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Abstract

We discuss existence and regularity results for multi-channel images in the setting of isotropic and anisotropic variants of the TV-model

1 Introduction

In our note we consider a multi-channel image

$$f : \Omega \to \mathbb{R}^N,\ f = (f^1, ..., f^N),\ N \geq 1,$$

defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, and try to denoise f by applying a minimization procedure

$$I[u] := \int_\Omega F(\nabla u)\,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |u - f|^2\,\mathrm{d}x \to \min, \tag{1.1}$$

where the minimizer $u : \Omega \to \mathbb{R}^N$ is sought in a suitable class of mappings $w : \Omega \to \mathbb{R}^N$ depending on the growth of the prescribed density F. We will mainly concentrate on more regular variants $F(\nabla u)$ (being convex and of linear growth) of the total variation TV-density $|\nabla u|$. More precisely, we consider two cases:

- The **isotropic** case: $F(\nabla u) = \varphi\big(\mathrm{trace}\,(\nabla u \nabla u^T)\big)$.
- The **anisotropic** case: $F(\nabla u) = \mathrm{trace}\,\varphi\left(\nabla u \nabla u^T\right)$.

The notions of isotropy and anisotropy are motivated by the corresponding gradient descent evolutions which are diffusion–reaction equations: the isotropic setting leads to a diffusion process with a scalar-valued diffusivity, while the anisotropic case uses a matrix-valued diffusion tensor [33].

Our goal is to prove existence, regularity and approximation results in both cases for functions $\varphi : [0, \infty) \to [0, \infty)$ of the principle form

$$\varphi(s) = \left\{ \begin{array}{c} \sqrt{\varepsilon^2 + s} - \varepsilon,\ \varepsilon > 0 \\ \text{or} \\ \Phi_\mu(\sqrt{s}),\ \mu > 1 \end{array} \right\},\ s \geq 0. \tag{1.2}$$

For details including an explanation of the terminology we refer to Section 2 and Section 3, respectively.

Let us now have a closer look on the history of the problem and its variants studied in image analysis. While there has been a long tradition of using regularization methods in the context of ill-posed problems [30], early quadratic regularization approaches for image analysis problems go back to the 1980s [7]. These concepts have been generalised to energies with non-quadratic regularizing functions F that are either convex [28] or nonconvex [25]. They can be related to the nonlinear diffusion filter of Perona and Malik [26]. It is fairly straightforward to extend this diffusion filter to vector-valued images in the isotropic case [23] and to establish corresponding energies. For matrix-valued data sets, isotropic nonquadratic models have been pioneered in [31]. Anisotropic regularization approaches for vector-valued images have been introduced in [33], and their matrix-valued counterparts have been considered first in [32].

The popular TV-regularization approach of Rudin et al. [27] uses the total variation seminorm as regularizing function $F(\nabla u)$. An early extension of the TV-regularizer to color images has been considered in [16], and numerous variations of this idea using different channel couplings have been proposed within the last two decades; see e.g. [19] and the references therein. A TV-regularization approach for matrix-valued images goes back to [17]. In [6] an anisotropic but rotationally invariant extension of the TV-regularizer has been introduced. For further references and a review on the large body of work on TV-regularization in image analysis we refer to [15].

2 Isotropic Regularization

In this section we discuss the following version of the variational problem (1.1)

$$J[u] := \int_\Omega \varphi\big(\mathrm{trace}\,(\nabla u \nabla u^T)\big)\,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |u-f|^2\,\mathrm{d}x \to \min \ \text{ in } W^{1,1}(\Omega,\mathbb{R}^N) \tag{2.1}$$

with a given function $f:\Omega\to\mathbb{R}^N$ for which we require

$$f \in L^2(\Omega,\mathbb{R}^N). \tag{2.2}$$

We recall that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and that λ denotes some positive number. In what follows, $|\cdot|$ is the Euclidean norm of vectors and matrices, in particular we have $|\nabla u| = \mathrm{trace}\,\big(\nabla u \nabla u^T\big)^{1/2}$ for $\nabla u = \big(\partial_\alpha u^i\big)_{1\le\alpha\le n}^{1\le i\le N}$. Hence we can write $J[u]$ as

$$J[u] = \int_\Omega \psi\big(|\nabla u|\big)\,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |u-f|^2\,\mathrm{d}x$$

with $\psi(s) := \varphi(s^2)$. On the data f we can even impose an extra side condition like

$$f(x) \in K \tag{2.3}$$

for a closed convex subset of $\mathbb{R}^N$, e.g. we can study the case ($N=m^2$, $\mathbb{R}^{m\times m}$:=space of $(m\times m)$-matrices)

$$K = \mathbb{S}^m := \Big\{A = (a_{i,j})_{1\le i,j\le m} \in \mathbb{R}^{m\times m} : a_{ij} = a_{ji},\ i,j = 1,...,m,$$
$$\sum_{i,j=1}^m a_{ij}\xi_i\xi_j \ge \alpha|\xi|^2 \text{ for all } \xi\in\mathbb{R}^m\Big\},$$

where $\alpha \ge 0$ is fixed. Thus K consists of all symmetric $(m\times m)$-matrices A being α-positive (semi-)definite. Concerning the density $\psi:\mathbb{R}\to[0,\infty)$ our assumptions are as follows (and of course partially can be weakened, compare section 3):

$$\psi \in C^2(\mathbb{R}), \psi(-y) = \psi(y), \psi(0) = 0, \tag{2.4}$$
$$|\psi'(y)| \le \nu_1, \tag{2.5}$$
$$\psi(y) \ge \nu_2|y| - \nu_3, \tag{2.6}$$
$$\psi''(y) > 0 \tag{2.7}$$

for all $y \in \mathbb{R}$ and with constants $\nu_1, \nu_2 > 0$, $\nu_3 \in \mathbb{R}$. Thus

$$F(p) := \psi\big(|p|\big), p \in \mathbb{R}^{n\times N}, \tag{2.8}$$

is a strictly convex energy density of linear growth including examples like ($\varepsilon > 0$)

$$F(p) := \sqrt{\varepsilon^2 + |p|^2} - \varepsilon,\ p \in \mathbb{R}^{n\times N}, \tag{2.9}$$

and ($\mu > 1$)

$$F(p) := \Phi_\mu\big(|p|\big),\ \Phi_\mu(t) := \int_0^t \int_0^s (1+r)^{-\mu}\, dr\, ds,\ p \in \mathbb{R}^{n\times N},\ t \geq 0. \tag{2.10}$$

