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MEASURE AND INTEGRATION: THE BASIC EXTENSION AND REPRESENTATION THEOREMS IN TERMS OF NEW INNER AND OUTER ENVELOPES

HEINZ KÖNIG

Dedicated to the Memory of Adriaan Cornelis Zaanen
on the occasion of the 100th Anniversary of his Birth

Abstract. The work of the author in measure and integration is based on new inner and outer envelope formations, which replace the traditional Carathéodory outer measure and certain simple suprema and infima. The new formations lead to essential improvements in both the extent and the adequacy of the basic results. However, they did not find entrance into the recent textbook literature. The present paper wants to demonstrate their power with the examples of the basic inner and outer extension and representation theorems for set functions and functionals.

The present paper returns to the foundations of the theory developed in the author's book [1] and in the subsequent 25 articles which recently have been collected in the volume [2]. We consider the basic inner and outer $\bullet$ extension theorems for set functions, as before in the versions $\bullet = \star\sigma\tau$ with $\star =$ finite, $\sigma =$ sequential, $\tau =$ nonsequential, and the basic inner and outer $\bullet$ representation theorems for functionals in terms of the Choquet integral, this time as before for $\bullet = \sigma\tau$. The decisive formations are the new inner and outer $\bullet$ *envelopes*, which for set functions and $\bullet = \sigma\tau$ have to take the place of the Carathéodory outer measure and of the $\bullet = \star$ envelopes in the traditional treatments. The relevant results in [1] and [2] were drastic improvements, to an extent that it appears mysterious why the new concepts were not widely adopted. In the present paper the first two sections are kind of summaries of the extension theories and of the initial part of the representation theories, while the third section is a further development of the final representation theories - all with the intention to illuminate the role of our inner and outer $\bullet$ envelopes. In contrast to the earlier papers the front versions will be the *inner* ones, because there are quite some indications that the inner situation is the superior one. For the comparison with the traditional treatments and for concrete situations we can refer to an abundance of places in [1] and [2].

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1. The Inner and Outer Extension Theorems

We start to recall the relevant traditional concepts. Our main references will be the [2] articles (10) and (24). The entire paper assumes a nonvoid set X, which carries the set systems under consideration. A nonvoid set system $\mathfrak{S}$ in X is called a *paving*. The most frequent ones are the *lattices* (with respect to $\cap\cup$) and the *rings* and *algebras*. For a paving $\mathfrak{S}$ we define $\mathfrak{S}_\star \subset \mathfrak{S}_\sigma \subset \mathfrak{S}_\tau$ and $\mathfrak{S}^\star \subset \mathfrak{S}^\sigma \subset \mathfrak{S}^\tau$ to consist of the intersections and unions of the finite/countable/arbitrary subpavings of $\mathfrak{S}$. One of the most fundamental concepts is that of the $\bullet$ *compact* pavings for $\bullet = \sigma\tau$, of which the counterpart $\bullet = \star$ is trivially fulfilled.

In the present context the usual set functions on a paving $\mathfrak{S}$ are the *isotone* $\varphi : \mathfrak{S} \to [0, \infty[$ or $[0, \infty]$. For these ones we recall the notions (*almost*) *downward /upward* $\bullet$ *continuous* for $\bullet = \star\sigma\tau$, of which $\bullet = \star$ is trivially fulfilled, and *inner/outer regular* $\mathfrak{M}$ for a subpaving $\mathfrak{M} \subset \mathfrak{S}$. For a set function $\varphi : \mathfrak{S} \to [0, \infty]$ on a lattice $\mathfrak{S}$ we recall the notions *modular* and *super/submodular*. In case $\varnothing \in \mathfrak{S}$ the function φ is called a *content* iff it is isotone with $\varphi(\varnothing) = 0$ and modular; this is the usual notion when $\mathfrak{S}$ is a ring.

We conclude the list of traditional concepts with the *Carathéodory class* for a set function $\vartheta : \mathfrak{P}(X) \to [0, \infty]$ with $\vartheta(\varnothing) = 0$, defined to be

$$\mathfrak{C}(\vartheta) := \{A \subset X : \vartheta(M) = \vartheta(M \cap A) + \vartheta(M \cap A') \text{ for all } M \subset X\} \subset \mathfrak{P}(X);$$

its members are called *measurable* ϑ. The basic properties of $\mathfrak{C}(\vartheta)$ are collected in [2](24) sect.2. Beyond $\vartheta(\varnothing) = 0$ the class $\mathfrak{C}(\vartheta)$ can be defined after [1] pp.40-42, but the explicit definition will not be needed in the sequel.

Next we recall from [2](24) 3.1 the basic concepts for set functions in [1] and [2]. Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$, and $\bullet = \star\sigma\tau$. We define an isotone $\varphi : \mathfrak{S} \to [0, \infty[$ with $\varphi(\varnothing) = 0$ to be an *inner* $\bullet$ *premeasure* iff it can be extended to a content $\alpha : \mathfrak{A} \to [0, \infty]$ on a ring $\mathfrak{A} \supset \mathfrak{S}_\bullet$ such that

$\alpha|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous (note that $\alpha|\mathfrak{S}_\bullet < \infty$) and
α is inner regular $\mathfrak{S}_\bullet$.

These contents α are called the *inner* $\bullet$ *extensions* of φ. Note that an inner $\bullet$ premeasure is downward $\bullet$ continuous and modular. Likewise we define an isotone $\varphi : \mathfrak{S} \to [0, \infty]$ with $\varphi(\varnothing) = 0$ to be an *outer* $\bullet$ *premeasure* iff it can be extended to a content $\alpha : \mathfrak{A} \to [0, \infty]$ on a ring $\mathfrak{A} \supset \mathfrak{S}^\bullet$ such that

$\alpha|\mathfrak{S}^\bullet$ is upward $\bullet$ continuous and
α is outer regular $\mathfrak{S}^\bullet$.

