Universität des Saarlandes

Fachrichtung Mathematik

Preprint Nr. 389

A remark on the denoising of greyscale images using energy densities with varying growth rates

Martin Fuchs and Jan Müller

Saarbrücken 2017

Fachrichtung Mathematik
Universität des Saarlandes

Preprint No. 389
submitted: July 10, 2017

A remark on the denoising of greyscale images using energy densities with varying growth rates

Martin Fuchs

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Jan Müller

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
jmueller@math.uni-sb.de

Edited by
FR Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

A remark on the denoising of greyscale images using energy densities with varying growth rates

M. Fuchs J. Müller

AMS-classification: 49N60, 49J45, 49Q20
Keywords: variational problems with nonstandard growth, TV-regularization, denoising and inpainting of images

Abstract

We prove the solvability in Sobolev spaces for a class of variational problems related to the TV-model proposed by Rudin, Osher and Fatemi in [1] for the denoising of greyscale images. In contrast to their approach we discuss energy densities with variable growth rates depending on $|\nabla u|$ in a rather general form including functionals of $(1, p)$-growth.

1 Introduction

In 1992 Rudin, Osher and Fatemi proposed (compare [1]) to study the variational problem

$$I_1[w] := \int_\Omega |\nabla w| \, \mathrm{d}x + \frac{\lambda}{2} \int_\Omega |f - w|^2 \, \mathrm{d}x \to \min \tag{1.1}$$

as a model for the restoration of a noisy greyscale image f. In this setting (and throughout our paper) Ω is a bounded Lipschitz region in $\mathbb{R}^2$, the function $f : \Omega \to \mathbb{R}$ represents the noisy data, for which we assume

$$0 \le f \le 1 \ \text{ a.e. on } \ \Omega, \tag{1.2}$$

and $\lambda > 0$ denotes a parameter being under our disposal. As a matter of fact, problem (1.1) has to be discussed in the space $BV(\Omega)$ of functions with finite total variation (see, e.g., [2] or [3] for a definition and further properties of this class) admitting a unique solution u which in addition satisfies (1.2). From the analytical point of view, the functional I_1 from (1.1) does not behave very nicely: the energy density $|\nabla w|$ is neither differentiable nor strictly convex ("elliptic") so that no additional information on the minimizer u are available. One common alternative used in the variational approach towards the denoising of images is to replace (1.1) by

$$I_p[w] := \int_\Omega |\nabla w|^p \, \mathrm{d}x + \frac{\lambda}{2} \int_\Omega |w - f|^2 \, \mathrm{d}x \to \min \tag{1.3}$$

for some power $p > 1$, where the choice $p = 2$ already occurs in the work of Arsenin and Tikhonov [4], we refer to the monograph [5] for more information on the subject including references. The natural space for problem (1.3) is the Sobolev class $W^{1,p}(\Omega)$ (compare [6] for details), and from nowadays standard results on nonlinear elliptic equations (see the references stated in Chapter 3.2 of [7]) going back to e.g. Uralt'seva, Uhlenbeck, Evans, Di Benedetto and many other prominent authors it follows that the unique solution of problem (1.3) is at least of class C^1 on the interior of the domain Ω. However, from the point of view of applications, a high degree of regularity of the minimizer is not always favourable ("effect of oversmoothing"), which means that in certain cases one should discuss a linear growth model but with better ellipticity properties in comparison to the functional I_1. This is the subject of the papers [8, 9, 10], in which we studied the problem

$$J_\mu[w] := \int_\Omega F_\mu(\nabla w)\,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |w-f|^2\,\mathrm{d}x \to \min \tag{1.4}$$

(including even inpainting) with density

$$F_\mu(\xi) := \Phi_\mu(|\xi|),\ \xi \in \mathbb{R}^2, \tag{1.5}$$

the function $\Phi_\mu : [0,\infty) \to [0,\infty)$ being defined through

$$\Phi_\mu(t) := \int_0^t \int_0^s (1+r)^{-\mu}\,\mathrm{d}r\,\mathrm{d}s,\ t \geq 0, \tag{1.6}$$

with explicit formula

$$\begin{cases} \Phi_\mu(t) = \dfrac{1}{\mu-1}t + \dfrac{1}{\mu-1}\dfrac{1}{\mu-2}(t+1)^{-\mu+2} - \dfrac{1}{\mu-1}\dfrac{1}{\mu-2},\ \mu \neq 2, \\ \Phi_2(t) = t - \ln(1+t),\ t \geq 0. \end{cases} \tag{1.7}$$

In the case $\mu > 1$ the density F_μ is of linear growth in the sense that

$$c_1\big(|\xi|-1\big) \leq F_\mu(\xi) \leq c_2\big(|\xi|+1\big)\,,\ \xi \in \mathbb{R}^2\,, \tag{1.8}$$

with constants $c_1, c_2 > 0$. Formally we can also consider values $\mu < 1$, but then (1.4) reduces to (1.3) for the choice $p = 2-\mu$. The density F_μ is of class C^2 satisfying in case $\mu > 1$ the condition of μ-ellipticity, i.e.

$$c_3\big(1+|\xi|\big)^{-\mu}|\eta|^2 \leq D^2F_\mu(\xi)(\eta,\eta) \leq c_4\big(1+|\xi|\big)^{-1}|\eta|^2 \tag{1.9}$$

with $c_3, c_4 > 0$ and for all $\xi, \eta \in \mathbb{R}^2$. From (1.7) it follows

$$\lim_{\mu\to\infty} (\mu-1)\,F_\mu\,(\xi) = |\xi|\ ,\ \xi \in \mathbb{R}^2\,, \tag{1.10}$$

and (1.9) together with (1.10) shows that "$(1-\mu)F_\mu(\nabla w)$" is a reasonable approximation of the TV-density "$|\nabla w|$" occurring in problem (1.1). Moreover, it turns out that the degree of regularity of the solution $u_\mu \in \mathrm{BV}(\Omega)$ of problem (1.4) can be controlled in terms of the parameter μ. Precisely it holds

THEOREM 1.1. *Let f satisfy (1.2), fix $\mu > 1$ and define F_μ according to (1.5), (1.6).*

a) If $\mu < 2$, then the solution u_μ of (1.4) belongs to the Sobolev space $W^{1,1}(\Omega)$ and is of class C^1 in the interior of Ω.

b) In case $\mu > 2$ there are simple examples of data f for which $u_\mu \notin W^{1,1}(\Omega)$.