Recall that we have the following explicit formulas for the functions Φ_μ

$$\begin{cases} \Phi_\mu(t) = \dfrac{1}{\mu-1} t + \dfrac{1}{\mu-1}\dfrac{1}{\mu-2}(t+1)^{-\mu+2} - \dfrac{1}{\mu-1}\dfrac{1}{\mu-2},\ \mu \neq 2, \\ \Phi_2(t) = t - \ln(1+t),\ t \geq 0, \end{cases} \tag{2.11}$$

and from (2.11) we infer that Φ_μ approximates the TV-density in the sense that

$$\lim_{\mu\to\infty} (\mu-1)\Phi_\mu\big(|p|\big) = |p|,\ p \in \mathbb{R}^{n\times N}. \tag{2.12}$$

As a matter of fact – under the above assumptions on the data – problem (2.1) in general fails to have a solution in the Sobolev space $W^{1,1}(\Omega, \mathbb{R}^N)$ and we therefore pass to the relaxed variant of (2.1) formulated in the space $BV(\Omega, \mathbb{R}^N)$ of vector-valued functions with finite total variation (see e.g. [2], [24] for a definition and further properties of this space). The relaxed variational problem then reads

$$\begin{aligned} K[w] := \int_\Omega \psi\big(|\nabla^a w|\big)\, \mathrm{d}x + \psi'_\infty \cdot |\nabla^s w|(\Omega) + \frac{\lambda}{2}\int_\Omega |w-f|^2\, \mathrm{d}x \\ \to \min\ \text{in}\ BV(\Omega, \mathbb{R}^N), \quad \psi'_\infty := \lim_{y\to\infty} \psi'(y) \in (0,\infty), \end{aligned} \tag{2.13}$$

where $\nabla w = \nabla^a w \mathcal{L}^n + \nabla^s w$ is the Lebesgue decomposition of the tensor-valued Radon measure ∇w in its regular and singular part w.r.t. Lebesgue's measure $\mathcal{L}^n$. For details concerning the relaxation procedure the reader is referred e.g. to [10–12, 20, 22, 29]. We wish to note that

$$J[w] = K[w] \text{ for all } w \in W^{1,1}(\Omega, \mathbb{R}^N), \tag{2.14}$$

moreover, by standard embedding theorems (compare [1] and [2]) the finiteness of $\int_\Omega |w-f|^2\, \mathrm{d}x$ for $w \in W^{1,1}(\Omega, \mathbb{R}^N)$ or $w \in BV(\Omega, \mathbb{R}^N)$ is only guaranteed if $n = 2$. Let us now state our first result:

Theorem 2.1
Assume that we have (2.2), (2.3) for the data f with $K \subset \mathbb{R}^N$ closed and convex. Then the minimization problem (2.13) *admits a unique solution $u \in BV(\Omega, \mathbb{R}^N)$. The minimizer respects the side-condition (2.3), i.e. we have the "maximum principle"*

$$u(x) \in K \text{ for almost all } x \in \Omega. \tag{2.15}$$

Moreover it holds

$$\inf_{u \in BV(\Omega, \mathbb{R}^N)} K[u] = \inf_{w \in W^{1,1}(\Omega, \mathbb{R}^N)} J[w] \tag{2.16}$$

with J defined in (2.1).

Remark 2.1
We emphasize that in (2.13) the unconstrained problem is considered, i.e. we do not impose the condition $w(x) \in K$ a.e. on the comparison functions $w \in BV(\Omega, \mathbb{R}^N)$. It just turns out that the unconstrained minimizer u satisfies a kind of maximum-principle better known as convex-hull-property.

Concerning the regularity of the minimizer we have

Theorem 2.2
Under the assumptions and with the notation from Theorem 2.1 we impose the following additional requirements on the data f and ψ:

$$f \in L^\infty(\Omega, \mathbb{R}^N), \tag{2.17}$$

$$\begin{cases} \nu_4(1+t)^{-\mu} \leq \min\left\{\dfrac{\psi'(t)}{t}, \psi''(t)\right\}, \\ \max\left\{\dfrac{\psi'(t)}{t}, \psi''(t)\right\} \leq \nu_5 \dfrac{1}{1+t} \end{cases} \tag{2.18}$$

for all $t > 0$ with positive constants ν_4, ν_5 and with exponent $\mu > 1$. Then, in the case

$$\mu < 2, \tag{2.19}$$

problem (2.1) has a solution in the space $W^{1,1}(\Omega, \mathbb{R}^N)$. Moreover, u has Hölder continuous first derivatives in the interior of Ω.

Remark 2.2 *i) From (2.18) it follows that the density F introduced in (2.8) is μ-elliptic in the sense of*

$$\nu_4\big(1+|p|\big)^{-\mu}|q|^2 \le D^2F(p)(q,q) \le \nu_5\frac{|q|^2}{1+|p|},\ p,q\in\mathbb{R}^{n\times N}. \tag{2.20}$$

We remark that example (2.9) satisfies (2.20) with exactly $\mu = 3$, whereas F from (2.10) satisfies (2.20) precisely with the given value of μ.

ii) W.r.t. regularity results the bound on μ stated in (2.19) is rather optimal, since even in the case $n = 1 = N$ there are counterexamples of singular solutions, if the case $\mu > 2$ is considered. We refer the reader to [21].

Concerning the proofs we just note that Theorem 2.2 is a direct consequence of the results obtained in [11], [13] and [29], whereas the existence part of Theorem 2.1 has been established in a very general framework in [22]. It therefore remains to justify (2.15) for the unique solution u of problem (2.13). We need the following observation.

Lemma 2.1
Consider a closed convex subset K of $\mathbb{R}^N$ and let $\pi : \mathbb{R}^N \to K$ denote the nearest-point-projection onto K, which means that $y_0 := \pi(y)$ is the unique solution of

$$|y-y_0| = \inf_{z\in K}|y-z|. \tag{2.21}$$

The point y_0 is characterized through the variational inequality

$$(y-y_0)\cdot(v-y_0)\le 0 \quad \forall v\in K. \tag{2.22}$$

Moreover, the mapping π is non-expansive, which means

$$|\pi(y)-\pi(y')| \le |y-y'| \quad \forall y,y'\in\mathbb{R}^N. \tag{2.23}$$

Let us briefly comment on (2.23): by (2.22) we have

$$(y-\pi(y))\cdot(\pi(y')-\pi(y))\le 0$$

as well as

$$(y'-\pi(y'))\cdot(\pi(y)-\pi(y'))\le 0,$$

thus

$$(y'-\pi(y')+\pi(y)-y)\cdot(\pi(y)-\pi(y'))\le 0$$

and therefore

$$|\pi(y)-\pi(y')|^2 \le (y-y')\cdot(\pi(y)-\pi(y')),$$

which implies (2.23).

