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Abstract. We solve a characterization problem for dual Hardy-space Toeplitz operators on the unit sphere $\mathbb{S}_n$ in $\mathbb{C}^n$ posed by Guediri in [9]. Our proof relies on the observation that dual Toeplitz operators on the orthogonal complement $H^2(\mathbb{S}_n)^\perp$ of the Hardy space in L^2 can be viewed as Toeplitz operators with respect to a suitable spherical isometry. This correspondence also allows us to determine the commutator ideal of the dual Toeplitz C^*-algebra.

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1 Introduction

Let $\mathbb{B}_n$ denote the open Euclidean unit ball in $\mathbb{C}^n$, $\mathbb{S}_n = \partial\mathbb{B}_n$ its boundary and σ the normalized surface measure on $\mathbb{S}_n$. Given an element $\varphi \in L^\infty(\sigma)$, the Toeplitz operator T_φ with symbol φ acting on the Hardy space

$$H^2(\sigma) = \overline{\mathbb{C}[z_1, \ldots, z_n]}^{\|\cdot\|_2} \subset L^2(\sigma)$$

is defined as the compression of the multiplication operator $M_\varphi : L^2(\sigma) \to L^2(\sigma)$, $f \mapsto \varphi f$ onto $H^2(\sigma)$, that is,

$$T_\varphi : H^2(\sigma) \to H^2(\sigma), \quad f \mapsto P_{H^2(\sigma)} M_\varphi f.$$

By definition a dual Toeplitz operator on $H^2(\sigma)^\perp$ is an operator of the form

$$S_\varphi : H^2(\sigma)^\perp \to H^2(\sigma)^\perp, \quad f \mapsto P_{H^2(\sigma)^\perp} M_\varphi f.$$

In contrast to the case of ordinary Toeplitz operators wich have been introduced more than half a century ago, the investigation of the dual case has just begun – at least in the Hardy space situation (see the recent work [9] of Guediri). On the Bergman space of the unit disc $\mathbb{D}$, these operators have been introduced and studied in detail by Stroethoff and Zheng [13] in 2002. A corresponding theory on the Hardy space $H^2(\partial\mathbb{D})$ over the unit disc contains nothing new, since there is a natural isomorphism $H^2(\partial\mathbb{D}) \cong H^2(\partial\mathbb{D})^\perp$, under which Toeplitz and dual Toeplitz operators are equivalent. But in complex dimension $n > 1$, new phenomena occur (see Proposition 1.3 and the subsequent remarks in [9], or [4] where it is shown that $T_z \in B(H^2(\sigma))^n$ is not even quasi-similar to $S_{\overline{z}}$ for $n > 1$).

From now on, we suppose that $n > 1$. Given $\varphi \in L^\infty(\sigma)$, the orthogonal decomposition $L^2(\sigma) = H^2(\sigma) \oplus H^2(\sigma)^\perp$ yields a representation of the multiplication operator $M_\varphi \in B(L^2(\sigma))$ as an operator-matrix of the form

$$M_\varphi = \begin{pmatrix} T_\varphi & H^*_{\overline{\varphi}} \\ H_\varphi & S_\varphi \end{pmatrix},$$

where $H_\varphi : H^2(\sigma) \to H^2(\sigma)^\perp, f \mapsto P_{H^2(\sigma)^\perp} M_\varphi f$, is the so-called Hankel operator with symbol φ. Various algebraic relations connecting the operators T_φ, S_φ and H_φ (see the equations (2.1) and Lemma 2.1 in [9]) result from this representation. Moreover, by $M^*_\varphi = M_{\overline{\varphi}}$ it follows immediately that $S^*_\varphi = S_{\overline{\varphi}}$.

The aim of this note is to demonstrate that the theory of dual Toeplitz operators fits into the more general context of Toeplitz operators with respect to spherical isometries. To be more specific, the tuple $T = (S_{\overline{z}_1}, \ldots, S_{\overline{z}_n}) \in B(H^2(\sigma)^\perp)^n$ is a spherical isometry, and the dual Toeplitz operators on $H^2(\sigma)^\perp$ are precisely the associated T-Toeplitz operators (see Proposition 2.1). This correspondence immediately leads to some known results on dual Toeplitz operators and allows us to establish a short exact sequence for the dual Toeplitz C^*-algebra $C^*(S_f : f \in C(\mathbb{S}_n))$ (see Section 3). As another application we solve a problem posed by Guediri ([9], Remark 3.3) concerning the characterization of dual Toeplitz operators (see Section 4). The following section contains the necessary background on spherical isometries and their associated Toeplitz operators.

