-\mu}{2}} dx \le \frac{c}{(R-r)^2} \int\limits_{A_\delta(k,R)} |D^2F_\delta(\nabla u_\delta)|(\Gamma_\delta - k)^2 dx + \frac{c}{R-r} \int\limits_{A_\delta(k,R)} \eta |\nabla u_\delta|^{2+\mu} dx$$

which proves (2.2). □

Considering the case $\mu \in (1, 2)$ we now deduce that ∇u_δ is locally uniformly bounded w.r.t. δ by adopting ideas as applied for example in [11], p.119-122. A major tool beside the lemma of Stampacchia (compare Lemma 5.1, p.219 in [21]) used during these arguments is the local uniform p-integrability of ∇u_δ for all finite p (see (2.1)). However, w.r.t. future problems it might be imaginable that one is only able to show local uniform p-integrability of ∇u_δ up to a fixed exponent p. As we are going to show in the following lemma, under the assumptions of Lemma 2.1 (i) we only need local uniform p-integrability of ∇u_δ for a certain exponent p to conclude that ∇u_δ is locally uniformly bounded w.r.t. δ whereby p is dependent of the elliptic parameter $\mu > 1$.

Now, we can state the main result of this section.

Lemma 2.4
Suppose that the assumptions of Lemma 2.1 (i) hold for some $\mu > 1$ and assume in addition that we have $\nabla u_\delta \in L^{2+2\mu+\varepsilon}_{loc}(\Omega, \mathbb{R}^2)$ uniformly in δ for a number $\varepsilon > 0$, i.e.

$$\sup_\delta \int\limits_{\Omega'} |\nabla u_\delta|^{2+2\mu+\varepsilon} dx = c(\mu, \varepsilon, \Omega') < \infty \tag{2.12}$$

where $\Omega' \Subset \Omega$. Then it holds $\nabla u_\delta \in L^\infty_{loc}(\Omega, \mathbb{R}^2)$ uniformly in δ.

As the first step for proving Lemma 2.4 we present a technical lemma which is of pure algebraic nature, its proof is given in the Appendix.

Lemma 2.5
Consider real numbers $\overline{p}, \nu, \mu > 1$ with

$$\mu + 3 < \overline{p} \qquad \textit{as well as} \qquad \mu + \nu < \overline{p}. \tag{2.13}$$

Then, there exist real numbers $s_1, s_2, s_3, s_4 > 1$ such that

(i) $2\dfrac{s_1}{s_1-1} < \overline{p}$, *(ii)* $\dfrac{2}{s_1} > 1$,

(iii) $\mu\dfrac{s_2}{s_2-1} < \overline{p}$, *(iv)* $3\dfrac{s_3}{s_3-1} < \overline{p}$,

(v) $\nu\dfrac{s_4}{s_4-1} < \overline{p}$, *(vi)* $\dfrac{1}{s_3} + \dfrac{1}{s_2} > 1$,

(vii) $\dfrac{1}{s_4} + \dfrac{1}{s_2} > 1$.

After these preparations we now come to the Proof of Lemma 2.4:

With the previous notation and using Sobolev's inequality we have

$$\int\limits_{A_\delta(k,r)} (\Gamma_\delta - k)^2 dx \leq \int\limits_{B_R(x_0)} (\eta(\Gamma_\delta - k)^+)^2 dx \leq c\Big(\int\limits_{B_R(x_0)} |\nabla[\eta(\Gamma_\delta - k)^+]| dx\Big)^2.$$

Moreover it holds

$$\begin{aligned} c\Big(\int\limits_{B_R(x_0)} |\nabla[\eta(\Gamma_\delta - k)^+]| dx\Big)^2 &= c\Big(\int\limits_{A_\delta(k,R)} |\nabla[\eta(\Gamma_\delta - k)]| dx\Big)^2 \\ &\leq c\Big(\int\limits_{A_\delta(k,R)} |\nabla\eta|(\Gamma_\delta - k) dx\Big)^2 \\ &+ c\Big(\int\limits_{A_\delta(k,R)} \eta|\nabla\Gamma_\delta| dx\Big)^2 \\ &=: c\Big[I_1^2 + I_2^2\Big]. \end{aligned}$$

As a consequence we can state

$$\int\limits_{A_\delta(k,r)} (\Gamma_\delta - k)^2 dx \leq c\Big[I_1^2 + I_2^2\Big]. \tag{2.14}$$

In Lemma 2.5 we now let $\mu > 1$ and define $\nu := 2 + \mu$ as well as $\overline{p} := 2 + 2\mu + \varepsilon$ whereby $\varepsilon > 0$ is a fixed number. Hence we get

$$\mu + 3 < 2\mu + 2 < \overline{p}$$

Therefore, (2.13) in Lemma 2.5 is satisfied and we get existence of real numbers $s_i > 1$, $i = 1, \ldots, 4$, for this choice of the parameters.