Now, if f satisfies (2.3), we obtain from Lemma 2.1

$$|\pi(w)-f| = |\pi(w)-\pi(f)| \le |w-f|$$

a.e. on Ω for any measurable function $w:\Omega\to\mathbb{R}^N$, thus $\pi(u)=u$ and thereby (2.15) holds for our BV-solution of problem (2.13) (recall that we have uniqueness), provided we can show that

$$\int\limits_\Omega \psi\big(|\nabla^a\pi(w)|\big)\,\mathrm{d}x + \psi'_\infty|\nabla^s\pi(w)|(\Omega) \le \int\limits_\Omega \psi\big(|\nabla^a w|\big)\,\mathrm{d}x + \psi'_\infty|\nabla^s w|(\Omega) \tag{2.24}$$

holds for $w\in BV(\Omega,\mathbb{R}^N)$. Inequality (2.24) can be obtained along the lines of the proof of Theorem 1 in [9], however, since the arguments used in this paper are rather technical, we prefer to give a more direct proof of (2.15). For $\delta>0$ let $F_\delta(p) := \frac{\delta}{2}|p|^2 + F(p)$, $p\in\mathbb{R}^{n\times N}$, with F from (2.8) and consider the unique solution of the problem

$$J_\delta[w] := \int\limits_\Omega F_\delta(\nabla w)\,\mathrm{d}x + \frac{\lambda}{2}\int\limits_\Omega |w-f|^2\,\mathrm{d}x \to \min \text{ in } W^{1,2}(\Omega,\mathbb{R}^N). \tag{2.25}$$

From [22], (4.14), it follows that u_δ is a K-minimizing sequence converging e.g. in $L^1(\Omega,\mathbb{R}^N)$ and a.e. on Ω to our K-minimizer u. As remarked above we deduce from (2.3) and (2.23) the validity of

$$\int\limits_\Omega |\pi(u_\delta)-f|^2\,\mathrm{d}x \le \int\limits_\Omega |u_\delta - f|^2\,\mathrm{d}x,$$

whereas from Lemma B.1 in [8] it follows

$$|\partial_\nu(\pi(u_\delta))| \le \mathrm{Lip}(\pi)|\partial_\nu u_\delta| = |\partial_\nu u_\delta|$$

a.e. on Ω, $\nu=1,...,n$. This yields ($\tilde{u}_\delta := \pi(u_\delta)$)

$$|\nabla\tilde{u}_\delta| = \left(\sum_{\nu=1}^n |\partial_\nu\tilde{u}_\delta|^2\right)^{1/2} \le \left(\sum_{\nu=1}^n |\partial_\nu u_\delta|^2\right)^{1/2} = |\nabla u_\delta|$$

and the structure of F_δ finally implies $J_\delta[\tilde{u}_\delta] \le J_\delta[u_\delta]$, thus $\tilde{u}_\delta = u_\delta$ by uniqueness. Recalling the convergence $u_\delta\to u$ a.e., $\pi(u_\delta)=u_\delta$ implies our claim $u=\pi(u)$. $\square$

Coming back to the convergence property of the functions $(\mu-1)\Phi_\mu$ stated in formula (2.12) we have the following approximation property of the regularized problems towards the TV-case.

Theorem 2.3
Let $\Psi := (\mu - 1)\Phi_\mu$ *with* Φ_μ *from (2.10) and let* $u_\mu \in BV(\Omega, \mathbb{R}^N)$ *denote the unique minimizer of the functional* K *defined in (2.13) corresponding to this choice of* Ψ *(note that* $\Psi'_\infty = 1$*), compare Theorem 2.1. Then it holds*

$$\|u_\mu - u\|_{L^p(\Omega,\mathbb{R}^N)} \to 0 \quad \forall p < \frac{n}{n-1} \tag{2.26}$$

and

$$u_\mu \rightharpoondown u \text{ in } L^2(\Omega, \mathbb{R}^N) \tag{2.27}$$

as $\mu \to \infty$*, where* u *is the unique minimizer ("TV-solution") of the problem*

$$\int_\Omega |\nabla u| + \frac{\lambda}{2} \int_\Omega |u - f|^2 \,\mathrm{d}x \to \min \text{ in } BV(\Omega, \mathbb{R}^N). \tag{2.28}$$

Remark 2.3
Clearly a version of Theorem 2.3 also holds for the choice $\Psi(s) = \sqrt{\varepsilon^2 + s^2} - \varepsilon$, $\varepsilon > 0$*, with corresponding solutions* u_ε *for which we have the convergences (2.26) and (2.27) as* $\varepsilon \downarrow 0$.

Remark 2.4
Adopting the ideas presented after formula (3.17) in [10] it might be possible to improve the convergences (2.26), (2.27) towards

$$\lim_{\mu\to\infty} \|u_\mu - u\|_{L^2(\Omega,\mathbb{R}^N)} = 0.$$

Proof of Theorem 2.3: It holds (see formula (2.11))

$$\begin{aligned} K[w] = &\int_\Omega |\nabla^a w| \,\mathrm{d}x - \frac{1}{\mu - 2} \int_\Omega \big(1 + |\nabla^a w|\big)^{-\mu+2} \,\mathrm{d}x - \frac{1}{\mu - 2}\mathcal{L}^n(\Omega) \\ &+ |\nabla^s w|(\Omega) + \frac{\lambda}{2} \int_\Omega |f - w|^2 \,\mathrm{d}x, \ w \in BV(\Omega, \mathbb{R}^N), \ \mu > 2, \end{aligned}$$

and from $K[u_\mu] \leq K[0]$ we directly infer

$$\sup_\mu \left\{ \int_\Omega |\nabla^a u_\mu| \,\mathrm{d}x + |\nabla^s u_\mu|(\Omega) + \int_\Omega |u_\mu - f|^2 \,\mathrm{d}x \right\} < \infty, \tag{2.29}$$

where we have used that

$$\frac{1}{\mu - 2} \int_\Omega \big(1 + |\nabla^a u_\mu|\big)^{-\mu+2} \,\mathrm{d}x \to 0 \text{ as } \mu \to \infty.$$

Clearly (quoting BV-compactness) we can deduce from (2.29) the existence of $\overline{u} \in BV(\Omega, \mathbb{R}^N)$ such that (at least for a subsequence)

$$\begin{cases} \|u_\mu - \overline{u}\|_{L^p(\Omega,\mathbb{R}^N)} \to 0, \ p < \dfrac{n}{n-1}, \\ u_\mu \rightharpoonup \overline{u} \text{ in } L^2(\Omega, \mathbb{R}^N) \text{ and} \\ u_\mu \to \overline{u} \text{ a.e.} \end{cases} \tag{2.30}$$

holds as $\mu \to \infty$. By lower semi-continuity of the total variation and by using Fatou's lemma or quoting $u_\mu \rightharpoonup \overline{u}$ in L^2 we find

$$\begin{aligned} \int_\Omega |\nabla \overline{u}| + \frac{\lambda}{2} \int_\Omega |\overline{u} - f|^2 \,\mathrm{d}x &\le \liminf_{\mu\to\infty} \left(\int_\Omega |\nabla u_\mu| + \frac{\lambda}{2} \int_\Omega |u_\mu - f|^2 \,\mathrm{d}x \right) \\ &= \liminf_{\mu\to\infty} K[u_\mu] \le \liminf_{\mu\to\infty} K[u] \\ &= \int_\Omega |\nabla u| + \frac{\lambda}{2} \int_\Omega |u - f|^2 \,\mathrm{d}x, \end{aligned}$$

where we have used the K-minimality of the u_μ. Thus $\overline{u}$ is a TV-minimizer, hence $u = \overline{u}$ by the unique solvability of (2.28) and (2.30) is true not only for a subsequence which proves (2.26) and (2.27). □