2 Spherical isometries and dual Toeplitz operators

Let H be a separable complex Hilbert space. A spherical isometry $T \in B(H)^n$ is a commuting tuple of operators satisfying the algebraic condition

$$\sum_{i=1}^n T_i^* T_i = 1_H$$

which is modelled after the relation $\sum_{i=1}^n |z_i|^2 = 1$ describing the unit sphere in $\mathbb{C}^n$. Perhaps the most important example of a spherical isometry is given by the Toeplitz tuple $T_z = (T_{z_1}, \ldots, T_{z_n}) \in B(H^2(\sigma))^n$ on the Hardy space of the sphere. The study of general spherical isometries has been initiated by Athavale in 1990 who proved that they are subnormal and that their minimal normal extension is again a spherical isometry (see [3]). Note that a normal tuple is spherically isometric if and only if its spectrum is contained in the unit sphere. It is called a spherical unitary in this case.

Inspired by the prototypical example T_z, Prunaru developed a general theory of Toeplitz operators for spherical isometries which – in many respects – parallels the classical theory of Hardy-space Toeplitz operators on the sphere. Following Prunaru, we call an operator $X \in B(H)$ a Toeplitz operator with respect to a given spherical isometry $T \in B(H)^n$ (T-Toeplitz operator, for short) if the identity

$$\sum_{i=1}^{n} T_i^* X T_i = X$$

holds. Let $U \in B(K)^n$ be the minimal normal extension of T and let

$$\Psi_U : L^\infty(\mu) \to W^*(U)$$

be the von Neumann algebra isomorphism associated with a scalar spectral measure μ of U. By a result of Prunaru (Theorem 1.2 in [11]) the set $\mathcal{T}_T$ of all T-Toeplitz operators is given by

$$\mathcal{T}_T = P_H(U)'|H,$$

where $(U)'$ is the commutant of U in $B(K)$ and P_H denotes the orthogonal projection from K onto H. In particular, for $f \in L^\infty(\mu)$, the operator

$$T_f = P_H \Psi_U(f)|H$$

is a T-Toeplitz operator. We call T_f the T-Toeplitz operator with symbol f. If $W^*(U)$ ist maximal abelian, then

$$\mathcal{T}_T = \{T_f;\ f \in L^\infty(\mu)\}.$$

Details can be found in [11] or [7].

Let $T \in B(H)^n$ again be an arbitrary spherical isometry. In analogy with the classical case, we define the Toeplitz C^*-algebra associated with T by

$$\mathcal{T}_C(T) = C^*(T_f : f \in C(\mathbb{S}_n)).$$

We now want to point out how dual Toeplitz operators fit into this context. Towards this end, we assume in addition that the spherical isometry $T \in B(H)^n$ is pure in the sense that there is no non-zero reducing subspace $M \subset H$ for T such that $T|M$ is normal. Generalizing a concept introduced by Conway [5] for a single subnormal operator, Athavale defines in [2] the dual $\widetilde{T} \in B(H^\perp)^n$ of a pure subnormal tuple $T \in B(H)^n$ with minimal normal extension $U \in B(K)^n$ as

$$\widetilde{T} = U^*|H^\perp \quad \text{where } H^\perp = K \ominus H.$$

The assumption on T to be pure guarantees that U^* is the minimal normal extension of $\widetilde{T}$ (see [2], Remark 3). As the restriction of the spherical unitary U^* to an invariant subspace, $\widetilde{T}$ is a also spherical isometry and hence possesses an associated space of Toeplitz operators $\mathcal{T}_{\widetilde{T}}$ in the sense defined above.