For part a) we refer to [8, 9, 10, 11, 12], a discussion of b) even for the one-dimensional case $\Omega = (0,1)$ can be found in [13]. Up to now, all our energy functionals are of uniform power growth in the sense that the regularizing part involving ∇w can be estimated from above and below by the quantity $\int_\Omega |\nabla w|^q \, dx$ for some power $q \in [1,\infty)$, and the purpose of the present paper is to introduce - at least to some extend - energy functionals and densities F, which allow some flexibility of the growth rate, which means that the growth rate of $F(\nabla w)$ can be prescribed in terms of $|\nabla w|$. To be precise, we consider a density $F : \mathbb{R}^2 \to [0,\infty)$ of class C^2 satisfying $F(0) = 0$ and $DF(0) = 0$. For numbers $c_5, c_6 > 0$ and for exponents

$$p, \mu \in (1,\infty) \tag{1.11}$$

we assume the validity of ($\eta, \xi \in \mathbb{R}^2$)

$$c_5 \,(1+|\xi|)^{-\mu}\, |\eta|^2 \le D^2F(\xi)(\eta,\eta) \le c_6\,(1+|\xi|)^{p-2}\,|\eta|^2\,, \tag{1.12}$$

and in Lemma 2.1 we will show that (1.12) yields the growth estimate ($c_7, \tilde{c}_7, c_8 > 0$)

$$c_7|\xi| - \tilde{c}_7 \le F(\xi) \le c_8\,(|\xi|^p + 1)\ . \tag{1.13}$$

The reader should note that (1.12) implies (1.9), if we allow the choice $p = 1$. An example of a density F with (1.12) is given by ($\varepsilon > 0$)

$$F(\xi) := \int_0^{|\xi|} \int_0^s (\varepsilon + r)^{\varphi(r)-2} \, dr \, ds\,,\ \xi \in \mathbb{R}^2\,, \tag{1.14}$$

for a continuous and decreasing function

$$\varphi : [0,\infty) \to [2-\mu, p], \quad \varphi(0) = p, \quad \lim_{r\to\infty} \varphi(r) = 2-\mu.$$

A discussion of (1.14) together with further examples can be found in Section 5. Assuming (1.12) we then look at the variational problem

$$J[w] := \int_\Omega F(\nabla w)\, dx + \frac{\lambda}{2} \int_{\Omega - D} |w - f|^2 \, dx\,, \tag{1.15}$$

where D is a measurable subset of Ω such that

$$0 \le \mathcal{L}^2(D) < \mathcal{L}^2(\Omega)\,, \tag{1.16}$$

i.e. we study an inpainting problem combined with simultaneous denoising, where D is the inpainting region and the choice $D = \emptyset$ corresponds to the case of pure denoising. We have the following results:

THEOREM 1.2. *Let (1.2), (1.11) and (1.12) hold together with (1.16). Assume in addition that*

$$\mu, p < 2\,. \tag{1.17}$$

Then the variational problem

$$J[w] \to \min \textit{ in } W^{1,1}(\Omega) \tag{1.18}$$

with J defined in (1.15) admits a unique solution u. This solution additionally satisfies $0 \leq u \leq 1$ a.e. on Ω as well as $u \in W^{1,s}_{\mathrm{loc}}(\Omega)$ for any finite s.

REMARK 1.1. *Once having established the local higher integrability result $|\nabla u| \in L^s_{\mathrm{loc}}(\Omega), s < \infty$, we think that actually $u \in C^{1,\alpha}(\Omega)$, $0 < \alpha < 1$, can be deduced along similar lines as in [11], where densities F satisfying (1.9) for some exponent $\mu \in (1, 2)$ are considered.*

REMARK 1.2. *Energy densities F, for which*

$$c_9|\nabla w|^s - \tilde{c}_9 \leq F(\nabla w) \leq c_{10}\left(|\nabla w|^q + 1\right) \tag{1.19}$$

holds or for which an appropriate variant of (1.12) is true, have been extensively discussed for instance in the papers [14, 15, 16, 17, 18, 19, 20, 21, 22] dealing even with the higher-dimensional case including vector-valued functions. Roughly speaking it is shown in the above mentioned papers and the references quoted therein, that (1.19) provides some additional regularity of (local) minimizers, provided $s > 1$ and q is not too far away from s, we refer to [23] for a survey. Recalling that (1.12) implies (1.13), Theorem 1.2 covers the case "$s = 1$", and (1.17) expresses the fact that the upper bound p satisfies "$p < 2s$". Note that the latter requirement turns out to be a sufficient condition for the regularity of minimizers in the setting of [21].

REMARK 1.3. *Variational problems of mixed linear/superlinear growth are the subject of Section 6 in [23]. Here the density F is of splitting form in the sense that*

$$F(\nabla w) = F(\partial_1 w\ \partial_2 w) = F_1(\partial_1 w) + F_2(\partial_2 w) \tag{1.20}$$

with F_1 growing linearly in $|\partial_1 w|$, whereas $F_2(\partial_2 w)$ behaves as $|\partial_2 w|^p$ with power $p > 1$. From the point of view of image restoration condition (1.20) seems to be unnatural, however, if F_1 satisfies (1.9) with $\mu \in (1, 2)$ and if $p < 2$, then regularity results are available, thus our hypothesis (1.17) naturally occurs in the splitting case (1.20).

Next let $\rho : [0, \infty) \to [0, \infty)$ denote a function of class C^1 being strictly increasing and strictly convex, e.g. $\rho(t) = \sqrt{1 + t^2} - 1$, and let

$$K[w] := \int_\Omega F(\nabla w)\,\mathrm{d}x + \int_{\Omega - D} \rho\left(|w - f|\right)\,\mathrm{d}x\,, \tag{1.21}$$

which means that we consider more general data terms.