Remark 2.5

We leave it as an exercise to the reader to show that the statements of Theorems 2.1 and 2.3 remain valid if the quantity $\frac{\lambda}{2}\int_\Omega |u-f|^2 \,\mathrm{d}x$ *is replaced by* $\frac{\lambda}{2}\int_\Omega \omega(u-f)\,\mathrm{d}x$ *with* $\omega : \mathbb{R}^N \to [0,\infty)$ *being strictly convex, e.g. we may choose*

$$\omega(y) := \sqrt{\varepsilon^2 + |y|^2} - \varepsilon, \ \varepsilon > 0, \ y \in \mathbb{R}^N,$$

or $\omega(y) := |y|^p$ *with exponent* $p > 1$. *Of course (2.27) then has to be replaced with* $u_\mu \rightharpoonup u$ *in* $L^{n/(n-1)}(\Omega, \mathbb{R}^N)$ *in the first case and* $u_\mu \rightharpoonup u$ *in* $L^q(\Omega, \mathbb{R}^N)$ *in the second case, where* $q := \max\{p, n/_{n-1}\}$.

Remark 2.6

If for a given set of data f *it is desirable to have smoothness of the regularizer* u *on a subset* Ω' *of* Ω, *whereas on the complement of* Ω' *non-smoothness of* u *seems to be natural, then such a behaviour can be generated by considering non-autonomous densities of the form*

$$F(x, \nabla u) = \eta(x)\Phi_\mu\big(|\nabla u|\big) + \big(1 - \eta(x)\big)\Phi_\nu\big(|\nabla u|\big)$$

with $\mu \in (1,2)$ *and* $\nu \in (2,\infty)$ *large. Here* η *is a smooth function on* Ω *such that* $0 \le \eta \le 1$ *and with the property* $\eta = 1$ *on* Ω'. *For details we refer to the paper [14].*

Remark 2.7
We note that our discussion can easily be extended to isotropic models of superlinear growth. To be precise we consider the problem (compare (2.1))

$$\int_\Omega \Phi_\mu\big(|\nabla u|\big)\,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |u-f|^2\,\mathrm{d}x \to \min \tag{2.31}$$

but now with the choice $\mu \leq 1$, where in case $\mu = 1$ the correct class for (2.31) is the Orlicz-Sobolev space $W^{1,h}(\Omega, \mathbb{R}^N)$ generated by the function $h(t) := t\ln(1+t)$, $t \geq 0$, (compare [1]) and for values $\mu < 1$ problem (2.31) is well posed in the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^N)$, $p := 2 - \mu > 1$. In both cases (2.31) admits a unique solution u satisfying $u(x) \in K$, if f has this property (with $K \subset \mathbb{R}^N$ closed and convex), moreover, it holds $u \in C^{1,\alpha}(\Omega, \mathbb{R}^N)$ for any $\alpha \in (0,1)$. Some details and further references concerning the superlinear case are presented in [14].

3 Anisotropic regularization

We start with some preliminaries concerning the definition of the densities F we now have in mind where for notational simplicity we consider the quadratic case for which $n = N$. The general situation is briefly discussed in Remark 3.1. For matrices $p \in \mathbb{R}^{n\times n}$ let

$$J(p) := pp^T\ \big((p^T)_{ij} = p_{ji}\big), \tag{3.1}$$

and observe that $J(p)$ is symmetric and positive semidefinite with eigenvalues $0 \leq \sigma_1(p) \leq ... \leq \sigma_n(p)$. We introduce the numbers

$$\lambda_i(p) := \sqrt{\sigma_i(p)} \tag{3.2}$$

which correspond to the eigenvalues of $\sqrt{J(p)}$ and are known as the *singular values* of the matrix p. The following observation of Ball (see Theorem 6.1 in [4]) and compare [3], Theorem 5.1 on p. 363 for a complete proof in any dimension n) is of crucial importance

Lemma 3.1
Consider a function $\rho : [0,\infty) \to [0,\infty)$ which is convex and increasing. Then the mapping

$$F : \mathbb{R}^{n\times n} \to \mathbb{R},\ p \mapsto \operatorname{trace}\rho\big(\sqrt{J(p)}\big) := \sum_{i=1}^{n} \rho\big(\lambda_i(p)\big),\ p \in \mathbb{R}^{n\times n}, \tag{3.3}$$

is a convex function on the space $\mathbb{R}^{n\times n}$.

Remark 3.1
For the sake of notational simplicity, we have restricted ourselves to the case of quadratic matrices. However, we would like to indicate how our results can be adapted to the general case of $n \times N$ matrices with $N \neq n$ with the help of of Lemma 3.1.

i) First we assume $N < n$. Let $p \in \mathbb{R}^{n\times N}$ and $J(p) := pp^T \in \mathbb{R}^{n\times n}$. As before, we denote the eigenvalues of $\sqrt{J(p)}$ by $\lambda_1(p), ..., \lambda_n(p)$ and now define $F : \mathbb{R}^{n\times N} \to \mathbb{R}$ through the formula

$$F : \mathbb{R}^{n\times N} \to \mathbb{R},\ p \mapsto \sum_{i=1}^{n} \rho\big(\lambda_i(p)\big)$$

where $\rho : [0,\infty) \to [0,\infty)$ is as in Lemma 3.1. Then we define $\widetilde{F} : \mathbb{R}^{n\times n} \to \mathbb{R}$ according to (3.3). Now consider the linear embedding $\mathcal{E} : \mathbb{R}^{n\times N} \to \mathbb{R}^{n\times n}$, which acts on an $(n \times N)$-matrix p by adding $(n - N)$ zero-columns. Then we observe $pp^T = \mathcal{E}(p)\mathcal{E}(p)^T$ for $p \in \mathbb{R}^{n\times N}$ and the convexity follows from the formula $F(p) = \widetilde{F}\big(\mathcal{E}(p)\big)$ and the convexity of $\widetilde{F}$.