These contents α are called the *outer* $\bullet$ *extensions* of φ. Note that an outer $\bullet$ premeasure is upward $\bullet$ continuous and modular. The deviation relative to the value ∞ is of course in order to avoid the difficulties known from the traditional treatment for $\bullet = \sigma$.

These definitions produce the natural tasks to *characterize* the inner and outer $\bullet$ premeasures and to *describe* their collections of inner/outer $\bullet$ extensions. The fundamental idea to solve these tasks is to introduce the inner and outer $\bullet$ *envelopes*: For an isotone set function $\varphi : \mathfrak{S} \to [0,\infty]$ on a paving $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ we define for $\bullet = \star\sigma\tau$ these envelopes $\varphi_\bullet$ and $\varphi^\bullet : \mathfrak{P}(X) \to [0,\infty]$ to be

$$\varphi_\bullet(A) = \sup\{\inf_{M\in\mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid} \bullet \text{ with } \mathfrak{M} \downarrow\subset A\},$$
$$\varphi^\bullet(A) = \inf\{\sup_{M\in\mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid} \bullet \text{ with } \mathfrak{M} \uparrow\supset A\},$$

in the usual notations and with $\inf\varnothing := \infty$. Thus $\varphi_\star \leqq \varphi_\sigma \leqq \varphi_\tau$ and $\varphi^\star \geqq \varphi^\sigma \geqq \varphi^\tau$. Their simplified forms for $\bullet = \star\sigma$ and basic properties are collected in [2](10) sect.2 and (24) sect.1.

The extension theorems in [2](10) sect.3 and (24) sects.3 and 4 then read as follows. Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$ and $\bullet = \star\sigma\tau$.

1.1 Inner Extension Theorem. *For an isotone set function $\varphi : \mathfrak{S} \to [0,\infty[$ with $\varphi(\varnothing) = 0$ the following are equivalent.*

1) *φ is an inner $\bullet$ premeasure.*
2) *$\varphi(B) = \varphi(A) + \varphi_\bullet(B\backslash A)$ for all $A \subset B$ in $\mathfrak{S}$.*
3) *$\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an extension of φ.*
4) *$\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an inner $\bullet$ extension of φ.*

In this case all inner $\bullet$ extensions of φ are restrictions of $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$. Moreover $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is a content on an algebra for $\bullet = \star$ and a measure on a σ algebra for $\bullet = \sigma\tau$.

1.2 Addendum. We have the equivalence

$$\begin{aligned} &\varphi(B) = \varphi(A) + \varphi_\bullet(B\backslash A) \text{ for all } A \subset B \text{ in } \mathfrak{S} \\ \iff &\varphi \text{ is downward} \bullet \text{ continuous and supermodular, and} \\ &\varphi(B) \leqq \varphi(A) + \varphi_\bullet(B\backslash A) \text{ for all } A \subset B \text{ in } \mathfrak{S}. \end{aligned}$$

The outer counterpart is somewhat more involved. For an isotone set function $\varphi : \mathfrak{S} \to [0,\infty]$ with $\varphi(\varnothing) = 0$ we shall need for $\bullet = \tau$ the condition

$$(\uparrow) \quad \varphi^\tau(A) = \sup\{\varphi^\tau(A\cap S) : S \in \mathfrak{S} \text{ with } \varphi(S) < \infty\}$$
$$\text{for all } A \subset X \text{ with } \varphi^\tau(A) < \infty,$$

which is of *inner regular* kind. It is superfluous for $\bullet = \star\sigma$ in view of [2](24) 3.4.

1.3 Outer Extension Theorem. *For an isotone set function $\varphi : \mathfrak{S} \to [0,\infty]$ with $\varphi(\varnothing) = 0$ the following are equivalent.*

1) *φ is an outer $\bullet$ premeasure.*
2) *$\varphi(B) = \varphi(A) + \varphi^\bullet(B\backslash A)$ for all $A \subset B$ in $\mathfrak{S}$, plus $(\uparrow)$ in case $\bullet = \tau$.*
3) *$\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ is an extension of φ.*
4) *$\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ is an outer $\bullet$ extension of φ.*

In this case all outer $\bullet$ extensions of φ are restrictions of $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$. Moreover $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ is a content on an algebra for $\bullet = \star$ and a measure on a σ algebra for $\bullet = \sigma\tau$.

1.4 Addendum. We have the equivalence

$$\varphi(B) = \varphi(A) + \varphi^\bullet(B\backslash A) \text{ for all } A \subset B \text{ in } \mathfrak{S}$$

$\iff$ φ is upward $\bullet$ continuous and submodular, and
$\varphi(B) \geqq \varphi(A) + \varphi^\bullet(B \backslash A)$ for all $A \subset B$ in $\mathfrak{S}$.

Moreover we want to recall the example [2](4) 5.4 based on [1] 4.11 with 5.6, which shows that condition ($\uparrow$) cannot be dispensed with.

1.5 Example. There exists an isotone $\varphi : \mathfrak{S} \to [0, \infty]$ on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ with the properties

$\varphi^\sigma(A) = \varphi^\tau(A)$ for all $A \subset X$ upward enclosable $\mathfrak{S}$,
$\varphi(B) = \varphi(A) + \varphi^\sigma(B \backslash A) = \varphi(A) + \varphi^\tau(B \backslash A)$ for all $A \subset B$ in $\mathfrak{S}$,

so that φ is an outer σ premeasure and upward τ continuous, but φ does NOT fulfil ($\uparrow$) and hence is not an outer τ premeasure.