THEOREM 1.3. *With ρ from above let f, F and D satisfy (1.2), (1.12) and (1.16), respectively, and assume in addition that* $\limsup_{t\to\infty} \frac{\rho(t)}{t^m} < \infty$ *for some $m \geq 1$. Moreover, let*

$$1 < \mu < 3/2\,, \tag{1.22}$$

$$1 < p < \mu\,. \tag{1.23}$$

Then the variational problem

$$K[w] \to \min \ \textit{in} \ \ W^{1,1}(\Omega) \tag{1.24}$$

with K from (1.21) has a unique solution u. It holds $0 \leq u \leq 1$ a.e. on Ω, moreover, $|\nabla u|$ is in $L^s_{\mathrm{loc}}(\Omega)$ for any finite s. If the density F is balanced in the sense that

$$\left|D^2F(\xi)\right| |\xi|^2 \leq c_{11}\left(F\left(\xi\right) + 1\right),\ \xi \in \mathbb{R}^2\,, \tag{1.25}$$

holds for some constant, then (1.23) can be replaced by the requirement $p \in (1, 2)$ (compare (1.17)).

REMARK 1.4. *We conjecture that in the balanced case (1.25) the results of Theorem 1.2 and 1.3 extend to any exponent $p \geq 2$, we refer to Remark 3.1.*

Our paper is organized as follows: in Section 2 we collect some preliminary material and discuss regularized problems approximating (1.18) and (1.24). Section 3 is devoted to the proof of Theorem 1.2, and Theorem 1.3 is established in Section 4. Finally, in Section 5 we present some examples of densities F satisfying (1.12) including the model from (1.14).

2 Some preliminary results and discussion of regularized problems

We start with a growth estimate for densities F satisfying (1.12).

Lemma 2.1. *Suppose that we have the ellipticity condition (1.12) for $F : \mathbb{R}^2 \to [0, \infty)$ with exponents p, μ according to (1.11). Then F is of $(1, p)$-growth in the sense of inequality (1.13).*

Proof. We just consider the case $p \geq 2$. For $p < 2$ the following arguments can be easily adjusted. We recall that F should satisfy $F(0) = 0$, $DF(0) = 0$, thus we obtain from Taylor's theorem (applied to $t \mapsto F(t\xi)$)

$$F(\xi) = \int_0^1 (1 - t)D^2F(t\xi)(\xi, \xi)\,\mathrm{d}t,\ \xi \in \mathbb{R}^2\,. \tag{2.1}$$

Applying (1.12) to the r.h.s. of (2.1) we find

$$(2.2)\qquad c_{12}\int_0^1 (1-t)(1+t|\xi|)^{-\mu}\,\mathrm{d}t|\xi|^2 \le F(\xi) \le c_{13}\int_0^1 (1-t)(1+t|\xi|)^{p-2}\,\mathrm{d}t|\xi|^2\,,$$

and from $(1+t|\xi|)^{p-2}|\xi|^2 \le (1+|\xi|)^p$ (in case $p \ge 2$) we immediately deduce the second inequality in (1.13). If $|\xi| \le 2$, then the first inequality in (1.13) is obvious by an appropriate choice of $c_7, \tilde{c}_7 > 0$. In case $|\xi| \ge 2$ we observe for the l.h.s. of (2.2)

$$\begin{aligned} c_{12}\int_0^1 (1-t)(1+t|\xi|)^{-\mu}\,\mathrm{d}t|\xi|^2 &\ge c_{12}\int_0^{1/|\xi|} (1-t)(1+t|\xi|)^{-\mu}\,\mathrm{d}t|\xi|^2 \\ \ge c_{12}\int_0^{1/|\xi|} (1-t)(1+1)^{-\mu}\,\mathrm{d}t|\xi|^2 &\ge c_{14}\int_{1/2|\xi|}^{1/|\xi|} (1-t)\,\mathrm{d}t|\xi|^2 \\ \ge c_{14}\int_{1/2|\xi|}^{1/|\xi|} \left(1-\frac{1}{|\xi|}\right)\mathrm{d}t|\xi|^2 &\ge c_{14}\int_{1/2|\xi|}^{1/|\xi|} \frac{1}{2}\,\mathrm{d}t|\xi|^2 = c_{15}|\xi|, \end{aligned}$$

thus the first inequality of (1.13) extends to the case $|\xi| \ge 2$ after adjusting $c_7, \tilde{c}_7$. □

REMARK 2.1. *The requirement $DF(0) = 0$ is essential for deducing the lower bound on F stated in (1.13) from the condition of μ-ellipticity, i.e. from the first inequality in (1.12).*

Lemma 2.2. *Under the conditions on the data stated in Theorem 1.2 and 1.3, respectively, but for arbitrary choices of $p, \mu \in (1,\infty)$, the variational problems (1.18) and (1.24) admit at most one solution $u \in W^{1,1,}(\Omega)$. We have*

$$(2.3)\qquad 0 \le u \le 1 \quad \textit{a.e. on } \Omega\,.$$

Proof. From "strict convexity" (for the density F this property follows from the first inequality in (1.12)) we get

$$\begin{cases} \nabla u = \nabla v & \text{a.e.}\quad \text{on } \Omega\,, \\ u = v & \text{a.e.}\quad \text{on } \Omega - D \end{cases}$$

for minimizers $u, v \in W^{1,1}(\Omega)$. But then $u = v$ is a consequence of (1.16). Replacing u by $\min(u,1)$ and $\max(u,0)$ we see by an elementary calculation (compare, e.g., [9]) that (2.3) holds for the minimizer u, since otherwise we could decrease the energy. □
During the proofs of Theorem 1.2 and 1.3 we will essentially benefit from

Lemma 2.3. *Suppose that we are in the situation of Theorem 1.2 or 1.3, where here we allow in both cases exponents $p \in (1,2)$ and $\mu \in (1,\infty)$. For $\delta > 0$ let $u_\delta \in W^{1,2}(\Omega)$ denote the solution of either*

$$(1.18)_\delta\qquad J_\delta[w] := \frac{\delta}{2}\int_\Omega |\nabla w|^2\,\mathrm{d}x + J[w] \to \min \quad \textit{in } W^{1,2}(\Omega)$$

or

$$(1.24)_\delta \qquad K_\delta[w] := \frac{\delta}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x + K[w] \to \min \ \textit{in } W^{1,2}(\Omega)$$

with J and K from (1.15) and (1.21), respectively. It holds:

i) $0 \le u_\delta \le 1$ *a.e. on* Ω.

ii) The functions u_δ are of class $W^{2,2}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

iii) We have the uniform bound $\sup\limits_{\delta>0} \|u_\delta\|_{W^{1,1}(\Omega)} < \infty$.

iv) Suppose that we can find an exponent $q > 1$ such that for each subdomain $\Omega^* \Subset \Omega$

$$(2.4) \qquad \sup_{\delta>0} \int_{\Omega^*} |\nabla u_\delta|^q \,\mathrm{d}x \le c_{16}(\Omega^*) < \infty \,.$$

Then $u_\delta \to: u$ in $L^1(\Omega) \cap W^{1,q}_{\mathrm{loc}}(\Omega)$ as $\delta \to 0$ for a function $u \in W^{1,1}(\Omega)$, and u solves the variational problem (1.18), respectively (1.24).