By (2.12) we have $\Gamma_\delta - k \in L^{\frac{\overline{p}}{2}}(B_R(x_0))$ uniformly in δ and in accordance with Lemma 2.5 (i), we may conclude $\Gamma_\delta - k \in L^{\frac{s_1}{s_1-1}}(B_R(x_0))$ uniformly in δ by (2.12). We get by using Hölder's inequality

$$\begin{aligned} I_1^2 &= \Bigg(\int\limits_{A_\delta(k,R)} |\nabla\eta|(\Gamma_\delta - k)dx \Bigg)^2 \\ &\le \frac{c}{(R-r)^2} (\mathcal{L}^2(A_\delta(k,R)))^{\frac{2}{s_1}} \Bigg(\int\limits_{A_\delta(k,R)} (\Gamma_\delta - k)^{\frac{s_1}{s_1-1}} dx \Bigg)^{\frac{2(s_1-1)}{s_1}} \\ &\le \frac{c}{(R-r)^2} (\mathcal{L}^2(A_\delta(k,R)))^{\frac{2}{s_1}}. \end{aligned} \tag{2.15}$$

Next we discuss I_2. We have $\Gamma_\delta^{\frac{\mu}{4}} \in L^2(B_R(x_0))$ uniformly in δ since $\overline{p} > \mu$ and an application of Hölder's inequality and (2.2) leads to

$$\begin{aligned} I_2^2 &\le \int\limits_{A_\delta(k,R)} \eta^2 |\nabla\Gamma_\delta|^2 \Gamma_\delta^{\frac{-\mu}{2}} dx \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\mu}{2}} dx \\ &\le \frac{c}{(R-r)^2} \int\limits_{A_\delta(k,R)} |D^2F_\delta(\nabla u_\delta)|(\Gamma_\delta - k)^2 dx \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\mu}{2}} dx \\ &+ \frac{c}{R-r} \int\limits_{A_\delta(k,R)} \eta |\nabla u_\delta|^{2+\mu} dx \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\mu}{2}} dx \\ &\le \Bigg[\frac{c}{(R-r)^2} \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{3}{2}} dx + \frac{c}{R-r} \int\limits_{A_\delta(k,R)} \Gamma_\delta^{1+\frac{\mu}{2}} dx \Bigg] \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\mu}{2}} dx \end{aligned} \tag{2.16}$$

where the last inequality follows by incorporating (1.5).
By (2.12) we have $\Gamma_\delta^{\frac{\mu}{2}} \in L^{\frac{\overline{p}}{\mu}}(B_R(x_0))$ uniformly in δ and according to Lemma 2.5 (iii), it follows $\Gamma_\delta^{\frac{\mu}{2}} \in L^{\frac{s_2}{s_2-1}}(B_R(x_0))$ uniformly in δ by (2.12). Hölder's inequality gives

$$\begin{aligned} \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\mu}{2}} dx &\le \Bigg(\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\mu s_2}{2(s_2-1)}} dx \Bigg)^{\frac{s_2-1}{s_2}} \mathcal{L}^2(A_\delta(k,R))^{\frac{1}{s_2}} \\ &\le c\mathcal{L}^2(A_\delta(k,R))^{\frac{1}{s_2}}. \end{aligned} \tag{2.17}$$

By (2.12) we deduce $\Gamma_\delta^{\frac{3}{2}} \in L^{\frac{\overline{p}}{3}}(B_R(x_0))$ uniformly in δ. Taking Lemma 2.5 (iv) into account we obtain $\Gamma_\delta^{\frac{3}{2}} \in L^{\frac{s_3}{s_3-1}}(B_R(x_0))$ uniformly in δ by using (2.12) again.