We conclude the present review with the fundamental Localization Principle [2](24) 4.4. It requires for a pair of pavings $\mathfrak{P}$ and $\mathfrak{Q}$ in X the concept $\mathfrak{P} \top \mathfrak{Q} := \{A \subset X : A \cap P \in \mathfrak{Q} \text{ for all } P \in \mathfrak{P}\}$, called their *transporter.*

1.6 Theorem ($\bullet = \star\sigma\tau$). *For an inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ we have $\mathfrak{S} \top \mathfrak{C}(\varphi_\bullet) \subset \mathfrak{C}(\varphi_\bullet)$. For an outer $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0, \infty]$ we have $[\varphi < \infty] \top \mathfrak{C}(\varphi^\bullet) \subset \mathfrak{C}(\varphi^\bullet)$.*

2. Preliminaries for the Representation Theorems

In the sequel the basic domains will be the nonvoid subsets $E \subset [0, \infty[^X$ of functions $f : X \to [0, \infty[$ and $E \subset [0, \infty]^X$ of functions $f : X \to [0, \infty]$. The most frequent E are positive-homogeneous with $0 \in E$ and *lattices* (with respect to the pointwise operations $\wedge\vee$). E is called *Stonean* iff $f \in E \Rightarrow f \wedge t, (f - t)^+ \in E$ for all $0 < t < \infty$; note that $f = f \wedge t + (f - t)^+$. We emphasize that E need not be stable under addition and under difference formation.

For a positive-homogeneous lattice $E \subset [0, \infty]^X$ with $0 \in E$ we define

$$\begin{aligned} \mathrm{Inn}(E) &:= \{[f \geqq t] : f \in E \text{ and } 0 < t < \infty\} \subset \mathfrak{P}(X), \\ \mathrm{Out}(E) &:= \{[f > t] : f \in E \text{ and } 0 < t < \infty\} \subset \mathfrak{P}(X), \end{aligned}$$

as in [2](19) sect.1 and 3 for $E \subset [0, \infty[^X$; both are lattices with $\varnothing$. Likewise for a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ we define

$$\mathrm{Inn}(\mathfrak{S}) := \{f \in [0, \infty]^X : [f \geqq t] \in \mathfrak{S} \text{ for all } 0 < t < \infty\},$$
$$\mathrm{Out}(\mathfrak{S}) := \{f \in [0, \infty]^X : [f > t] \in \mathfrak{S} \text{ for all } 0 < t < \infty\},$$

as in [2](20) sect.5; both are positive-homogeneous Stonean lattices with 0 in $[0, \infty]^X$. We also mention the earlier notations in [1] and [2](4) Intr. and [2](10) sect.7.1

$$\mathrm{Inn}(E) = \mathrm{um}(E) = \geqq(E) \quad \text{and} \quad \mathrm{Out}(E) = \mathrm{lm}(E) = >(E),$$
$$\mathrm{Inn}(\mathfrak{S}) = \mathrm{UM}(\mathfrak{S}) \quad \text{and} \quad \mathrm{Out}(\mathfrak{S}) = \mathrm{LM}(\mathfrak{S}).$$

One verifies for these E and $\mathfrak{S}$ the equivalences

$$\text{(Inn)} \quad E \subset \mathrm{Inn}(\mathfrak{S}) \ \Leftrightarrow\ \mathrm{Inn}(E) \subset \mathfrak{S},$$
$$\text{(Out)} \quad E \subset \mathrm{Out}(\mathfrak{S}) \ \Leftrightarrow\ \mathrm{Out}(E) \subset \mathfrak{S}.$$

The most basic concept is the *Choquet integral*, in [1] sect.11 called the *horizontal* integral. Let $\mathfrak{S}$ be a lattice in X with $\varnothing \in \mathfrak{S}$ and $\varphi : \mathfrak{S} \to [0, \infty]$ be isotone with $\varphi(\varnothing) = 0$. The *Choquet integral* $⨍ f d\varphi \in [0, \infty]$ is then defined to be

$$= \int_{0\leftarrow}^{\to\infty} \varphi\left([f \geqq t]\right) dt \text{ for } f \in \operatorname{Inn}(\mathfrak{S}) \text{ and } = \int_{0\leftarrow}^{\to\infty} \varphi\left([f > t]\right) dt \text{ for } f \in \operatorname{Out}(\mathfrak{S}),$$

both times as an improper Riemann integral of a decreasing function $\geqq 0$. One verifies that for $f \in \operatorname{Inn}(\mathfrak{S}) \cap \operatorname{Out}(\mathfrak{S})$ the two second members are equal. The basic properties of the Choquet integral are collected in [2](10) subsect.5.3. In particular it coincides with the usual integral $\int f d\varphi$ for a measure φ on a σ algebra $\mathfrak{S}$ and a φ measurable function $f : X \to [0, \infty]$.

For the remainder of the section we fix a positive-homogeneous lattice $E \subset [0, \infty]^X$ with $0 \in E$. We consider isotone functionals $I : E \to [0, \infty]$ with $I(0) = 0$. The relevant basic notions are listed in [2](10) subsect.5.2. As above we define for $\bullet = \star\sigma\tau$ the inner and outer $\bullet$ *envelopes* $I_\bullet$ and $I^\bullet : [0, \infty]^X \to [0, \infty]$ to be

$$\begin{aligned} I_\bullet(f) &= \sup\{\inf_{u\in M} I(u) : M \subset E \text{ nonvoid } \bullet \text{ with } M \downarrow\leqq f\}, \\ I^\bullet(f) &= \inf\{\sup_{u\in M} I(u) : M \subset E \text{ nonvoid } \bullet \text{ with } M \uparrow\geqq f\}, \end{aligned}$$

in the usual notations and with $\inf \varnothing := 0$. Thus $I_\star \leqq I_\sigma \leqq I_\tau$ and $I^\star \geqq I^\sigma \geqq I^\tau$. The simplified forms for $\bullet = \star\sigma$ and the basic properties are collected in [2](10) sect.5.