Proof. i) follows as inequality (2.3) in Lemma 2.2, ii) is immediate from elliptic regularity theory, and iii) is a consequence of the first inequality in (1.13). Let us discuss iv): from i), iii) and assumption (2.4) we deduce the existence of $u \in \mathrm{BV}(\Omega) \cap W^{1,q}_{\mathrm{loc}}(\Omega) \subset W^{1,1}(\Omega)$ such that

$$(2.5) \qquad u_\delta \to u \text{ in } L^1(\Omega) \text{ and a.e. },$$

$$(2.6) \qquad u_\delta \rightharpoondown u \text{ in } W^{1,q}_{\mathrm{loc}}(\Omega)$$

(at least for a subsequence) as $\delta \to 0$. From De Giorgi's theorem on lower semicontinuity (see, e.g., [24] Theorem 2.3, p.18) we see that (2.5) and (2.6) yield

$$(2.7) \qquad J[u] \le \liminf_{\delta\to 0} J[u_\delta]\,,$$

if we are in the situation of Theorem 1.2, whereas

$$(2.8) \qquad K[u] \le \liminf_{\delta\to 0} K[u_\delta]$$

in the setting of Theorem 1.3. Since for $v \in W^{1,2}(\Omega)$ it holds

$$J_\delta[u_\delta] \le J_\delta[v] \overset{\delta\to 0}{\longrightarrow} J[v]\,,$$

we obtain from (2.7) (recall the definition of J_δ in $(1.18)_\delta$)

$$(2.9) \qquad J[u] \le J[v]\,,$$

and by approximation ($W^{1,2}(\Omega) \ni v_k \to v$ in $W^{1,1}(\Omega)$), inequality (2.9) extends to $v \in W^{1,1}(\Omega)$. If the u_δ are the solutions of problem $(1.24)_\delta$, then by the same arguments it follows

$$K[u] \le K[v],\ v \in W^{1,2}(\Omega)\,. \tag{2.10}$$

Consider $v \in W^{1,1}(\Omega)$. In case $K[v] = +\infty$, i.e.

$$\int_{\Omega - D} \rho(|v - f|)\, \mathrm{d}x = +\infty\,,$$

there is nothing to prove. In the other case, due to the growth of ρ at infinity and by (1.2), we see that v is in the space $L^m(\Omega - D)$ and according to [25], Lemma 2.1, we find a sequence $v_k \in C^\infty(\overline{\Omega})$ such that

$$\|v_k - v\|_{W^{1,1}(\Omega)} + \|v_k - v\|_{L^m(\Omega - D)} \longrightarrow 0$$

as $k \to \infty$, hence $K[v_k] \to K[v]$, and since $K[u] \le K[v_k]$ by (2.10), we finally have shown that u solves (1.24). □

3 Proof of Theorem 1.2

In this section we assume that all the hypotheses of Theorem 1.2 are valid and define u_δ as in Lemma 2.3 as the unique solution of problem $(1.18)_\delta$. Let $F_\delta(\xi) := \frac{\delta}{2}|\xi|^2 + F(\xi),\ \xi \in \mathbb{R}^2$. For $\eta \in C^1_0(\Omega)$ with $0 \le \eta \le 1$ we have (by passing to the differentiated version of the Euler equation associated to $(1.18)_\delta$ and by quoting Lemma 2.3 ii))

$$\int_\Omega D^2 F_\delta\left(\nabla u_\delta\right)\left(\partial_\alpha \nabla u_\delta, \nabla[\eta^2 \partial_\alpha u_\delta]\right)\, \mathrm{d}x = \lambda \int_{\Omega - D} (u_\delta - f)\partial_\alpha\left(\eta^2 \partial_\alpha u_\delta\right)\, \mathrm{d}x\,, \tag{3.1}$$

where here and it what follows the sum in taken w.r.t. $\alpha = 1, 2$. It holds

$$\begin{aligned}\text{r.h.s. of (3.1)} &= \lambda \int_\Omega u_\delta \partial_\alpha\left(\eta^2 \partial_\alpha u_\delta\right)\, \mathrm{d}x - \lambda \int_D u_\delta \partial_\alpha\left(\eta^2 \partial_\alpha u_\delta\right)\, \mathrm{d}x \\ &\quad -\lambda \int_{\Omega - D} f \partial_\alpha\left(\eta^2 \partial_\alpha u_\delta\right)\, \mathrm{d}x =: T_1 - T_2 - T_3,\end{aligned}$$

$$T_1 = -\lambda \int_\Omega \eta^2 |\nabla u_\delta|^2\, \mathrm{d}x,$$

$$|T_2| + |T_3| \le c_{17}\left\{\int_\Omega \eta|\nabla\eta||\nabla u_\delta|\, \mathrm{d}x + \int_\Omega \eta^2 |\nabla^2 u_\delta|\, \mathrm{d}x\right\},$$

where we have used (1.2) as well as Lemma 2.3 i), c_k denoting a positive constant independent of δ. Recalling in addition Lemma 2.3 iii) we get from (3.1)

$$\begin{aligned}&\int_\Omega D^2 F_\delta(\nabla u_\delta)\left(\partial_\alpha \nabla u_\delta, \nabla\left[\eta^2 \partial_\alpha u_\delta\right]\right)\, \mathrm{d}x + \lambda \int_\Omega \eta^2 |\nabla u_\delta|^2\, \mathrm{d}x \\ &\qquad \le c_{18}\left\{\|\nabla\eta\|_{L^\infty(\Omega)} + \int_\Omega \eta^2\left|\nabla^2 u_\delta\right|\, \mathrm{d}x\right\}.\end{aligned} \tag{3.2}$$