Let us now, for later use, determine a formula for $\widetilde{T}_f$ in terms of the functional calculus Ψ_U. For a set $A \subset \mathbb{C}^n$, let $A^* = \{\overline{z} : z \in A\}$ denote the set obtained from

A by complex conjugation. If we define a Borel measure $\tilde{\mu}$ on $\sigma(U^*) = \sigma(U)^*$ by setting $\widetilde{\mu}(A) = \mu(A^*)$ for every Borel subset $A \subset \sigma(U^*)$, then the map

$$L^\infty(\widetilde{\mu}) \to L^\infty(\mu), \quad f \mapsto \widetilde{f} \qquad \text{where } \widetilde{f}(z) = \overline{f(\overline{z})}$$

is a conjugate linear isomorphism of von Neumann algebras, and hence the composition

$$L^\infty(\widetilde{\mu}) \ni f \mapsto \Psi_U(\widetilde{f})^* \in W^*(U)$$

is an isomorphism of von Neumann algebras. Since it maps z_i to U_i^* for $i = 1, \ldots, n$, the measure $\widetilde{\mu}$ is (mutually absolutely continuous with respect to) a scalar-valued spectral measure of U^* and the map described above is the canonical L^∞-functional calculus $\Psi_{U^*} : L^\infty(\widetilde{\mu}) \to W^*(U^*) = W^*(U)$ of the normal tuple U^*. Therefore, the $\widetilde{T}$-Toeplitz operator with symbol $f \in L^\infty(\widetilde{\mu})$ is given by the formula

$$\widetilde{T}_f = P_{H^\perp} \Psi_U(\widetilde{f})^* | H^\perp.$$

Applying the above observations to the Hardy space multiplication tuple $T = T_z \in B(H^2(\sigma))^n$, which is a pure spherical isometry, we are now able to identify the dual Toeplitz operators as the Toeplitz operators associated with the dual tuple $\widetilde{T_z}$. To simplify the notation, we set $T = T_z$, $\widetilde{T} = \widetilde{T_z}$ and write $\widetilde{T}_f$ for the Toeplitz operator with symbol $f \in L^\infty(\widetilde{\sigma})$ with respect to the spherical isometry $\widetilde{T}$. Note that in this case the measures $\widetilde{\sigma}$ and σ coincide.

2.1 Proposition. *(a) The dual $\widetilde{T}$ of T_z is the tuple $S_{\overline{z}} = (S_{\overline{z}_1}, \ldots, S_{\overline{z}_n})$ on $H^2(\sigma)^\perp$, which is a spherical isometry with empty point spectrum.*

(b) We have the identity $\widetilde{T}_f = S_{f^c}$ where $f^c(z) = f(\overline{z})$ for every $f \in L^\infty(\sigma)$.

(c) An operator X is a dual Toeplitz operator if and only if $\sum_{i=1}^n S_{z_i} X S_{\overline{z}_i} = X$.

Proof. It is well known that the minimal normal extension U of $T = T_z$ is the multiplication tuple $M_z = (M_{z_1}, \ldots, M_{z_n}) \in B(L^2(\sigma))^n$. Since the scalar-valued spectral measure $\widetilde{\sigma}$ of U^* does not have one-point atoms, we have $\emptyset = \sigma_p(U^*) \supset \sigma_p(S_{\overline{z}})$, as desired. Moreover, from the very definition of the dual tuple, we have

$$\widetilde{T} = U^* | H^\perp = M_{\overline{z}} | H^2(\sigma)^\perp = S_{\overline{z}} \in B(H^2(\sigma)^\perp)^n,$$

as stated in part (a). More generally, since the canonical L^∞-functional calculus of U is given by $\Psi_U(f) = M_f$ $(f \in L^\infty(\sigma))$, it follows by the considerations preceding the proposition that $\widetilde{T}_f = P_{H^2(\sigma)^\perp} M_{\widetilde{f}}^* | H^2(\sigma)^\perp = S_{f^c}$ $(f \in L^\infty(\widetilde{\sigma}))$. This proves part (b). To prove part (c), recall that $S_{\overline{z}_i}^* = S_{z_i}$ for $i = 1, \ldots, n$. Hence by definition an operator $X \in B(H^2(\sigma)^\perp)$ is a $\widetilde{T}$-Toeplitz operator with respect to the spherical isometry $\widetilde{T} = S_{\overline{z}}$ if

$$X = \sum_{i=1}^n S_{z_i} X S_{\overline{z}_i}.$$

Since $W^*(U^*) = W^*(U) = \{M_f;\ f \in L^\infty(\sigma)\}$ is maximal abelian, the set of all $\widetilde{T}$-Toeplitz operators is given by

$$\mathcal{T}_{\widetilde{T}} = \{\widetilde{T}_f;\ f \in L^\infty(\widetilde{\sigma})\}.$$