2.1 Remark. Let $\varphi : \operatorname{Inn}(E)/\operatorname{Out}(E) \to [0, \infty]$ be isotone with $\varphi(\varnothing) = 0$ and $I(f) = ⨍ f d\varphi$ for $f \in E$ (which makes sense after (Inn) and (Out) above). Then $I < \infty$ implies that $\varphi < \infty$. This is a simple consequence of the definitions.

After this we turn to the *representation theorems* in their initial versions. First comes the Greco type theorem [2](4) 2.12, and then the inner and outer $\bullet$ theorems [2](10) 7.2 and 7.1.

2.2 Theorem. *Assume that $E \subset [0, \infty]^X$ is a positive-homogeneous Stonean lattice with $0 \in E$ and $I : E \to [0, \infty]$ isotone with $I(0) = 0$. Then the following are equivalent.*

1) *There exists an isotone $\varphi : \operatorname{Inn}(E)/\operatorname{Out}(E) \to [0, \infty]$ with $\varphi(\varnothing) = 0$ such that $I(f) = ⨍ f d\varphi$ for all $f \in E$.*
2) *I is* Stonean*: $I(f) = I(f \wedge t) + I((f - t)^+)$ for $f \in E$ and $0 < t < \infty$, and* truncable*: $I(f) = \sup\{I\left((f - a)^+ \wedge (b - a)\right) : 0 < a < b < \infty\}$ for $f \in E$.*

In this case the set functions φ in 1) are precisely those with $I_\star(\chi_A) \leqq \varphi(A) \leqq I^\star(\chi_A)$ for all $A \in \operatorname{Inn}(E)/\operatorname{Out}(E)$.

2.3 Inner Theorem ($\bullet = \sigma\tau$). *Assume that* $E \subset [0,\infty[^X$ *is a positive-homogeneous Stonean lattice with* $0 \in E$ *and* $I : E \to [0,\infty[$ *isotone with* $I(0) = 0$*. Then the following are equivalent.*

1) *There exists an isotone* $\varphi : \mathrm{Inn}(E) \to [0,\infty[$ *with* $\varphi(\varnothing) = 0$ *which is downward* $\bullet$ *continuous and fulfils* $I(f) = \fint f d\varphi$ *for all* $f \in E$.
2) I *is Stonean and downward* $\bullet$ *continuous.*

In this case the set function φ *in* 1) *is unique and* $\varphi := I^\star(\chi.)|\mathrm{Inn}(E)$*. It fulfils* $\varphi_\bullet = I_\bullet(\chi.)$*. Likewise the following are equivalent.*

1') *There exists an isotone* $\varphi : \mathrm{Inn}(E) \to [0,\infty[$ *with* $\varphi(\varnothing) = 0$ *which is downward* $\bullet$ *continuous and supermodular and fulfils* $I(f) = \fint f d\varphi$ *for all* $f \in E$.
2') I *is Stonean and downward* $\bullet$ *continous and supermodular.*

In this case the unique set function φ *in* 1') *fulfils* $I_\bullet(f) = \fint f d\varphi_\bullet$ *for all* $f \in [0,\infty]^X$.

We add the next consequence which is remarkable because it depends on the deep theorem [2](10) 5.3.

2.4 Inner Consequence ($\bullet = \sigma\tau$). *In the situation of* 2.3 *under the assumptions* 1')2') *the functional* $I_\bullet$ *is superadditive.*

Proof. In view of [2](10) 2.8.1 $\varphi_\bullet$ is supermodular. Thus [2](10) 5.6.iv)$\Rightarrow$ i) applied to $I_\bullet(f) = \fint f d\varphi_\bullet$ for $f \in [0,\infty]^X$ asserts that $I_\bullet$ is superadditive. □

2.5 Outer Theorem ($\bullet = \sigma\tau$). *Assume that* $E \subset [0,\infty]^X$ *is a positive-homogeneous Stonean lattice with* $0 \in E$ *and* $I : E \to [0,\infty]$ *isotone with* $I(0) = 0$*. Then the following are equivalent.*

1) *There exists an isotone* $\varphi : \mathrm{Out}(E) \to [0,\infty]$ *with* $\varphi(\varnothing) = 0$ *which is upward* $\bullet$ *continuous and fulfils* $I(f) = \fint f d\varphi$ *for all* $f \in E$.
2) I *is Stonean and upward* $\bullet$ *continuous.*

In this case the set function φ *in* 1) *is unique and* $\varphi := I_\star(\chi.)|\mathrm{Out}(E)$*. It fulfils* $\varphi^\bullet = I^\bullet(\chi.)$*. Likewise the following are equivalent.*

1') *There exists an isotone* $\varphi : \mathrm{Out}(E) \to [0,\infty]$ *with* $\varphi(\varnothing) = 0$ *which is upward* $\bullet$ *continuous and submodular and fulfils* $I(f) = \fint f d\varphi$ *for all* $f \in E$.
2') I *is Stonean and upward* $\bullet$ *continuous and submodular.*

In this case the unique set function φ *in* 1') *fulfils* $I^\bullet(f) = \fint f d\varphi^\bullet$ *for all* $f \in [0,\infty]^X$.

2.6 Outer Consequence ($\bullet = \sigma\tau$). *In the situation of* 2.5 *under the assumptions* 1')2') *the functional* $I^\bullet$ *is subadditive.*

The proof corresponds to that of 2.4. Thus we see that up to this point the inner and outer procedures are quite parallel. However, the next section will show a different picture.