ii) The case $N > n$ can be treated in the same manner: let now $\mathcal{E} : \mathbb{R}^{n\times N} \to \mathbb{R}^{N\times N}$ denote the embedding which adds $N - n$ zero-rows to a matrix $p \in \mathbb{R}^{n\times N}$, define $F : \mathbb{R}^{n\times N} \to \mathbb{R}$ as above and $\widetilde{F} : \mathbb{R}^{N\times N} \to \mathbb{R}$ according to (3.3) (with "n" replaced by "N"). Then $\mathcal{E}(p)\mathcal{E}(p)^T = pp^T \oplus \mathbf{0}$, where $\mathbf{0}$ denotes the $(N-n)\times(N-n)$-zero matrix and $\widetilde{F}(\mathcal{E}(p)) = F(p) + (N-n)\rho(0)$ is a convex function by Lemma 3.1, and hence so is F.

iii) Since the linear map $\mathcal{E}$ is smooth in both cases, we can apply this strategy to extend our results concerning differentiability in Section 4 to the non-quadratic case.

Remark 3.2
Note, that the general version of Lemma 3.1 as it is found in [3] states, that if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is symmetric and convex, then $\Phi : \mathbb{R}^{n\times n} \to \mathbb{R}$, $p \mapsto \varphi(\lambda_1(p), ..., \lambda_n(p))$ is also convex. The necessity of symmetry is the reason, why we have to apply the same function ρ to each of the eigenvalues λ_i in (3.3).

Definition 3.1 (anisotropic energy densities of linear growth)**.** Let $\psi : [0,\infty) \to [0,\infty)$ denote an increasing and convex function satisfying in addition

$$c_1 t - c_2 \le \psi(t) \le c_3 t + c_4 \tag{3.4}$$

with constants $c_1, c_3 > 0$, $c_2, c_4 \in \mathbb{R}$. Then the mapping (recall (3.1)-(3.3))

$$F_\psi : \mathbb{R}^{n\times n} \to \mathbb{R},\ F_\psi := \operatorname{trace} \psi\big(\sqrt{J}\big), \tag{3.5}$$

is termed the anisotropic energy density of linear growth generated by ψ.

This terminology is justified by

Lemma 3.2
In the notation of Definition 3.1 the convex function $F_\psi : \mathbb{R}^{n\times n} \to [0,\infty)$ *satisfies*

$$c_1^*|p| - c_2^* \le F_\psi(p) \le c_3^*|p| + c_4^*,\ p \in \mathbb{R}^{n\times n}, \tag{3.6}$$

with constants $c_1^*, c_3^* > 0$, $c_2^*, c_4^* \in \mathbb{R}$, $|p|$ *denoting the Euclidean (=Frobenius) norm of the matrix* p.

Proof of Lemma 3.2: From (3.5) together with (3.4) it follows

$$\sum_{i=1}^n \big(c_1\lambda_i(p) - c_2\big) \le \sum_{i=1}^n \psi\big(\lambda_i(p)\big) \le \sum_{i=1}^n \big(c_3\lambda_i(p) + c_4\big),$$

hence

$$c_1\left(\sum_{i=1}^n \lambda_i(p)\right) - nc_2 \le F_\psi(p) \le c_3\left(\sum_{i=1}^n \lambda_i(p)\right) + nc_4.$$

We further observe

$$\sum_{i=1}^n \lambda_i^2(p) = \sum_{i=1}^n \sigma_i(p) = \operatorname{trace}(pp^T) = |p|^2,$$

which means

$$c_5 \sum_{i=1}^n \lambda_i(p) \le |p| \le c_6 \sum_{i=1}^n \lambda_i(p)$$

with positive numbers c_5, c_6. This immediately implies (3.6). □

Example 3.1 (anisotropic TV-density)**.** Letting $\psi(t) := t$, $t \ge 0$, in formula (3.5) we obtain

$$F_{TV}(p) = \sum_{i=1}^n \lambda_i(p),\ p \in \mathbb{R}^{n\times n}. \tag{3.7}$$

Note that the isotropic TV-density is just the quantity $|p| = \big(\sum_{i=1}^n \lambda_i(p)^2\big)^{1/2}$.

Example 3.2 (regularized TV-densities)**.** For $\mu > 1$ we let $\psi(t) := \Phi_\mu(t)$, $t \ge 0$, with Φ_μ from (2.10) and define

$$F_\mu := (\mu - 1)\operatorname{trace}\Phi_\mu\big(\sqrt{J}\big). \tag{3.8}$$

With a slight abuse of notation we can also consider

$$F_\varepsilon := \operatorname{trace}\sqrt{\varepsilon^2 + J},\ \varepsilon > 0 \tag{3.9}$$

which means that $\psi_\varepsilon(t) := \sqrt{\varepsilon^2 + t^2}$ in formula (3.5).

Let us now discuss variational problems in the anisotropic linear growth setting: as usual we consider data

$$f \in L^2(\Omega, \mathbb{R}^n) \tag{3.10}$$

for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. For $u : \Omega \to \mathbb{R}^n$ we let

$$\nabla u := \left(\nabla u^1 ... \nabla u^n\right) = \begin{pmatrix} \partial_1 u^1 & \dots & \partial_1 u^n \\ \vdots & & \vdots \\ \partial_n u^1 & \dots & \partial_n u^n \end{pmatrix}$$

whenever this $(n \times n)$-matrix is defined (in a weak sense). We have (compare (3.1))

$$J(\nabla u) = \nabla u \nabla u^T = \left(\partial_i u \cdot \partial_j u\right)_{1 \le i,j \le n},$$

"·" denoting the scalar product in $\mathbb{R}^n$, and by Lemma 3.2 the variational problem

$$J_\psi[u] := \int_\Omega F_\psi(\nabla u)\,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |u - f|^2\,\mathrm{d}x \to \min \tag{3.11}$$

is well defined on the Sobolev space $W^{1,1}(\Omega, \mathbb{R}^N)$ for any function ψ as in Definition 3.1 and for arbitrary choice of $\lambda > 0$. As explained in Section 2 we have to pass to the relaxed version of (3.11) which reads ($\frac{\nabla^s w}{|\nabla^s w|}$ denoting the density of the measure $\nabla^s w$ with respect to the measure $|\nabla^s w|$)

$$K_\psi[w] := \int_\Omega F_\psi(\nabla^a w)\,\mathrm{d}x + \int_\Omega F_\psi^\infty\left(\frac{\nabla^s w}{|\nabla^s w|}\right)\mathrm{d}|\nabla^s w| + \frac{\lambda}{2}\int |w - f|^2\,\mathrm{d}x \to \min \tag{3.12}$$

in $BV(\Omega, \mathbb{R}^n)$. Here our notation is introduced after (2.13), and we refer the reader to Theorem 5.47 (and the subsequent remarks) in [2], in particular,

$$F_\psi^\infty(p) := \lim_{t\to\infty} \frac{F_\psi(tp)}{t},\ p \in \mathbb{R}^{n\times n},$$

is the recession function of F_ψ, which here takes the form (compare (3.7))

$$F_\psi^\infty(p) = \lim_{t\to\infty}\frac{\psi(t)}{t}\sum_{i=1}^n \lambda_i(p) = \lim_{t\to\infty}\frac{\psi(t)}{t} F_{TV}(p). \tag{3.13}$$