Applying the Cauchy-Schwarz inequality to the bilinear form $D^2F_\delta(\nabla u_\delta)$ and using Young's inequality, the estimate (3.2) yields

$$\int_\Omega \eta^2 D^2F_\delta(\nabla u_\delta)\,(\partial_\alpha \nabla u_\delta, \partial_\alpha \nabla u_\delta)\,\mathrm{d}x + \int_\Omega \eta^2 |\nabla u_\delta|^2\,\mathrm{d}x$$
$$\leq c_{19}\left\{\int_\Omega D^2F_\delta(\nabla u_\delta)\,(\nabla\eta,\nabla\eta)\,|\nabla u_\delta|^2\,\mathrm{d}x + \int_\Omega \eta^2\left|\nabla^2 u_\delta\right|\,\mathrm{d}x + \|\nabla\eta\|_{L^\infty(\Omega)}\right\},$$

hence using (1.12) for D^2F (dropping the δ-term on the l.h.s.)

$$\int_\Omega \eta^2\,(1+|\nabla u_\delta|)^{-\mu}\left|\nabla^2 u_\delta\right|^2\,\mathrm{d}x + \int_\Omega \eta^2|\nabla u_\delta|^2\,\mathrm{d}x$$
$$\leq c_{20}\left\{\|\nabla\eta\|^2_{L^\infty(\Omega)}\delta\int_\Omega |\nabla u_\delta|^2\,\mathrm{d}x + \|\nabla\eta\|^2_{L^\infty(\Omega)}\int_{\operatorname{spt}\eta}(1+|\nabla u_\delta|)^p\,\mathrm{d}x\right.$$
$$\left. + \|\nabla\eta\|_{L^\infty(\Omega)} + \int_\Omega \eta^2\left|\nabla^2 u_\delta\right|\,\mathrm{d}x\right\}.$$

We remark the validity of $\sup\limits_{\delta>0}\delta\int_\Omega |\nabla u_\delta|^2\,\mathrm{d}x < \infty$ and assume w.l.g. $\|\nabla\eta\|_{L^\infty(\Omega)} \geq 1$. Then we obtain

$$\text{(3.3)}\qquad \int_\Omega \eta^2\,(1+|\nabla u_\delta|)^{-\mu}\left|\nabla^2 u_\delta\right|^2\,\mathrm{d}x + \int_\Omega \eta^2|\nabla u_\delta|^2\,\mathrm{d}x$$
$$\leq c_{21}\left\{\|\nabla\eta\|^2_{L^\infty(\Omega)}\int_{\operatorname{spt}\eta}(1+|\nabla u_\delta|)^p\,\mathrm{d}x + \int_\Omega \eta^2\left|\nabla^2 u_\delta\right|\,\mathrm{d}x + \|\nabla\eta\|^2_{L^\infty(\Omega)}\right\}.$$

On the r.h.s. of (3.3) we use Young's inequality twice recalling (1.17) and (1.11):

$$\|\nabla\eta\|^2_{L^\infty(\Omega)}\int_{\operatorname{spt}\eta}(1+|\nabla u_\delta|)^p\,\mathrm{d}x \leq \tau\int_{\operatorname{spt}\eta}|\nabla u_\delta|^2\,\mathrm{d}x + c_{22}(\tau)\,\|\nabla\eta\|^{\frac{4}{2-p}}_{L^\infty(\Omega)},$$
$$\int_\Omega \eta^2\left|\nabla^2 u_\delta\right|\,\mathrm{d}x \leq \varepsilon\int_\Omega \eta^2\,(1+|\nabla u_\delta|)^{-\mu}\left|\nabla^2 u_\delta\right|^2\,\mathrm{d}x + c_{23}(\varepsilon)\int_\Omega (1+|\nabla u_\delta|)^\mu\,\eta^2\,\mathrm{d}x$$
$$\leq \varepsilon\int_\Omega \eta^2\,(1+|\nabla u_\delta|)^{-\mu}\left|\nabla^2 u_\delta\right|^2\,\mathrm{d}x + \varepsilon\int_\Omega \eta^2|\nabla u_\delta|^2\,\mathrm{d}x + c_{24}(\varepsilon).$$

Inserting these estimates into (3.3), choosing η such that $\eta\equiv 1$ on $B_{r_1}(x_0), \eta\equiv 0$ outside $B_{r_2}(x_0)$, $B_{r_1}(x_0)\subset B_{r_2}(x_0)\Subset\Omega$, we obtain after appropriate choice of ε and τ

$$\text{(3.4)}\qquad \int_{B_{r_1}(x_0)}|\nabla u_\delta|^2\,\mathrm{d}x \leq \frac{1}{2}\int_{B_{r_2}(x_0)}|\nabla u_\delta|^2\,\mathrm{d}x + c_{25}\left((r_2-r_1)^{-\alpha}+1\right),$$

where for the moment we just neglect $\int_\Omega \eta^2(1+|\nabla u_\delta|)^{-\mu}|\nabla^2 u_\delta|^2\,\mathrm{d}x$ and α denotes a suitable positive number. Applying Lemma 3.1, p.161, from [24] to estimate (3.4) we find that (2.4) from Lemma 2.3 holds with the choice $q=2$, and we can quote iv) of Lemma 2.3 yielding a unique $W^{1,1}(\Omega)$- solution u of (1.18).

Going back to (3.3), recalling the estimates stated after (3.3) and applying our bound (2.4) valid for $q = 2$, it follows

$$\int_{\Omega^*} \left|\nabla^2 u_\delta\right|^2 (1 + |\nabla u_\delta|)^{-\mu} \,\mathrm{d}x \le c_{26}(\Omega^*) < \infty$$

for any $\Omega^* \Subset \Omega$, thus ($\varphi_\delta := (1 + |\nabla u_\delta|)^{1-\mu/2}$)

$$\|\varphi_\delta\|_{W^{1,2}(\Omega^*)} \le c_{27}(\Omega^*) < \infty\,,$$

which by Sobolev's theorem implies

$$\|\nabla u_\delta\|_{L^s(\Omega^*)} \le c_{28}(s, \Omega^*) \tag{3.5}$$

for any $s < \infty$. This proves the last claim of Theorem 1.2. □

REMARK 3.1. *Suppose that F satisfies the condition (1.25). In this case we estimate*

$$D^2 F_\delta(\nabla u_\delta)(\nabla\eta, \nabla\eta)\, |\nabla u_\delta|^2 \le c_{29}\,(F_\delta(\nabla u_\delta) + 1)$$