3. The Inner and Outer Representation Theorems

We start with the inner situation. This part of the section assumes a positive-homogeneous Stonean lattice $E \subset [0,\infty[^X$ with $0 \in E$ and an isotone functional $I : E \to [0,\infty[$ with $I(0) = 0$, that is the situation of the inner theorem 2.3.

3.1 Remark ($\bullet = \sigma\tau$). We have the equivalence

$$\begin{aligned} & I(v) = I(u) + I_\bullet(v-u) \text{ for all } u \leqq v \text{ in } E \\ \iff & I \text{ is Stonean and downward } \bullet \text{ continuous and supermodular, and} \\ & I(v) \leqq I(u) + I_\bullet(v-u) \text{ for all } u \leqq v \text{ in } E. \end{aligned}$$

Proof of $\Rightarrow$. I is Stonean, because for $f \in E$ and $0 < t < \infty$ we have $f - f \wedge t = (f-t)^+ \in E$ and hence

$$I(f) = I(f \wedge t) + I_\bullet(f - f \wedge t) = I(f \wedge t) + I((f-t)^+).$$

I is downward $\bullet$ continuous by [2](10) 5.10.5. And I is even modular, because for $u, v \in E$ we have

$$\begin{aligned} I(u \vee v) + I(u \wedge v) &= I(u) + I_\bullet(u \vee v - u) + I(u \wedge v) \\ &= I(u) + I_\bullet(v - u \wedge v) + I(u \wedge v) = I(u) + I(v). \end{aligned}$$

Proof of $\Leftarrow$. We combine 2.4 with $I_\bullet|E = I$ from [2](10) 5.10.5 to obtain $I(v) \geqq I(u) + I_\bullet(v-u)$ for all $u \leqq v$ in E. $\square$

3.2 Inner Representation Theorem ($\bullet = \sigma\tau$). *The following are equivalent.*

1) *There exists an inner $\bullet$ premeasure $\varphi : \mathrm{Inn}(E) \to [0,\infty[$ with $I(f) = ⨍ f d\varphi$ for all $f \in E$.*
2) *$I(v) = I(u) + I_\bullet(v-u)$ for all $u \leqq v$ in E.*

In this case the inner $\bullet$ premeasure φ in 1) *is unique and $\varphi := I^\star(\chi_\cdot)|\mathrm{Inn}(E)$. It fulfils $I_\bullet(f) = ⨍ f d\varphi_\bullet$ for all $f \in [0,\infty]^X$. Moreover I is additive* (for $u, v \in E$ with $u + v \in E$ as in [2](10) sect.5.2) *and modular.*

Proof. Combine the former version [2](10) 7.6 with the above 3.1. The final assertions are clear from 2): For $u, v \in E$ with $u + v \in E$ one has $I(u+v) = I(u) + I_\bullet(v) = I(u) + I(v)$, while the previous proof $\Rightarrow$ shows that I is modular. $\square$

We turn to the outer situation. We need two lemmata.

3.3 Lemma. *Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$ and $\mathrm{R}(\mathfrak{S})$ the generated ring. For each pair of functions $u \leqq v$ in $\mathrm{Out}(\mathfrak{S})$ with $u < \infty$ then $v - u \in \mathrm{Out}((\mathrm{R}(\mathfrak{S}))^\sigma)$.*

Proof. By assumption $[v > t], [u > t] \in \mathfrak{S}$ for $0 < t < \infty$, and the assertion is $[v - u > t] \in (\mathrm{R}(\mathfrak{S}))^\sigma$ for $0 < t < \infty$. We claim for $0 < t < \infty$ that

$$[v - u > t] = [v > t + u] = \bigcup_{0 < s \,\mathrm{rat} < \infty} \big([v > t + s] \setminus ([v > t + s] \cap [u > s])\big),$$

which implies the assertion. Proof of $\supset$:

$$[v > t + s] \setminus ([v > t + s] \cap [u > s]) = [v > t + s] \cap [s \geqq u] \subset [v > t + u].$$

Proof of $\subset$: Let $x \in X$ be in the left side, that is $v(x) - t > u(x)$, and take some rational $0 < s < \infty$ with $v(x) - t > s > u(x)$. Then

$$x \in [v > t + s] \cap [s \geqq u] = [v > t + s] \backslash ([v > t + s] \cap [u > s]),$$

which is contained in the right side. $\square$

3.4 Lemma ($\bullet = \sigma\tau$). *Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$, and assume that the isotone $\varphi : \mathfrak{S} \to [0, \infty]$ with $\varphi(\varnothing) = 0$ fulfils $\varphi(B) = \varphi(A) + \varphi^\bullet(B \backslash A)$ for all $A \subset B$ in $\mathfrak{S}$. For the sublattice $\mathfrak{T} := [\varphi < \infty] \subset \mathfrak{S}$ then $\varphi^\bullet|(\mathrm{R}(\mathfrak{T}))^\sigma$ is modular.*

Proof. It is clear that φ is modular and upward $\bullet$ continuous; hence [2](10) 2.8 asserts that $\varphi^\bullet$ is submodular and upward σ continuous. i) By [1] 3.4 (which for the present situation is due to Hausdorff) the restriction $\varphi|\mathfrak{T}$ has a unique extension to a content $\Phi : \mathrm{R}(\mathfrak{T}) \to [0, \infty[$. We claim that $\Phi = \varphi^\bullet|\mathrm{R}(\mathfrak{T})$, so that $\varphi^\bullet|\mathrm{R}(\mathfrak{T})$ is seen to be modular. Thus let $M \in \mathrm{R}(\mathfrak{T})$. From [1] 1.21 we obtain a finite sequence $S_1 \subset T_1 \subset ... \subset S_r \subset T_r$ in $\mathfrak{T}$ such that

$$M = \bigcup_{l=1}^{r} (T_l \backslash S_l) \text{ and hence } T_r \backslash S_1 = M \cup \bigcup_{l=2}^{r} (S_l \backslash T_{l-1}).$$