By part (b) this set consists precisely of all dual Toeplitz operators on $H^2(\sigma)^\perp$. □

3 An exact sequence for the dual Toeplitz C^*-algebra

Since there is an extensive theory of Toeplitz operators associated with spherical isometries (see e.g. [11], [6], [7]), Proposition 2.1 has many direct consequences. In the following proposition, we state only a few of them (see Prunaru [11] as well as Corollary 3.6 in [7]). Here, as in the sequel, $\mathcal{K}(H)$ denotes the ideal of all compact operators on the Hilbert space H. The symbol $\mathcal{R}(f)$ occurring in part (a) denotes the essential range of a given function f. Note that $\mathcal{R}(f) = \mathcal{R}(f^c)$.

3.1 Proposition. *(a) For $f \in L^\infty(\sigma)$, the inclusion $\sigma(S_f) \supset \mathcal{R}(f)$ holds.*

(b) The map $L^\infty(\sigma) \to B(H^2(\sigma)^\perp),\ f \mapsto S_f$, is an isometry.

(c) For every $f \in L^\infty(\sigma)$, the equality $\|S_f\| = \inf_{K \in \mathcal{K}(H^2(\sigma)^\perp)} \|S_f - K\|$ holds. In particular, the only compact dual Toeplitz operator is the zero-operator. □

All parts of the proposition follow directly from Corollary 3.6 in [7]. For a direct proof, not using the theory of Toeplitz operators associated with spherical isometries, see the recent article of Guediri (Corollary 2.6 and Remark 2.11 in [9]).

A classical problem in the context of Toeplitz operators is to determine the structure of operator algebras associated with special symbol classes, first of all $C(\mathbb{S}_n)$ which–in our context–gives rise to the dual Toeplitz C^*-algebra

$$\mathcal{T}_C(S_{\overline{z}}) = C^*(S_f : f \in C(\mathbb{S}_n)).$$

The following theorem shows that the situation is very similar to the classical case.

3.2 Theorem. *There is a unique short exact sequence of C^*-algebras*

$$0 \longrightarrow \mathcal{K}(H^2(\sigma)^\perp) \longrightarrow \mathcal{T}_C(S_{\overline{z}}) \stackrel{s}{\longrightarrow} C(\mathbb{S}_n) \longrightarrow 0,$$

where the symbol homomorphism s maps the dual Toeplitz operator S_f to its symbol f for every $f \in C(\mathbb{S}_n)$.

Proof. Prunaru's work [11] on spherical isometries applied to $\widetilde{T}_z = S_{\overline{z}}$ implies the existence of a short exact sequence of the form

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{T}_C(S_{\overline{z}}) \stackrel{\tau}{\longrightarrow} C(\mathbb{S}_n) \longrightarrow 0$$

with $\tau(\widetilde{T}_f) = f$ for every $f \in C(\mathbb{S}_n)$, where $\mathcal{C}$ is the commutator ideal (see Corollary 3.8 in [7] for a detailed proof). Replacing the map τ by $s = \tau(\cdot)^c$, the sequence remains exact and $s(S_f) = \tau(\widetilde{T}_{f^c})^c = f$ for all $f \in C(\mathbb{S}_n)$.

Since the tuple $T = T_z$ on $H^2(\sigma)$ is essentially normal (i.e., all self-commutators $[T_i, T_i^*]$ are compact), so is its dual $S_{\overline{z}}$ (see Athavale [2], Remark 2). The essential normality of $S_{\overline{z}}$ guarantees that the commutator ideal $\mathcal{C}$ is contained in the compact operators (see Lemma 3.9 in [7]). To finish the proof, it suffices to show that $\mathcal{T}_C(S_{\overline{z}})$ is irreducible (see [7], Proposition 3.10). This follows from the next lemma. □