and observe $\sup\limits_{\delta>0} \displaystyle\int_\Omega F_\delta(\nabla u_\delta)\,\mathrm{d}x < \infty$. Thus we can replace $\int_{\operatorname{spt}\eta} (1 + |\nabla u_\delta|)^p \,\mathrm{d}x$ in (3.3) through a constant ending up with

$$\int_\Omega \eta^2\,(1 + |\nabla u_\delta|)^{-\mu} \left|\nabla^2 u_\delta\right|^2 \mathrm{d}x + \int_\Omega \eta^2\, |\nabla u_\delta|^2 \,\mathrm{d}x \le c_{30}(\eta)\,,$$

hence we obtain (3.5) just assuming $\mu \in (1, 2)$. Thus the bound (1.17) imposed on p at this stage does not enter, however during our proof we work with the quadratic regularization $(1.18)_\delta$, which requires $p \le 2$. In other words: under the assumption (1.25) the claims of Theorem 1.2 extend to exponents $p > 2$ (keeping the bound $1 < \mu < 2$) and a proof can be carried out by working with the regularization

$$\delta \int_\Omega \left(1 + |\nabla w|^2\right)^{\overline{p}/2} \mathrm{d}x + J[w] \to \min \ \text{ in } W^{1,\overline{p}}(\Omega)$$

for some exponent $\overline{p} > p$. We leave the details to the reader.

4 Proof of Theorem 1.3

Let the assumptions of Theorem 1.3 hold. In place of equation (3.1) we have

$$\begin{aligned} &\int_\Omega D^2 F_\delta(\nabla u_\delta)\left(\partial_\alpha \nabla u_\delta, \nabla\left[\eta^2 \partial_\alpha u_\delta\right]\right) \mathrm{d}x \\ &\qquad = \int_{\Omega - D} \rho'\left(|u_\delta - f|\right) \frac{u_\delta - f}{|u_\delta - f|} \partial_\alpha\left(\eta^2 \partial_\alpha u_\delta\right) \mathrm{d}x\,, \end{aligned} \tag{4.1}$$

where u_δ is the solution of problem $(1.24)_\delta$ (see Lemma 2.3). From (1.2) and Lemma 2.3 i) it follows

$$\text{r.h.s. of (4.1) } \leq c_{31} \int_\Omega \left|\partial_\alpha \left(\eta^2 \partial_\alpha u_\delta\right)\right| \, \mathrm{d}x$$

and clearly (recall Lemma 2.3 iii))

$$\int_\Omega \left|\partial_\alpha \left(\eta^2 \partial_\alpha u_\delta\right)\right| \, \mathrm{d}x \leq c_{32}(\eta) + c_{33} \int_\Omega \eta^2 \left|\nabla^2 u_\delta\right| \, \mathrm{d}x \, , \tag{4.2}$$

where we use the symbol $c_k(\eta)$ to denote constants proportional to $\|\nabla \eta\|^\alpha_{L^\infty(\Omega)}$ for some positive exponent α. Applying Young's inequality to the integral on the r.h.s. of (4.2) and discussing the l.h.s. of (4.1) as done after (3.2) we find

$$\begin{aligned} &\int_\Omega \eta^2 \left(1 + |\nabla u_\delta|\right)^{-\mu} \left|\nabla^2 u_\delta\right|^2 \, \mathrm{d}x \\ &\qquad \leq c_{34} \int_\Omega \eta^2 \left(1 + |\nabla u_\delta|\right)^{\mu} \, \mathrm{d}x + c_{35}(\eta) \int_{\operatorname{spt}\eta} \left(1 + |\nabla u_\delta|\right)^p \, \mathrm{d}x \, . \end{aligned} \tag{4.3}$$

We specify η as in Section 3 and let

$$\varphi_\delta := \left(1 + |\nabla u_\delta|\right)^{1-\mu/2} , \ \Psi_\delta := \left(1 + |\nabla u_\delta|\right)^{\mu/2} \, .$$

Then (4.3) shows (with suitable $\alpha_1 > 0$)

$$\int_\Omega \eta^2 |\nabla \varphi_\delta|^2 \, \mathrm{d}x \leq c_{36} \left\{ \int_\Omega \eta^2 \Psi_\delta^2 \, \mathrm{d}x + (r_2 - r_1)^{-\alpha_1} \int_{B_{r_2}(x_0)} \left(1 + |\nabla u_\delta|\right)^p \, \mathrm{d}x \right\} . \tag{4.4}$$

Next we observe (quoting Sobolev's inequality)

$$\begin{aligned} \int_\Omega (\eta\Psi)^2 \, \mathrm{d}x \leq c_{37} \left(\int_\Omega |\nabla(\eta\Psi_\delta)| \, \mathrm{d}x \right)^2 &\leq c_{38} \left[\int_\Omega |\nabla\eta| \Psi_\delta \, \mathrm{d}x + \int_\Omega \eta \, |\nabla \Psi_\delta| \, \mathrm{d}x \right]^2 \\ &\leq c_{39}(\nabla\eta) + c_{40} \left(\int_\Omega |\nabla \Psi_\delta| \eta \, \mathrm{d}x \right)^2 , \end{aligned} \tag{4.5}$$

where we have used that $\sup_{\delta>0} \int_\Omega \Psi_\delta \, \mathrm{d}x < \infty$ on account of Lemma 2.3 iii). We discuss the remaining integral on the r.h.s. of (4.5) observing that $\Psi_\delta = \varphi_\delta^{\mu/(2-\mu)}$ and using Hölder's inequality:

$$\begin{aligned} &\int_\Omega \eta |\nabla \Psi_\delta| \, \mathrm{d}x \leq c_{41} \int_\Omega \eta |\nabla \varphi_\delta| \varphi_\delta^{\frac{\mu}{2-\mu} - 1} \, \mathrm{d}x \\ &\leq c_{42} \left(\int_\Omega \eta^2 \left|\nabla \varphi_\delta\right|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_{B_{r_2}(x_0)} \varphi_\delta^{2\frac{2\mu-2}{2-\mu}} \, \mathrm{d}x \right)^{1/2} . \end{aligned}$$