We obtain

$$\begin{aligned}
\varphi^\bullet(M) &\leqq \sum_{l=1}^{r} \varphi^\bullet(T_l \backslash S_l) \text{ and} \\
\varphi(T_r) - \varphi(S_1) &= \varphi^\bullet(T_r \backslash S_1) \leqq \varphi^\bullet(M) + \sum_{l=2}^{r} \varphi^\bullet(S_l \backslash T_{l-1}) \\
&\leqq \sum_{l=1}^{r} \varphi^\bullet(T_l \backslash S_l) + \sum_{l=2}^{r} \varphi^\bullet(S_l \backslash T_{l-1}) \\
&= \sum_{l=1}^{r} \big(\varphi(T_l) - \varphi(S_l)\big) + \sum_{l=2}^{r} \big(\varphi(S_l) - \varphi(T_{l-1})\big) = \varphi(T_r) - \varphi(S_1),
\end{aligned}$$

and hence

$$\varphi^\bullet(M) = \sum_{l=1}^{r} \varphi^\bullet(T_l \backslash S_l) = \sum_{l=1}^{r} \big(\varphi(T_l) - \varphi(S_l)\big) = \sum_{l=1}^{r} \Phi(T_l \backslash S_l) = \Phi(M),$$

as claimed. ii) Now let $A, B \in (\mathrm{R}(\mathfrak{T}))^\sigma$, and $A_l, B_l \in \mathrm{R}(\mathfrak{T})$ for $l \in \mathbb{N}$ with $A_l \uparrow A$ and $B_l \uparrow B$. Then $A_l \cup B_l \uparrow A \cup B$ and $A_l \cap B_l \uparrow A \cap B$. Thus $\varphi^\bullet(A_l \cup B_l) + \varphi^\bullet(A_l \cap B_l) = \varphi^\bullet(A_l) + \varphi^\bullet(B_l)$ implies that $\varphi^\bullet(A \cup B) + \varphi^\bullet(A \cap B) = \varphi^\bullet(A) + \varphi^\bullet(B)$ since $\varphi^\bullet$ is upward σ continuous. $\square$

For the remainder of the section we assume a positive-homogeneous Stonean lattice $E \subset [0, \infty]^X$ with $0 \in E$ and an isotone functional $I : E \to [0, \infty]$ with $I(0) = 0$, that is the situation of the outer theorem 2.5.

3.5 Remark ($\bullet = \sigma\tau$). We have the equivalence

$I(v) = I(u) + I^\bullet(v - u)$ for all $u \leqq v$ in E with $u < \infty$
$\iff$ I is Stonean and upward $\bullet$ continuous and submodular, and
$I(v) \geqq I(u) + I^\bullet(v - u)$ for all $u \leqq v$ in E with $u < \infty$.

Proof of $\Rightarrow$. The proofs that I is Stonean and upward $\bullet$ continuous are as in the proof of 3.1. To see that I is modular first consider $u, v \in E$ with $u < \infty$. Then as before

$$\begin{aligned} I(u \vee v) + I(u \wedge v) &= I(u) + I^\bullet(u \vee v - u) + I(u \wedge v) \\ &= I(u) + I^\bullet(v - u \wedge v) + I(u \wedge v) = I(u) + I(v). \end{aligned}$$

Now consider arbitrary $u, v \in E$. For $0 < t < \infty$ then

$$I\big((u \wedge t) \vee v\big) + I\big((u \wedge t) \wedge v\big) = I(u \wedge t) + I(v),$$

and hence for $t \uparrow \infty$ the assertion since I is upward $\bullet$ continuous.

Proof of $\Leftarrow$. This part is as in the proof of 3.1, with 2.6 instead of 2.4. $\square$

The next result is the fundamental new fact in the outer representation context. The decisive point is that no condition of the type ($\uparrow$) in the earlier outer 1.3 is involved.

3.6 Fundamental Lemma ($\bullet = \sigma\tau$). *In the situation of* 2.5 *and under the assumptions* 1')2') *we have the equivalence*

$$\begin{aligned} &\varphi(B) \geqq \varphi(A) + \varphi^\bullet(B \backslash A) \text{ for all } A \subset B \text{ in } \mathrm{Out}(E) \\ \Longleftrightarrow\ &I(v) \geqq I(u) + I^\bullet(v - u) \text{ for all } u \leqq v \text{ in } E \text{ with } u < \infty. \end{aligned}$$

Proof of $\Rightarrow$. i) We have by assumption the unique isotone $\varphi : \mathrm{Out}(E) \to [0, \infty]$ with $\varphi(\varnothing) = 0$ which is upward $\bullet$ continuous and submodular and fulfils $I(f) = \oint f d\varphi$ for all $f \in E$. By [2](10) 2.8.1 $\varphi^\bullet$ is submodular, which combined with the present assumption furnishes $\varphi(B) = \varphi(A) + \varphi^\bullet(B \backslash A)$ for all $A \subset B$ in $\mathrm{Out}(E) =: \mathfrak{S}$. For the sublattice $\mathfrak{T} := [\varphi < \infty] \subset \mathfrak{S}$ then 3.4 asserts that $\varphi^\bullet|(\mathrm{R}(\mathfrak{T}))^\sigma$ is modular.