Hölder's inequality implies

$$\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{3}{2}} dx \le \Bigg[\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{3}{2}\frac{s_3}{s_3-1}} dx \Bigg]^{\frac{s_3-1}{s_3}} \mathcal{L}^2(A_\delta(k,R))^{\frac{1}{s_3}} \le c\mathcal{L}^2(A_\delta(k,R))^{\frac{1}{s_3}}. \tag{2.18}$$

In addition, by incorporating (2.12) we get $\Gamma_\delta^{1+\frac{\mu}{2}} \in L^{\frac{\overline{p}}{2+\mu}}(\Omega)$ uniformly in δ. Due to Lemma 2.5 (v), it follows $\Gamma_\delta^{1+\frac{\mu}{2}} \in L^{\frac{s_4}{s_4-1}}(\Omega)$ uniformly in δ by taking (2.12) into account. An application of Hölder's inequality results in

$$\begin{aligned} \int\limits_{A_\delta(k,R)} \Gamma_\delta^{1+\frac{\mu}{2}} dx &\le \Bigg[\int\limits_{A_\delta(k,R)} \Gamma_\delta^{(1+\frac{\mu}{2})\frac{s_4}{s_4-1}} dx \Bigg]^{\frac{s_4-1}{s_4}} \mathcal{L}^2(A_\delta(k,R))^{\frac{1}{s_4}} \\ &\le c\mathcal{L}^2(A_\delta(k,R))^{\frac{1}{s_4}}. \end{aligned} \tag{2.19}$$

Recalling $R_0 < 1$, (2.16) - (2.19) lead to

$$I_2^2 \le \frac{c}{(R-r)^2}\Bigg(\mathcal{L}^2(A_\delta(k,R))^{\frac{1}{s_2}+\frac{1}{s_3}} + \mathcal{L}^2(A_\delta(k,R))^{\frac{1}{s_2}+\frac{1}{s_4}}\Bigg). \tag{2.20}$$

Applying Lemma 2.5 (vi) and (vii) to (2.20) we get existence of a real number $\widetilde{\beta} > 1$ with

$$I_2^2 \le \frac{c}{(R-r)^2}\mathcal{L}^2(A_\delta(k,R))^{\widetilde{\beta}} \tag{2.21}$$

Thus (2.15), (2.21) and Lemma 2.5 (ii) imply existence of a real number $\beta > 1$ such that we may deduce from (2.14) (with R_0 sufficiently small)

$$\int\limits_{A_\delta(k,r)} (\Gamma_\delta - k)^2 dx \le \frac{c}{(R-r)^2}\mathcal{L}^2(A_\delta(k,R))^{\beta}. \tag{2.22}$$

At this point we define the following quantities for $k \ge 2$ and $r < R$:

$$\tau(k,r) := \int\limits_{A_\delta(k,r)} (\Gamma_\delta - k)^2 dx, \quad a(k,r) := \mathcal{L}^2(A_\delta(k,r)).$$

Now, suppose that there are given two real numbers h, k with $h > k > 2$, hence $\frac{\Gamma_\delta - k}{h-k} \ge 1$ on $A_\delta(h,R)$. We get

$$a(h,R) = \int\limits_{A_\delta(h,R)} 1 dx \le \int\limits_{A_\delta(h,R)} (\Gamma_\delta - k)^2 (h-k)^{-2} dx,$$

thus

$$a(h,R) \leq \frac{1}{(h-k)^2}\tau(k,R) \tag{2.23}$$

From (2.22) and (2.23) it follows

$$\tau(h,r) \leq \frac{c}{(R-r)^\gamma (h-k)^\alpha}(\tau(k,R))^\beta$$

where

$$\gamma := 2,\ \alpha := 2\beta > 0,\ \beta > 1. \tag{2.24}$$

At this point, we apply Stampacchia's lemma (see Lemma 5.1, p.219 in [21] or Lemma B.1, p. 63 in [18]) to deduce that ∇u_δ is locally uniformly bounded in Ω w.r.t. δ. Observe that there exists a positive quantity d such that

$$\tau(d+k_0, R_0 - \sigma R_0) = 0$$

for all $\sigma \in (0,1)$ with

$$d^\alpha = \frac{2^{\frac{(\alpha+\beta)\beta}{\beta-1}} C}{\sigma^\gamma R_0^\gamma}[\tau(k_0,R_0)]^{\beta-1}.$$

Choosing $k_0 = 2$ and $\sigma = \frac{1}{2}$ we arrive at

$$\tau(d+2, R_0/2) = 0. \tag{2.25}$$

Moreover, d is uniformly bounded w.r.t. δ since we may use (2.12) with $\overline{p} > 4$, i.e.

$$d^\alpha \leq c(R_0).$$

As a consequence we obtain

$$|\nabla u_\delta| \leq c \tag{2.26}$$

a.e. on $B_{R_0/2}(x_0)$ for all $\delta \in (0,1)$ where c in particular is independent of δ. Using a covering argument, we get

$$||\nabla u_\delta||_{L^\infty(\omega,\mathbb{R}^2)} \leq c(\omega)$$

for all $\omega \Subset \Omega$ and $\delta \in (0,1)$, i.e. u_δ is locally uniformly Lipschitz continuous with Lipschitz constant $c(\omega) > 0$. This completes the proof of Lemma 2.4.