We have

$$\varphi_\delta^{2\frac{2\mu-2}{2-\mu}} = \left(1 + |\nabla u_\delta|\right)^{2\mu-2}$$

with exponent $2\mu - 2 \in (0,1)$, which follows from (1.22). Quoting Lemma 2.3 iii) one more time, another application of Hölder's inequality gives (for some $\alpha_2 > 0$)

$$\int_\Omega \eta |\nabla \Psi_\delta| \, \mathrm{d}x \le c_{43} \, r_2^{\alpha_2} \left(\int_\Omega \eta^2 \, |\nabla \varphi_\delta|^2 \, \mathrm{d}x \right)^{1/2} . \tag{4.6}$$

We insert (4.6) into (4.5) giving the bound

$$\int_\Omega (\eta \Psi_\delta)^2 \, \mathrm{d}x \le c_{44} \, (\nabla \eta) + c_{45} \, r_2^{2\alpha_2} \int_\Omega \eta^2 |\nabla \varphi_\delta|^2 \, \mathrm{d}x \, . \tag{4.7}$$

With (4.7) we return to (4.4) and assume that the radius r_2 is sufficiently small, thus

$$\int_\Omega \eta^2 |\nabla \varphi_\delta|^2 \, \mathrm{d}x \le c_{46} (r_2 - r_1)^{-\alpha_3} \left(1 + \int_{B_{r_2}(x_0)} (1 + |\nabla u_\delta|)^p \, \mathrm{d}x \right) . \tag{4.8}$$

Up to now we have not used our hypothesis (1.23), which enters next:

$$\begin{aligned} \int_{B_{r_1}(x_0)} (1 + |\nabla u_\delta|)^\mu \, \mathrm{d}x &\le \int_\Omega (\eta \Psi_\delta)^2 \, \mathrm{d}x \\ &\overset{(4.7),\,(4.8)}{\le} c_{47} \left[(r_2 - r_1)^{-\alpha_4} + (r_2 - r_1)^{-\alpha_5} \int_{B_{r_2}(x_0)} (1 + |\nabla u_\delta|)^p \, \mathrm{d}x \right] \\ &\le c_{48} \, (r_2 - r_1)^{-\alpha_6} + \frac{1}{2} \int_{B_{r_2}(x_0)} (1 + |\nabla u_\delta|)^\mu \, \mathrm{d}x \, , \end{aligned}$$

where in the last estimate we applied Hölder's inequality and use the smallness of r_2 to get the factor 1/2. As outlined after (3.4) we deduce (2.4) with value $q := \mu$. Moreover, using this information in (4.4), we see

$$\sup_{\delta > 0} \|\varphi_\delta\|_{W^{1,2}(\Omega^*)} < \infty$$

for any subdomain $\Omega^* \Subset \Omega$, thus (3.5) holds, and we get all the results of Theorem 1.3 as described in Section 3, where for the balancing case we refer to Remark 3.1. □

5 Examples

In this section we focus on energy densities depending on the modulus of ∇u, a situation for which the following observations are helpful.

Proposition 5.1. *Let $g : [0,\infty) \to [0,\infty)$ denote a C^2-function for which $g(0) = g'(0) = 0,\ g'' \ge 0$. Then*

$$G : \mathbb{R}^2 \to [0,\infty),\ G(\xi) := g(|\xi|) \, , \tag{5.1}$$

is a convex function of class C^2 *for which* $G(0)=0$, $DG(0)=0$ *and*

$$\min\left\{g''(|\xi|)\,,\frac{1}{|\xi|}g'(|\xi|)\right\}|\eta|^2 \le D^2G(\xi)(\eta,\eta) \tag{5.2}$$
$$\le \max\left\{g''(|\xi|)\,,\frac{1}{|\xi|}g'(|\xi|)\right\}|\eta|^2,\ \ \xi,\eta\in\mathbb{R}^2.$$

Proof. We just note that (5.2) follows from the formula

$$D^2G(\xi)(\eta,\eta)=\frac{1}{|\xi|}g'(|\xi|)\left[|\eta|^2-\frac{(\eta\cdot\xi)^2}{|\xi|^2}\right]+g''(|\xi|)\frac{(\eta\cdot\xi)^2}{|\xi|^2}\,.$$

□

If $G:\mathbb{R}^2\to[0,\infty)$ is a non-negative function of class C^2, we recall the balancing condition (see (1.25)):

$$\left|D^2G(\xi)\right||\xi|^2\le c_{49}\left(G\left(\xi\right)+1\right),\ \ \xi\in\mathbb{R}^2\,. \tag{5.3}$$

Proposition 5.2. *Let* g *satisfy the assumptions of Proposition 5.1. Assume further that*

$$t^2\max\left\{g''(t),\frac{1}{t}g'(t)\right\}\le c_{50}\left(g(t)+1\right),\ \ t\ge 0\,. \tag{5.4}$$

Then G *from (5.1) satisfies (5.3).*

Proof. This is an immediate consequence of (5.2) and (5.4). □

Proposition 5.3. *If* g *is a function as in Proposition 5.1 such that*

$$tg''(t)\le c_{51}\ g'(t),\ \ t\ge 0\,, \tag{5.5}$$

and if g *satisfies the* $(\Delta 2)$*-condition, i.e.*

$$g(2t)\le c_{52}\ g(t),\ \ t\ge 0\,, \tag{5.6}$$

then inequality (5.4) holds. Thus $G(\xi):=g(|\xi|)$, $\xi\in\mathbb{R}^2$, *is balanced in the sense of (5.3).*

Proof. We have (recalling $g(0)=0$ as well as $g'(t)\ge 0$)

$$g(t)=\int_0^t g'(s)\,\mathrm{d}s\ge\int_{t/2}^t g'(s)\,\mathrm{d}s\ge\frac{t}{2}g'(t/2)\,,$$

where the last inequality follows from the fact that g' is increasing on account of our hypothesis $g''\ge 0$. Thus we get

$$t\,g'(t)\le g(2t)\le c_{52}\ g(t)$$

(see (5.6)), and together with (5.5) we arrive at (5.4). □

All our energy densities "$G(\nabla u)$" discussed below are of the principal form (compare (1.6) and (1.14))

$$G(\xi) := g(|\xi|) := \int_0^{|\xi|}\int_0^s \omega(r)\,\mathrm{d}r\,\mathrm{d}s,\ \xi\in\mathbb{R}^2\,, \tag{5.7}$$

for a continuous function $\omega : [0,\infty) \to [0,\infty)$ such that $\omega(s) > 0$ for $s > 0$. Note that g is strictly increasing and strictly convex implying the strict convexity of G on $\mathbb{R}^2$.