ii) Next $I^\bullet(f) = \oint f d\varphi^\bullet$ for all $f \in [0, \infty]^X$ from 2.5. We conclude as in the proofs of 2.4 and 2.6 from [2](10) 5.6.iv)$\Rightarrow$i) applied to $\varphi^\bullet|(\mathrm{R}(\mathfrak{T}))^\sigma$ and the above i) that $I^\bullet|\mathrm{Out}\big((\mathrm{R}(\mathfrak{T}))^\sigma\big)$ is additive.

iii) Now to be shown is $I(v) \geqq I(u) + I^\bullet(v - u)$ for all $u \leqq v$ in E with $u < \infty$. We can assume that $I(v) < \infty$ and hence $I(u) < \infty$. Thus by definition the $[u > t], [v > t] \in \mathrm{Out}(E) = \mathfrak{S}$ for $0 < t < \infty$ have $\varphi([u > t]), \varphi([v > t]) < \infty$ and hence are $\in \mathfrak{T}$. Hence $u, v \in \mathrm{Out}(\mathfrak{T})$, and we conclude from 3.3 that $v - u \in \mathrm{Out}\big((\mathrm{R}(\mathfrak{T}))^\sigma\big)$. Thus ii) implies that $I^\bullet(v) = I^\bullet(u) + I^\bullet(v - u)$, and hence even $I(v) = I(u) + I^\bullet(v - u)$ from [2](10) 5.10.5.

Proof of $\Leftarrow$. i) Each $A \in \mathrm{Out}(E)$ is of the form $A = [f > 0]$ for some $f \in E$ with $f \leqq 1$. In fact,

$$\begin{aligned} A &= [f > t] \text{ for some } f \in E \text{ and } 0 < t < \infty \\ &= [(f - t)^+ > 0] = [g > 0] \text{ with } g := (f - t)^+ \in E \\ &= [g \wedge 1 > 0] = [h > 0] \text{ with } h := g \wedge 1 \in E \text{ and } h \leqq 1. \end{aligned}$$

ii) Now let $A \subset B$ in $\mathrm{Out}(E)$, that is $A = [f > 0]$ and $B = [g > 0]$ with $f, g \in E$ and $f, g \leqq 1$. Then $A = [f > 0] \cap [g > 0] = [f \wedge g > 0]$, so that we can assume $f \leqq g \leqq 1$. For $n \in \mathbb{N}$ we have $nf \wedge 1 \in E$, with $nf \wedge 1 \uparrow \chi_A$ for $n \to \infty$ and hence $I(nf \wedge 1) = I^\bullet(nf \wedge 1) \uparrow I^\bullet(\chi_A) = \varphi^\bullet(A) = \varphi(A)$, since $I^\bullet$

is upward σ continuous by [2](10) 5.10.8. Likewise $ng\wedge 1\in E$ with $ng\wedge 1\uparrow \chi_B$ and $I(ng\wedge 1)\uparrow \varphi(B)$ for $n\to\infty$.

iii) For $n\geqq m$ in $\mathbb{N}$ we have by the present assumption

$$I(ng\wedge 1)\geqq I(mf\wedge 1)+I^\bullet(ng\wedge 1-mf\wedge 1),$$

and hence for $n\to\infty$ from ii)

$$\varphi(B)\geqq I(mf\wedge 1)+I^\bullet(\chi_B-mf\wedge 1)\geqq I(mf\wedge 1)+I^\bullet(\chi_B-\chi_A),$$

so that for $m\to\infty$ it follows that

$$\varphi(B)\geqq \varphi(A)+I^\bullet(\chi_{B\backslash A})=\varphi(A)+\varphi^\bullet(B\backslash A).\ \square$$

We come to the main theorems.

3.7 New Outer Representation Theorem ($\bullet=\sigma\tau$). *In the situation of* 2.5 *the following are equivalent.*

1-) *There exists an isotone* $\varphi:\mathrm{Out}(E)\to[0,\infty]$ *with* $\varphi(\varnothing)=0$ *and* $\varphi(B)=\varphi(A)+\varphi^\bullet(B\backslash A)$ *for all* $A\subset B$ *in* $\mathrm{Out}(E)$ *which fulfils* $I(f)=⨍ f d\varphi$ *for all* $f\in E$.

2-) $I(v)=I(u)+I^\bullet(v-u)$ *for all* $u\leqq v$ *in* E *with* $u<\infty$.

In this case the set function φ *in* 1-) *is unique and* $\varphi:=I_\star(\chi.)|\mathrm{Out}(E)$. *It fulfils* $I^\bullet(f)=⨍ f d\varphi^\bullet$ *for all* $f\in[0,\infty]^X$.

Proof of 1-)⇒2-). The set function φ satisfies 1'), and hence I satisfies 2'). Then the implications 3.6 ⇒ and 3.5 ⇐ furnish the assertion 2-).

Proof of 2-)⇒1-). The implication 3.5 ⇒ furnishes 2') and hence 1') and $I(v)\geqq I(u)+I^\bullet(v-u)$ for all $u\leqq v$ in E with $u<\infty$, and then 3.6 ⇐ furnishes $\varphi(B)\geqq\varphi(A)+\varphi^\bullet(B\backslash A)$ for all $A\subset B$ in $\mathrm{Out}(E)$. Combined with the fact that $\varphi^\bullet$ is submodular this leads to the assertion 1-). $\square$

However, there is an essential difference between the conditions 1-) in the present 3.7 and 1) in the inner representation theorem 3.2: the present condition 1-) is NOT for the outer $\bullet$ premeasures $\varphi:\mathrm{Out}(E)\to[0,\infty]$, but for the isotone set functions $\varphi:\mathrm{Out}(E)\to[0,\infty]$ with $\varphi(\varnothing)=0$ and $\varphi(B)=\varphi(A)+\varphi^\bullet(B\backslash A)$ for all $A\subset B$ in $\mathrm{Out}(E)$. The outer extension theorem 1.3 asserts that the two classes coincide in case $\bullet=\sigma$, but example 1.5 shows that they can be different in case $\bullet=\tau$. In that case our former outer $\bullet$ representation theorem [2](10) 7.3 introduced a certain counterpart to the earlier condition (↑) and thus was able to involve the outer $\bullet$ premeasures $\varphi:\mathrm{Out}(E)\to[0,\infty]$ in both $\bullet=\sigma\tau$. This procedure will be our final step below. But we want to emphasize that the new outer $\bullet$ representation theorem 3.7 is a fundamental equivalence assertion as well.