Remark 2.6
If we have $\mu \in (1,2)$, then we clearly get $\nabla u_\delta \in L^\infty_{loc}(\Omega,\mathbb{R}^2)$ uniformly in δ by taking (2.1) into account. Moreover it holds $u_\delta \to u$ in $L^1_{loc}(\Omega)$ as $\delta \to 0$ (see Lemma 2.1 (ii)) and since u_δ is locally uniformly Lipschitz continuous (w.r.t δ) we may apply Arzelà-Ascoli's theorem to see that $u \in C^{0,1}(\Omega)$.

Finally, we are going to show that u has locally Hölder continuous first partial derivatives in Ω completing the proof of Theorem 1.4.
Let $\omega \Subset \Omega$ be arbitrary again. First, we observe that u is a solution of the Euler equation

$$\int\limits_{\Omega} DF(\nabla u)\nabla\varphi dx = \int\limits_{\Omega} g\varphi dx$$

for all $\varphi \in C_0^\infty(\Omega)$ where $g := \lambda\chi_{\Omega-D}(u-f)$.
Setting $\varphi = \partial_\alpha\psi$, $\alpha \in \{1,2\}$, with $\psi \in C_0^\infty(\Omega)$ we arrive at

$$\int\limits_{\Omega} DF(\nabla u)\partial_\alpha\nabla\psi dx = \int\limits_{\Omega} g\partial_\alpha\psi dx$$

for all $\psi \in C_0^\infty(\Omega)$.
Since u is Lipschitz continuous, we may argue with the standard difference quotient technique to get $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$. Moreover, we have $DF(\nabla u) \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^2)$ by applying the chain rule for Sobolev functions. By means of these results, an integration by parts leads to

$$-\int\limits_{\omega} D^2F(\nabla u)(\partial_\alpha\nabla u,\nabla\psi)dx = \int\limits_{\omega} g\partial_\alpha\psi dx.$$

Setting $v := \partial_\alpha u$, we get

$$\int\limits_{\omega} D^2F(\nabla u)(\nabla v,\nabla\psi)dx = -\int\limits_{\omega} g\partial_\alpha\psi dx.$$

The coefficients $a_{\alpha\beta}(x) := \frac{\partial^2 F}{\partial p_\alpha \partial p_\beta}(\nabla u)$ are strictly elliptic and bounded on ω. This fact follows immediately from (1.5) and from the local Lipschitz continuity of u. Finally, Theorem 8.22, p.200, of [17] ensures interior Hölder continuity of v and therefore of $\partial_\alpha u$ for all $\alpha \in \{1,2\}$, i.e. u has locally Hölder continuous first partial derivatives in Ω. This completes the proof of Theorem 1.4.

3 Appendix: Proof of Lemma 2.5

At first we choose $\widetilde{p} < \overline{p}$ such that (2.13) still holds for $\widetilde{p}$ instead of $\overline{p}$.
Due to (2.13) it holds $\widetilde{p} > 3$ and $\widetilde{p} > \nu$. Thus (i), (iv) and (v) are obvious by setting $s_1 := \frac{\widetilde{p}}{\widetilde{p}-2} > 1$, $s_3 := \frac{\widetilde{p}}{\widetilde{p}-3} > 1$ and $s_4 := \frac{\widetilde{p}}{\widetilde{p}-\nu} > 1$.
Using (2.13) and $\mu > 1$ we also get (ii).
To prove (vi) and (vii) we observe that we have

$$1 - \frac{1}{s_3} = \frac{3}{\widetilde{p}} < 1$$

as well as

$$1 - \frac{1}{s_4} = \frac{\nu}{\widetilde{p}} < 1.$$

Setting $m_1 := 1 - \frac{1}{s_3}$ and $m_2 := 1 - \frac{1}{s_4}$ we have shown that