Noting that $\lambda_i\left(\frac{p}{|p|}\right) = \frac{1}{|p|}\lambda_i(p)$, we may therefore write for $w \in BV(\Omega, \mathbb{R}^n)$

$$\begin{aligned} \int_\Omega F_\psi^\infty\left(\frac{\nabla^s w}{|\nabla^s w|}\right)\mathrm{d}|\nabla^s w| &= \lim_{t\to\infty}\frac{\psi(t)}{t}\int_\Omega \sum_{i=1}^n \lambda_i\left(\frac{\nabla^s w}{|\nabla^s w|}\right)\mathrm{d}|\nabla^s w| \\ = \lim_{t\to\infty}\frac{\psi(t)}{t}\int_\Omega\left(\sum_{i=1}^n \lambda_i\right)(\nabla^s w) &= \lim_{t\to\infty}\frac{\psi(t)}{t}\left(\sum_{i=1}^n \lambda_i\right)(\nabla^s w)(\Omega), \end{aligned} \tag{3.14}$$

where in the last line we apply the convex function F_{TV} (compare (3.7)) to the matrix-valued measure $\nabla^s w$ in the sense of [18] and calculate the total mass of the resulting nonnegative measure. For the particular case $\psi(t) = t$ the functional (3.12) reduces to

$$K_{TV}[w] = \int_\Omega \sum_{i=1}^n \lambda_i(\nabla^a w)\,\mathrm{d}x + \int_\Omega \left(\sum_{i=1}^n \lambda_i\right)(\nabla^s w) + \frac{\lambda}{2}\int_\Omega |w-f|^2\,\mathrm{d}x, \qquad (3.15)$$

$w \in BV(\Omega, \mathbb{R}^n)$.

We further like to remark that in formulas (3.13) and (3.14) the quantity $\lim_{t\to\infty} \frac{\psi(t)}{t}$ can be replaced by (compare (2.13))

$$\psi'_\infty := \lim_{t\to\infty} \psi'(t),$$

provided ψ satisfies (2.4)-(2.7). After these preparations we can state

Theorem 3.1
Let $\psi : [0,\infty) \to [0,\infty)$ denote a convex and increasing function of linear growth as stated in (3.4) and consider the density

$$F_\psi(p) := \operatorname{trace}\psi\big(\sqrt{pp^T}\big),\ p \in \mathbb{R}^{n\times n},$$

being defined in formulas (3.3) and (3.5).

a) The variational problem (see (3.12))

$$K_\psi \to \min \ \textit{in}\ BV(\Omega,\mathbb{R}^n)$$

admits a unique solution $u \in BV(\Omega,\mathbb{R}^n)$. It holds

$$K[u] = \inf_{v\in W^{1,1}(\Omega,\mathbb{R}^n)} J_\psi[v]$$

with J_ψ from (3.11).

b) Let $\psi := (\mu-1)\Phi_\mu$, i.e. $F_\psi = F_\mu$ with F_μ from (3.8), where $\mu > 1$. Consider the corresponding version of (3.12), i.e.

$$K_\mu[w] := \int_\Omega (\mu-1)\sum_{i=1}^n \Phi_\mu\big(\lambda_i(\nabla^a w)\big)\,\mathrm{d}x + \int_\Omega \left(\sum_{i=1}^n \lambda_i\right)(\nabla^s w) + \frac{\lambda}{2}\int_\Omega |w-f|^2\,\mathrm{d}x$$

$$\to \min \ \textit{in}\ BV(\Omega,\mathbb{R}^n)$$

with unique solution u_μ. Then it holds

$$\|u_\mu - u\|_{L^p(\Omega,\mathbb{R}^n)} \to 0,\ p < \frac{n}{n-1},$$

$$u_\mu - u \rightharpoonup 0 \ in \ L^2(\Omega, \mathbb{R}^n) \ and \ a.e.$$

as $\mu \to \infty$, where $u \in BV(\Omega, \mathbb{R}^n)$ is the unique TV-solution, i.e. the unique minimizer of the energy K_{TV} defined in (3.15).

c) For $\varepsilon > 0$ let $\psi(t) := \psi_\varepsilon(t) := \sqrt{\varepsilon^2 + t^2}$, $t \geq 0$, in (3.12), i.e. we look at the problem

$$K_\varepsilon[w] := \int_\Omega \sum_{i=1}^n \sqrt{\varepsilon^2 + \lambda_i(\nabla^a w)^2} \, \mathrm{d}x + \int_\Omega \left(\sum_{i=1}^n \lambda_i \right) (\nabla^s w) + \frac{\lambda}{2} \int_\Omega |w - f|^2 \, \mathrm{d}x$$
$$\to \min \ in \ BV(\Omega, \mathbb{R}^n)$$

with corresponding solution u_ε. Then we have

$$\|u_\varepsilon - u\|_{L^p(\Omega, \mathbb{R}^n)} \to 0, \ p < \frac{n}{n-1},$$

$$u_\varepsilon - u \rightharpoonup 0 \ in \ L^2(\Omega, \mathbb{R}^n) \ and \ a.e.$$

as $\varepsilon \to 0$, where u is the solution of (see (3.15))

$$K_{TV} \to \min \ in \ BV(\Omega, \mathbb{R}^n).$$

Proof of Theorem 3.1: a) Let u_k denote a K_ψ-minimizing sequence from $BV(\Omega, \mathbb{R}^n)$. Lemma 3.2 (compare inequality (3.6)) in combination with the definition of K_ψ then yields

$$\sup_k |\nabla u_k|(\Omega), \ \sup_k \|u_k\|_{L^2(\Omega, \mathbb{R}^n)} < \infty,$$

hence, quoting BV-compactness, $u_k \to: \overline{u}$ in $L^1(\Omega, \mathbb{R}^n)$ for some $\overline{u} \in BV(\Omega, \mathbb{R}^n)$ and a subsequence of u_k. Moreover, we may assume that $u_k \to \overline{u}$ a.e. on Ω and

$$\int_\Omega |\overline{u} - f|^2 \, \mathrm{d}x \leq \liminf_{k \to \infty} \int_\Omega |u_k - f|^2 \, \mathrm{d}x$$

follows from Fatou's Lemma (or from $u_k \rightharpoonup \overline{u}$ in $L^2(\Omega)$). According to Theorem 5.47 in [2] and the remarks stated after this theorem the functional

$$w \mapsto \int_\Omega F_\psi(\nabla^a w) \, \mathrm{d}x + \int_\Omega F_\psi^\infty \left(\frac{\nabla^s w}{|\nabla^s w|} \right) \mathrm{d}|\nabla^s w|$$

is lower semi-continuous with respect to $L^1(\Omega, \mathbb{R}^n)$-convergence. Here we make essential use of Ball's convexity result Lemma 3.1 implying the convexity of F_ψ. Altogether we have

$$K_\psi[\overline{u}] \leq \liminf_{k \to \infty} K_\psi[u_k],$$

thus $\overline{u}$ is K_ψ-minimizing. Uniqueness of the minimizer is immediate, all other claims follow along the lines of Theorem 2.1.