We define for an isotone functional $I:E\to[0,\infty]$ on a positive-homogeneous Stonean lattice $E\subset[0,\infty]^X$ with $0\in E$ and $I(0)=0$ the condition

$$(\Uparrow)\quad I^\tau(f)=\sup\{I^\tau(f\wedge u):u\in E \text{ with } I(u)<\infty\}\quad\text{for all } f\in[0,\infty]^X \text{ with } I^\tau(f)<\infty,$$

which as before is of *inner regular* kind. And as before its counterpart for $\bullet=\sigma$ is superfluous. One then obtains the final result which follows.

3.8 Outer Representation Theorem ($\bullet = \sigma\tau$). *In the situation of* 2.5 *the following are equivalent.*

1+) *There exists an outer* $\bullet$ *premeasure* $\varphi : \mathrm{Out}(E) \to [0,\infty]$ *which fulfils* $I(f) = ⨍ f d\varphi$ *for all* $f \in E$.

2+) $I(v) = I(u) + I^\bullet(v-u)$ *for all* $u \leqq v$ *in* E *with* $u < \infty$, *plus* ($\Uparrow$) *in case* $\bullet = \tau$.

In this case the outer $\bullet$ *premeasure* φ *in* 1+) *is unique and* $\varphi := I_\star(\chi.)|\mathrm{Out}(E)$. *It fulfils* $I^\bullet(f) = ⨍ f d\varphi^\bullet$ *for all* $f \in [0,\infty]^X$.

This theorem is an immediate combination of the former outer representation theorem [2](4) 5.3 = (10) 7.3 with the above 3.5.

In conclusion we note a remarkable consequence from the comparison of the two representation theorems 3.7 and 3.8 via the outer extension theorem 1.3 in case $\bullet = \tau$: *If* $I : E \to [0,\infty]$ *and* $\varphi : \mathrm{Out}(E) \to [0,\infty]$ *is a pair as in* 3.7, *then condition* ($\Uparrow$) *for* I *is equivalent to condition* ($\uparrow$) *for* φ.

Note (added 17 August 2013). We want to complement the above inner representation theorem 3.2 in that we recall the former comprehensive inner result [2](19) theorem 1.3. It has been described in [2](19) sect.4 that this result is a far simultaneous extension of the former Daniell-Stone and Riesz representation theorems. As before we assume a positive-homogeneous Stonean lattice $E \subset [0,\infty[^X$ with $0 \in E$ and an isotone functional $I : E \to [0,\infty[$ with $I(0) = 0$. Then let $\mathfrak{S}$ be a lattice in X with $\varnothing \in \mathfrak{S}$ and

$$(\bullet) \quad \mathfrak{S} \subset (\mathrm{Inn(E)})_\bullet \text{ and } \mathrm{Inn(E)} \subset \mathfrak{S}\top\mathfrak{S}_\bullet.$$

We introduce the two related conditions

$\uparrow (I,\mathfrak{S})$: for any $f \in E$ and $\epsilon > 0$ there exists $S \in \mathfrak{S}$ such that all $u \in E$ with $u \leqq f$ and $u|S = f|S$ fulfil $I(f) \leqq I(u) + \epsilon$,

$\updownarrow (I,\mathfrak{S})$: for any $f \in E$ and $\epsilon > 0$ there exists $S \in \mathfrak{S}$ such that all $u,v \in E$ with $u \leqq v \leqq f$ and $u|S = v|S$ fulfil $I(v) \leqq I(u) + \epsilon$,

in order to express that the functional I is *concentrated on* $\mathfrak{S}$. The result in question then reads as follows.

3.9 Theorem ($\bullet = \sigma\tau$). *The following are equivalent.*

1) *There exists an inner* $\bullet$ *premeasure* $\vartheta : \mathfrak{S} \to [0,\infty[$ *which fulfils* $I(f) = ⨍ f d\vartheta_\bullet$ *for all* $f \in E$.

2.$\uparrow$) $I(v) = I(u) + I_\bullet(v-u)$ *for all* $u \leqq v$ *in* E, *and* $\uparrow (I,\mathfrak{S})$.

2.$\updownarrow$) $I(v) = I(u) + I_\bullet(v-u)$ *for all* $u \leqq v$ *in* E, *and* $\updownarrow (I,\mathfrak{S})$.

In this case the inner $\bullet$ *premeasure* ϑ *in* 1) *is unique and* $\vartheta := I^*|\mathfrak{S}$. *Moreover* $\vartheta_\bullet = \varphi_\bullet$ *for the inner* $\bullet$ *premeasure* $\varphi := I^*|\mathrm{Inn}(E)$ *of theorem* 3.2. *Hence* $I_\bullet(f) = ⨍ f d\vartheta_\bullet$ *for all* $f \in [0,\infty]^X$, *and the two extensions* $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ *and* $\Theta = \vartheta_\bullet|\mathfrak{C}(\vartheta_\bullet)$ *are equal.*

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