3.3 Lemma. *The tuple $S_{\overline{z}} \in B(H^2(\sigma)^\perp)^n$ has no proper reducing subspaces.*

Proof. Suppose that $P \in B(H^2(\sigma)^\perp)$ is a projection commuting with $S_{\overline{z}_i}$ for all $i = 1, \ldots, n$. Then we have

$$\sum_{i=1}^n S_{z_i} P S_{\overline{z}_i} = \sum_{i=1}^n S_{z_i} S_{\overline{z}_i} P = \sum_{i=1}^n S_{|z_i|^2} P = P \qquad (i = 1, \ldots, n).$$

It follows that $P = S_g$ is a dual Toeplitz operator (see Proposition 2.1) whose symbol $g \in L^\infty(\sigma)$ is self-adjoint and satisfies $g^2 = g$ (here we use Proposition 3.1 (a)). By using the elementary identity $S_{fg} = H_f H_{\overline{g}}^* + S_f S_g$ $(f, g \in L^\infty)$ with $f = g = \overline{g}$, we obtain that

$$H_g H_g^* = S_{g^2} - S_g^2 = S_g - S_g = 0.$$

From $0 = H_g 1 = P_{H^2(\sigma)^\perp} g$ it follows that $g \in L^\infty(\sigma) \cap H^2(\sigma) = H^\infty(\sigma)$. Since the boundary value map $r : H^\infty(\mathbb{B}_n) \to H^\infty(\sigma)$, $f \mapsto f^*$, is an algebra isomorphism with inverse given by the Poisson transform $H^\infty(\sigma) \to H^\infty(\mathbb{B}_n)$, $f \mapsto P[f]$ (see [12], Theorem 4.3.3 and Theorem 5.6.8), it follows that $G = P[g] \in H^\infty(\mathbb{B}_n)$ is an idempotent. But then $G = 0$ or $G = 1$ and the same holds for g in $L^\infty(\sigma)$. Thus $P = S_g$ is either the zero operator or the identity on $H^2(\sigma)^\perp$ as was to be shown. □

It should be remarked that the assertion of the above lemma also follows from the Brown-Halmos-type theorem for dual Toeplitz operators due to Guediri (see Theorem 5.1 in [9] and Corollary 5.6 in [9]). For another argument showing the irreducibility of duals of irreducible pure subnormal tuples see Athavale [2].

4 A characterization of (dual) Toeplitz operators

In [9], Guediri poses a question concerning a characterization of dual Toeplitz operators using automorphisms of the unit ball. In order to formulate this problem here, we have to introduce some notations. Let $\mathrm{Aut}(\mathbb{B}_n)$ be the automorphism group of the unit ball. For $w \in \mathbb{B}_n$, we denote by $\varphi_w \in \mathrm{Aut}(\mathbb{B}_n)$ the standard automorphism mapping w to 0 which is explicitly given by the formula

$$\varphi_w(z) = \frac{w - P_w(z) - \sqrt{1 - |w|^2} Q_w(z)}{1 - \langle z, w \rangle} \qquad (z \in \mathbb{B}_n).$$

Here P_w denotes the orthogonal projection onto the one-dimensional subspace of $\mathbb{C}^n$ spanned by w if $w \neq 0$ and $P_0 = 0$, and $Q_w = 1 - P_w$. Motivated by the fact that the reproducing kernel K_w of the Hardy space $H^2(\mathbb{B}_n)$ satisfies the identity

$$K_w^{-1}(z) = (1 - \langle z, w\rangle)^n = \sum_{j=0}^{n} \sum_{|m|=j} \binom{n}{j} (-1)^j \frac{j!}{m!} z^m \overline{w}^m \qquad (z, w \in \mathbb{B}_n),$$

Guediri defines for every $w \in \mathbb{B}_n$ the operator $\mathcal{S}_w : B(H^2(\sigma)^\perp) \to B(H^2(\sigma)^\perp)$ by

$$\mathcal{S}_w(X) = \sum_{j=0}^{n} \sum_{|m|=j} \binom{n}{j} (-1)^j \frac{j!}{m!} S_{\varphi_w^m} X S_{\overline{\varphi_w^m}}$$

and observes (see [9], Proposition 3.2) that, given any $w \in \mathbb{B}_n$,

$$\mathcal{S}_w(X) = 0 \quad \text{holds for every dual Toeplitz operator } X \in B(H^2(\sigma)^\perp).$$

In Remark 3.3 of the cited paper, he asks if this condition (even for some fixed $w \in \mathbb{B}_n$) is also sufficient for X to be a dual Toeplitz operator.