$$m := \max(m_1, m_2) < 1 \tag{3.1}$$

Due to (3.1) we may choose $s_2 > 1$ in such a way that

$$m < \frac{1}{s_2} < 1. \tag{3.2}$$

Using (3.2) we directly get

$$\frac{1}{s_3} + \frac{1}{s_2} > \frac{1}{s_3} + m_1 = 1$$

and

$$\frac{1}{s_4} + \frac{1}{s_2} > \frac{1}{s_4} + m_2 = 1$$

which proves (vi) as well as (vii).
To prove (iii) we are going to show that

$$\frac{1}{s_2} < 1 - \frac{\mu}{\widetilde{p}}. \tag{3.3}$$

In order to show that (3.3) holds we observe that we have $m_1 < 1-\frac{\mu}{\widetilde{p}}$ in accordance with the first inequality in (2.13). Furthermore it also holds $m_2 < 1-\frac{\mu}{\widetilde{p}}$ according to the second inequality in (2.13).
Thus it follows

$$m < 1 - \frac{\mu}{\widetilde{p}}. \tag{3.4}$$

Therefore we may choose $s_2 > 1$ in addition to (3.2) in such a way that

$$m < \frac{1}{s_2} < 1 - \frac{\mu}{\widetilde{p}}$$

which proves (iii) and altogether Lemma 2.5.

References

[1] R.A. Adams, *Sobolev spaces*, Academic Press, New York-London, 1975, Pure and Applied Mathematics, Vol. 65.

[2] P. Arias, V. Caselles, G. Facciolo, V. Lazcano and R. Sadek, *Nonlocal variational models for inpainting and interpolation*, Math. Models Methods Appl. Sci. 22 (2012), no. suppl. 2.

[3] P. Arias, V. Casseles and G. Sapiro, *A variational framework for non-local image inpainting*, IMA Preprint Series No. 2265 (2009).

[4] P. Arias, G. Facciolo, V. Casseles and G. Sapiro, *A variational framework for exemplar-based image inpainting*, Int. J. Comput. Vis. 93 (2011), no. 3, 319–347.

[5] G.Aubert and P. Kornprobst, *Mathematical problems in image processing*, Applied Mathematical Sciences, Vol. 147, Springer-Verlag, New York, 2002.

[6] M. Bertalmio,C. Ballester, G. Sapiro and V. Caselles , *Image inpainting*, Proceedings of the 27th annual conference on Computer graphics and interactive techniques ACM press/Addison-Wesley Publishing Co. (2000), 417–424.

[7] M. Bildhauer and M. Fuchs, *A variational approach to the denoising of images based on different variants of the TV-regularization*, Appl. Math. Optim. 66 (2012), no. 3, 331–361.

[8] M. Bildhauer and M. Fuchs, *On some perturbations of the total variation image inpainting method. Part 1: Regularity theory*, Preprint Nr. 328 Saarland University.

[9] M. Bildhauer and M. Fuchs, *On some perturbations of the total variation image inpainting method. Part 2: Relaxation and dual variational formulation*, Preprint Nr. 332 Saarland University.

[10] M. Burger, L. He and C.-B. Schönlieb, *Cahn-Hilliard inpainting and a generalization for grayvalue images* SIAM J. Imaging Sci 2 (2009), no. 4, 1129–1167.

[11] M. Bildhauer, *Convex variational problems* Lecture Notes in Mathematics, vol. 1818, Springer-Verlag, Berlin.2003, Linear, nearly linear and anisotropic growth conditions.

[12] T.F. Chan, S.H. Kang and J. Shen, *Euler's elastica and curvature based inpaintings*, SIAM J. Appl. Math. 63 (2002), no. 2, 564–592.

[13] T.F. Chan and J. Shen, *Nontexture inpainting by curvature-driven diffusions*, Journal of Visual Communication and Image Representation 12 (2001), No. 4, 436–449.

[14] T.F. Chan and J. Shen, *Mathematical models for local nontexture inpaintings*, SIAM J. Appl. Math. 62 (2001/02), no. 3, 1019–1043.

[15] S. Esedoglu and J. Shen, *Digital inpainting based on the Momford-Shah-Euler image model*, European Journal of Applied Mathematics 13 (2002), no. 4, 353–370.

[16] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, Vol. 80, Birkhäuser Verlag, Basel, 1984.

[17] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order, second ed., Grundlehren der Mathematischen Wissenschaften, vol. 224, Springer-Verlag, Berlin, 1983.

[18] D. Kinderlehrer and G. Stampacchia, An introduction to variational inequalities and their applications, Academic Press, vol. 88, 1980.

[19] K. Papafitsoros, B. Sengul and C.-B. Schönlieb, *Combined first and second order total variation impainting using split bregman*, IPOL Preprint (2012).

[20] J. Shen, J. *Inpainting and the fundamental problem of image processing*, SIAM News 36, No. 5 (2003), 1–4.

[21] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) 15 (1965), no. fasc. 1, 189-258.