For part *b*) and *c*) we refer to Theorem 2.3 and Remark 2.3. □

4 Differentiable models

Concerning the regularity properties of the minimizer $u \in BV(\Omega, \mathbb{R}^n)$ from Theorems 2.1 and 3.1, it is desirable to consider energy densities F which are sufficiently smooth. Namely we would like to have $F \in C^2(\mathbb{R}^{n\times n})$. To this end, we consider a slight modification of the function F from (3.3) by setting

$$F^*(p) := \sum_{i=1}^{n} \psi\left(\sqrt[4]{\varepsilon^2 + \sigma_i(p)^2}\right) \tag{4.1}$$

for some $\varepsilon > 0$, with σ_i as usual denoting the eigenvalues of pp^T and $\psi : \mathbb{R} \to [0, \infty)$ is a convex and increasing function which satisfies (cf. (2.4)-(2.7))

$$\begin{cases} \psi \in C^2(\mathbb{R}), \\ \psi(-y) = \psi(y),\ \psi(0) = 0, \\ |\psi'(y)| \leq \nu_1, \\ \psi(y) \geq \nu_2|y| - \nu_3,\ \psi''(y) > 0, \end{cases} \tag{4.2}$$

with $\nu_1, \nu_2, \nu_4 > 0$, $\nu_3 \in \mathbb{R}$. As for the map $p \mapsto (\sigma_1(p), ..., \sigma_n(p))$, which is not immediately seen to be differentiable, we can once more benefit from a result by John Ball in [5] which gives us the desired smoothness. Precisely we have

Theorem 4.1
The density F^* being defined in (4.1) is convex and C^2 on $\mathbb{R}^{n\times n}$.

Remark 4.1
As we have already mentioned in Remark 3.1 iii), the above result can easily be adjusted to the non-quadratic case $f : \Omega \to \mathbb{R}^N$ for $N \neq n$.

Proof of Theorem 4.1:

With the notation from (3.3) and (3.5) we have

$$F^*(p) = \operatorname{trace} \widetilde{\psi}\big(\sqrt{J}\big),\ J = J(p) = pp^T,$$

i.e. $F^*(p) = \sum_{i=1}^n \widetilde{\psi}\big(\lambda_i(p)\big)$, if we set

$$\widetilde{\psi}(t) := \psi\left(\sqrt[4]{\varepsilon^2 + t^4}\right).$$

Since $\widetilde{\psi}$ fulfills the requirements imposed on ρ of Lemma 3.1, the convexity of F^* follows. We use the notation from [5], Section 5. Let

$$\left.\begin{aligned}
&E := \mathbb{S}^n \quad \text{(symmetric } (n\times n)\text{-matrices)},\\
&\Gamma_E := \{\text{diagonal matrices in } E\},\\
&v_i(A) := \text{i-th eigenvalue of } A \in E,\\
&H : \Gamma_E \cong \mathbb{R}^n \ni (t_1, ..., t_n) \mapsto \sum_{i=1}^n \psi\left(\sqrt[4]{\varepsilon^2 + t_i^2}\right).
\end{aligned}\right\} \tag{4.3}$$

Obviously, $H \in C^2(\mathbb{R}^n)$. But then, Theorem 5.5 on p. 717 in [5], implies that also

$$h : \mathbb{S}^n \ni A \mapsto H(v_1(A), ..., v_n(A))$$

is of class C^2 on $\mathbb{S}^n$ ($\cong \mathbb{R}^{n(n-1)/2}$). Now note that

$$F^*(p) = h(pp^T),\ p \in \mathbb{R}^{n\times n}$$

and since the map $p \mapsto pp^T$ is obviously smooth, this shows $F^* \in C^2(\mathbb{R}^{n\times n})$. □

Remark 4.2
The symmetry of the function H is essential for establishing both convexity and differentiability of our models. In particular we cannot generalize our model to $\sum_{i=1}^n \psi_i\left(\sqrt[4]{\varepsilon + t_i^2}\right)$ with distinct ψ_i's for each $i \in \{1, ..., n\}$.

Theorem 4.2
Let ψ satisfy (4.2) and define F^* according to (4.1).

a) F^* grows linearly in the sense of inequality (3.6).

b) The relaxation of

$$\int_\Omega F^*(\nabla u)\,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |u-f|^2\,\mathrm{d}x \to \min\ \text{ in } W^{1,1}(\Omega, \mathbb{R}^n) \tag{4.4}$$

with $f \in L^2(\Omega, \mathbb{R}^n)$, $\lambda > 0$, is given by

$$\int_\Omega F^*(\nabla^a u)\,\mathrm{d}x + \psi'_\infty \cdot \left(\sum_{i=1}^n \lambda_i\right)(\nabla^s u)(\Omega) + \frac{\lambda}{2}\int_\Omega |u-f|^2\,\mathrm{d}x \to \min \tag{4.5}$$

in $BV(\Omega, \mathbb{R}^n)$

and is uniquely solvable. (Here we have abbreviated $\psi'_\infty := \lim_{s\to\infty} \psi'(s) = \lim_{s\to\infty} \frac{\psi(s)}{s}$.)

c) Let us in addition assume that F^* satisfies

$$\left|D^2F^*(p)\right| \le \nu_4 \frac{1}{1+|p|} \tag{4.6}$$

for some constant $\nu_4 > 0$. Then, if $\Omega' \subset \Omega$ and $f \in W^{1,2}_{\mathrm{loc}}(\Omega', \mathbb{R}^n)$, we have $u \in W^{1,2}_{\mathrm{loc}}(\Omega', \mathbb{R}^n)$ for the unique solution u of (4.5).

Corollary 4.1
If the data f are chosen from the space $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and F^* satisfies (4.6), then (4.4) is solvable in $W^{1,1}(\Omega, \mathbb{R}^n)$.