The aim of this section is to answer this question in the affirmative.

Given a spherical isometry $T \in B(H)^n$ and a function $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$, the tuple $\varphi(T)$ can be defined in the following way. As an automorphism of the unit ball, φ extends to a holomorphic map $\varphi_U : U \to \mathbb{C}^n$, where $U \supset \overline{\mathbb{B}_n}$ is an open neighborhood. Since any two such extensions φ_U and φ_V coincide on the connected component of $U \cap V$ containing $\overline{\mathbb{B}}_n$, and since the holomorphic functional calculus is compatible with restrictions, it makes perfect sense to define

$$\varphi(T) = \varphi_U(T),$$

where φ_U is any extension of φ as above and $\varphi_U(T)$ is formed using Taylor's holomorphic functional calculus. If $\Psi : L^\infty(\mu) \to B(K)$ denotes the canonical L^∞-functional calculus of the minimal normal extension of T and φ has the components $\varphi_i : \mathbb{B}_n \to \mathbb{C}$, one can show that $\varphi(T) = (\Psi(\varphi_1)|H, \ldots, \Psi(\varphi_n)|H)$ using the continuity of the holomorphic functional calculus and the Taylor expansions of the functions φ_i at the origin. The identity

$$\sum_{i=1}^{n} \|\Psi(\varphi_i)x\|^2 = \sum_{i=1}^{n} \langle \Psi(|\varphi_i|^2)x, x\rangle = \|x\|^2 \qquad (x \in H)$$

shows that the tuple $\varphi(T)$ is a spherical isometry again.

For a given commuting tuple $S \in B(H)^n$, we write $\mathcal{A}_S$ for the dual algebra generated by the components of S and the identity, that is,

$$\mathcal{A}_S = \overline{\mathbb{C}[S]}^{w^*} \subset B(H).$$

4.1 Lemma. *For $\varphi \in Aut(\mathbb{B}_n)$, the dual algebras $\mathcal{A}_T$ and $\mathcal{A}_{\varphi(T)}$ coincide.*

Proof. Fix an arbitrary $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$ as well as a holomorphic extension $\varphi_U : U \to \mathbb{C}^n$ as described above. Choose $r > 1$ such that $\overline{B}_r(0) = \{z \in \mathbb{C}^n;\ |z| \leq r\} \subset U$. Since the components of φ_U are uniform limits of polynomials on $\overline{B}_r(0)$, the components of the tuple $\varphi(T) = \varphi_U(T)$ belong to the norm closed subalgebra of $B(H)$ generated by T. Consequently, $\mathcal{A}_{\varphi(T)} \subset \mathcal{A}_T$. For the reverse inclusion, it suffices to verify that $\varphi^{-1}(\varphi(T)) = T$. To see this, write $\psi = \varphi^{-1}$ and choose extensions ψ_V and φ_U in such a way that $\psi_V \circ \varphi_U$ is well defined. By the identity theorem the identity $\psi_V \circ \phi_U = id$ holds on the connected component of U containing $\overline{\mathbb{B}}_n$. Since the holomorphic functional calculus is compatible with compositions (see Theorem 5.2.3 in [8]), we have $\varphi^{-1}(\varphi(T)) = \psi_V(\varphi_U(T)) = \psi_V \circ \varphi_U(T) = T$. Thus we obtain that $\mathcal{A}_T = \mathcal{A}_{\varphi^{-1}(\varphi(T))} \subset \mathcal{A}_{\varphi(T)}$ from the first part of the proof. □

By definition, T-Toeplitz operators are the fixed points of the unital and completely positive mapping

$$\Sigma_T : B(H) \to B(H), \quad X \mapsto \sum_{i=1}^{n} T_i^* X T_i.$$

Based on this observation, Prunaru was able to define a completely positive projection $\Phi_T : B(H) \to B(H)$ whose range is precisely the set $\mathcal{T}_T = \ker(1 - \Sigma_T)$ of all T-Toeplitz operators (see Lemma 2.7 in [11]). From the construction of Φ_T it follows immediately that $A^* \Phi_T(X) B = \Phi_T(A^* X B)$ for $A, B \in (T)'$. As a simple consequence we obtain the following property of the completely positive mapping Σ_T.

4.2 Lemma. *The identity* $\ker(1 - \Sigma_T) = \ker(1 - \Sigma_T)^2$ *holds.*

Proof. For the non-trivial inclusion, fix an operator $X \in \ker(1 - \Sigma_T)^2$. This means precisely that $Y = (1 - \Sigma_T)(X)$ is a T-Toeplitz operator. Using the Toeplitz projection Φ_T, we deduce that

$$Y = \Phi_T(Y) = \Phi_T\left(X - \sum_{i=1}^{n} T_i^* X T_i\right) = \Phi_T(X) - \sum_{i=1}^{n} T_i^* \Phi_T(X) T_i = 0,$$

as desired. □

Combining the above observations with results proved earlier by Prunaru in [11] and the authors in [6], we obtain the following characterizations of Toeplitz operators for spherical isometries.

4.3 Theorem. *Let $T \in B(H)^n$ be a spherical isometry with minimal normal extension $U \in B(K)^n$, and let $\varphi \in Aut(\mathbb{B}_n)$ be an arbitrary automorphism of the unit ball. The following conditions on an operator $X \in B(H)$ are equivalent:*

(a) *X is a T-Toeplitz operator (in the sense that $\sum_{i=1}^n T_i^* X T_i = X$);*

(b) *there exists an operator $Y \in (U)'$ such that $X = P_H Y|H$;*

(c) *for every isometry $J \in \mathcal{A}_T$, we have $J^* X J = X$;*

(d) *for some, or equivalently, every natural number $N \geq 1$, the identity*

$$\sum_{j=0}^{N} \sum_{|m|=j} \binom{N}{j} (-1)^j \frac{j!}{m!} T^{m*} X T^m = 0$$

holds;

(e) *X is a $\varphi(T)$-Toeplitz operator;*

(f) *for some, or equivalently, every natural number $N \geq 1$, the identity*

$$\sum_{j=0}^{N} \sum_{|m|=j} \binom{N}{j} (-1)^j \frac{j!}{m!} \widetilde{\varphi}^m(T)^* X \widetilde{\varphi}^m(T) = 0$$

holds.

Proof. The equivalence of (a) – (c) follows from the two references cited above. Lemma 4.1 guarantees the equivalence of (c) and (e). Since

$$(1 - \Sigma_T)^n(X) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \Sigma_T^j(X) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \sum_{|m|=j} \frac{j!}{m!} T^{m*} X T^m,$$

Lemma 4.2 shows that (a) and (d) are equivalent. To complete the proof, note that $\widetilde{\varphi} \in \mathrm{Aut}(\mathbb{B}_n)$ and that $\widetilde{\varphi}(T)^m = \widetilde{\varphi}^m(T)$ for all $m \in \mathbb{N}^n$. □

As an immediate consequence we obtain the characterization of dual Toeplitz operators that Guediri asked for in Remark 3.3 of [9].

4.4 Corollary. *Let $w \in \mathbb{B}_n$ be arbitrary. An operator $X \in B(H^2(\sigma)^\perp)$ is a dual Toeplitz operator if and only if $\mathcal{S}_w(X) = 0$.*

Proof. By Proposition 2.1, the dual Toeplitz operators are precisely the $S_{\overline{z}}$-Toeplitz operators. Define $\widetilde{T} = S_{\overline{z}}$ and fix an automorphism $\varphi = \varphi_w \in \mathrm{Aut}(\mathbb{B}_n)$. By Theorem 4.3 an operator $X \in B(H^2(\sigma)^\perp)$ is a dual Toeplitz operator if and only if

$$\sum_{j=0}^{n} \sum_{|m|=j} \binom{n}{j} (-1)^j \frac{j!}{m!} \widetilde{\varphi}^m(\widetilde{T})^* X \widetilde{\varphi}^m(\widetilde{T}) = 0.$$

The observation that (see Proposition 2.1 and the remarks preceding Lemma 4.1)

$$\widetilde{\varphi}^m(\widetilde{T}) = \widetilde{T}_{\widetilde{\varphi}^m} = S_{(\widetilde{\varphi}^m)^c} = S_{\overline{\varphi^m}}$$

completes the proof. □

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