Remark 4.3
As usual Theorem 4.2 and Corollary 4.1 extend to the non-quadratic case $f : \Omega \to \mathbb{R}^N$ for $N \neq n$ via Remark 3.1.

Remark 4.4
In $\left(\sum_{i=1}^n \lambda_i\right)(\nabla^s u)$ the convex function $p \mapsto \sum_{i=1}^n \lambda_i(p)$ is applied to the matrix-valued measure $\nabla^s u$ which yields a positive Radon measure on Ω, whose total mass enters in (4.5). We refer to the comments after formula (3.14).

Proof of Theorem 4.2: Ad a): cf. the proof of Lemma 3.2;

Ad b): see Theorem 3.1 and note (cf. (3.13)) that

$$\left(F^\infty\right)(p) := \lim_{t\to\infty} \frac{1}{t} F^*(tp) = \psi'_\infty \sum_{i=1}^n \psi\left(\sqrt[4]{\sigma_i^2(p)}\right) = \psi'_\infty \left(\sum_{i=1}^n \lambda_i\right)(p)$$

$$\left(\lambda_i(p) := \text{ eigenvalues of } \sqrt{pp^T} = \sqrt{\sigma_i(p)}\right).$$

This implies (cf. (3.7))

$$\left(F^*\right)^\infty(p) = \psi'_\infty F_{TV}(p).$$

Ad c): let w.l.o.g. $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. In all the following calculations we have to replace u with the sequence of regularizers u_δ (cf. (2.25) and compare [22] for more details), however, for notational simplicity we drop the index δ, i.e. $F^*(p) = F^*_\delta(p) := \frac{\delta}{2}|p|^2 + F^*(p)$ and $u = u_\delta$ is the unique solution of

$$\int_\Omega F^*_\delta(\nabla w)\,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |w - f|^2\,\mathrm{d}x \to \min \text{ in } W^{1,2}(\Omega, \mathbb{R}^n).$$

From the minimality of u along with $F^* \in C^2$ it follows (using summation convention w.r.t. the index α)

$$\int_\Omega D^2F^*(\nabla u)\big(\partial_\alpha \nabla u, \nabla(\eta^2\partial_\alpha u)\big)\,\mathrm{d}x = \lambda \int_\Omega \partial_\alpha(\eta^2\partial_\alpha u)\cdot(u - f)\,\mathrm{d}x,$$

where $\eta \in C^\infty_0(\Omega)$, spt $\eta \subset B_{2R}(x_0)$ with $0 \le \eta \le 1$ and $\eta \equiv 1$ on $B_R(x_0)$ for some $x_0 \in \Omega$ and some radius $R > 0$ s.t. $B_{2R}(x_0) \subset \Omega$. Hence

$$\int_\Omega D^2F^*(\nabla u)\big(\eta\partial_\alpha\nabla u, \eta\partial_\alpha\nabla u\big)\,\mathrm{d}x + \int_\Omega D^2F^*(\nabla u)\big(\partial_\alpha\nabla u, \nabla\eta^2 \otimes \partial_\alpha u\big)\,\mathrm{d}x$$
$$+ \lambda\int_\Omega \eta^2|\nabla u|^2\,\mathrm{d}x = \lambda\int_\Omega \eta^2\partial_\alpha u\cdot\partial_\alpha f\,\mathrm{d}x$$

and thus

$$\begin{aligned}
&\int_\Omega D^2F^*(\nabla u)\big(\eta\partial_\alpha\nabla u, \eta\partial_\alpha\nabla u\big)\,\mathrm{d}x + \lambda\int_\Omega \eta^2|\nabla u|^2\,\mathrm{d}x \\
&= \lambda\int_\Omega \eta^2\partial_\alpha u\cdot\partial_\alpha f\,\mathrm{d}x - \int_\Omega D^2F^*(\nabla u)\big(\partial_\alpha\nabla u, \nabla\eta^2\otimes\partial_\alpha u\big)\,\mathrm{d}x \\
&=: T_1 + T_2.
\end{aligned} \tag{4.7}$$

The integral T_1 can be estimated by Young's inequality through

$$|T_1| \le c(\varepsilon,\lambda)\int_\Omega \eta^2|\nabla f|^2\,\mathrm{d}x + \lambda\varepsilon\int_\Omega \eta^2|\nabla u|^2\,\mathrm{d}x \le c(\varepsilon,\lambda,R) + \lambda\varepsilon\int_\Omega \eta^2|\nabla u|^2\,\mathrm{d}x.$$

Choosing $\varepsilon = 1/2$ and absorbing terms on the left-hand side of (4.7), we obtain

$$\int_\Omega D^2F^*(\nabla u)\big(\eta\partial_\alpha\nabla u, \eta\partial_\alpha\nabla u\big)\,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega \eta^2|\nabla u|^2\,\mathrm{d}x \le c(R) + T_2.$$

Now, for T_2, we apply the Cauchy-Schwarz inequality to the bilinear form $D^2F^*(\nabla u)$ observing

$$D^2F^*(\nabla u)\big(\partial_\alpha \nabla u, \nabla \eta^2 \otimes \partial_\alpha u\big) = 2D^2F^*(\nabla u)\big(\eta\partial_\alpha \nabla u, \nabla \eta \otimes \partial_\alpha u\big)$$

and obtain after an application of Young's inequality the following result:

$$\begin{aligned} |T_2| \leq & c(\varepsilon) \int_\Omega D^2F^*(\nabla u)\big(\nabla \eta \otimes \partial_\alpha u, \nabla \eta \otimes \partial_\alpha u\big)\,\mathrm{d}x \\ & + \varepsilon \int_\Omega D^2F^*(\nabla u)\big(\eta\partial_\alpha \nabla u, \eta\partial_\alpha \nabla u\big)\,\mathrm{d}x. \end{aligned} \tag{4.8}$$

Choosing $\varepsilon = 1/2$, the second term on the right-hand side of (4.8) can be absorbed in the left-hand side of (4.7). For estimating the first term on the right-hand side of (4.8), we need our additional assumption (4.6) on D^2F^* which yields:

$$\begin{aligned} & \int_\Omega D^2F^*(\nabla u)\big(\nabla \eta \otimes \partial_\alpha u, \nabla \eta \otimes \partial_\alpha u\big)\,\mathrm{d}x \\ & \qquad \leq c \int_\Omega D^2F^*(\nabla u)\big(\nabla u, \nabla u\big)\,\mathrm{d}x \leq c \int_\Omega \big(1 + |\nabla u|\big)\,\mathrm{d}x \end{aligned} \tag{4.9}$$

with a suitable constant c uniformly with respect to the (invisible) parameter δ. Consequently, (4.7) yields a uniform (in δ) bound for $\int_\Omega |\nabla u|^2\,\mathrm{d}x$, which concludes the proof of Theorem 4.2. □

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