Example 5.1. *Consider a continuous function $\eta : [0,\infty) \to [0,1]$ and define G according to (5.7) with the choice*

$$\omega(t) := \eta(t)(1+t)^{-\mu} + (1-\eta(t))\,(1+t)^{p-2},\ t\ge 0\,, \tag{5.8}$$

for exponents $p,\mu \in (1,\infty)$.

Proposition 5.4. *The density G satisfies (1.12).*

Proof. It holds on account of $\eta(t) \in [0,1]$ and $(1+t)^{-\mu} \le (1+t)^{p-2}$ for any $t \ge 0$

$$g''(t) \overset{(5.7)}{=} \omega(t) \overset{(5.8)}{\le} (1+t)^{p-2}\,,$$

$$\frac{1}{t}g'(t) \overset{(5.7)}{=} \frac{1}{t}\int_0^t \omega(s)\,\mathrm{d}s \overset{(5.8)}{\le} \frac{1}{t}\int_0^t (1+s)^{p-2}\,\mathrm{d}s = \frac{1}{t}\frac{1}{p-1}\left\{(t+1)^{p-1}-1\right\}\,.$$

In case $t\ge 1$ we observe $\{\dots\} \le (2t)^{p-1}$, thus $\frac{1}{t}g'(t) \le c_{53}\,t^{p-2} \le c_{54}(1+t)^{p-2}$, whereas in case $t \le 1$ we use

$$\frac{1}{t}\left\{(t+1)^{p-1}-1\right\} = (p-1)(T+1)^{p-2}$$

for some $T \in (0,1)$, hence

$$\frac{1}{t}g'(t) \le c_{55} \le c_{56}(1+t)^{p-2}\,,$$

and the second inequality in (5.2) gives the upper bound

$$D^2G(\xi)(\eta,\eta) \le c_{57}\,(1+|\xi|)^{p-2}\,|\eta|^2\,. \tag{5.9}$$

With analogous calculations we obtain a lower bound:

$$g''(t) = \omega(t) \ge (1+t)^{-\mu},\ t\ge 0\,,$$

$$\frac{1}{t}g'(t) = \frac{1}{t}\int_0^t \omega(s)\,\mathrm{d}s \ge \frac{1}{t}\int_0^t (1+s)^{-\mu}\,\mathrm{d}s = \frac{1}{t(1-\mu)}\left\{\,(1+t)^{1-\mu}-1\right\}.$$

Case 1: $t \ge t(\mu)$ (≥ 1 sufficiently large). Then we have after appropriate choice of $t(\mu)$

$$\frac{1}{t}g'(t) \ge c_{58}\frac{1}{t} \ge c_{58}(1+t)^{-\mu}\,.$$

Case 2: $t \le t(\mu)$. Here we observe

$$\frac{1}{t}\big\{(1+t)^{1-\mu}-1\big\} = (1-\mu)(1+\widetilde{T})^{-\mu}$$

for a suitable $\widetilde{T} \in (0, t(\mu))$, hence

$$\frac{1}{t}g'(t) \ge (1+\widetilde{T})^{-\mu} \ge c_{59} \ge c_{59}(1+t)^{-\mu} .$$

Recalling (5.9) and (5.2), the above estimates imply (1.12) for our density G. □

REMARK 5.1. *An equivalent form of (5.8) is given by*

$$\widetilde{(5.8)} \qquad \omega(t) := \Theta(t)(1+t)^{-\mu}, \ t \ge 0 ,$$

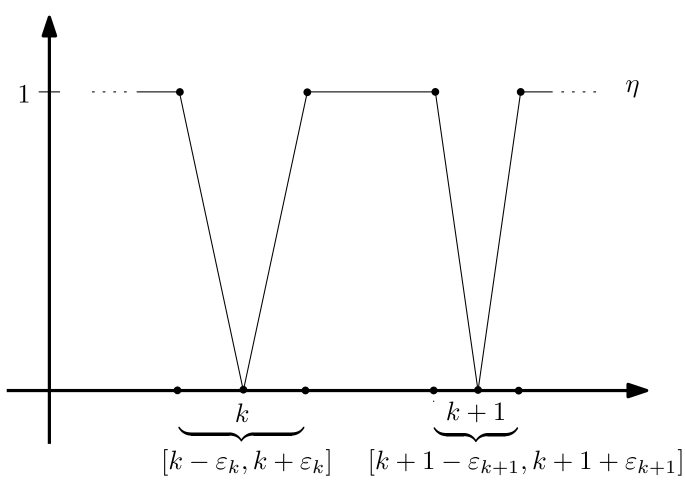
for a continuous function $\Theta : [0,\infty) \to [0,\infty)$ *such that* $1 \le \Theta(t) \le (1+t)^{\mu+p-2}, \ t \ge 0$.
In fact, if η *is given, let*

$$\Theta(t) := \eta(t) + (1-\eta(t))(1+t)^{p+\mu-2} ,$$

and if we start from $\widetilde{(5.8)}$ *we obtain (5.8) by defining*

$$\eta(t) := \left((1+t)^{p-2} - \Theta(t)(1+t)^{-\mu}\right)\left((1+t)^{p-2} - (1+t)^{-\mu}\right)^{-1} .$$

The density G defined in (5.7) with ω as in (5.8) in general does not satisfy the balancing condition (5.3): consider $\eta : [0,\infty) \to [0,1]$ as indicated in the picture below:



Here ε_k denotes a suitable sequence going to zero, we let $\eta \equiv 1$ on each interval $[k+\varepsilon_k, k+1-\varepsilon_{k+1}]$ with linear interpolation on $[k-\varepsilon_k, k+\varepsilon_k]$ such that $\eta(k) = 0$ for each k. Then it holds for $t \in \mathbb{N}$

$$t^2 g''(t) = t^2(1+t)^{p-2}$$

and at the same time (after appropriate choice of ε_k)

$$g(s) \le c_{60}\, s \tag{5.10}$$

for $s \ge 0$ sufficiently large, hence (5.4) is violated. We discuss (5.10): it holds

$$g(t) = \int_0^t \int_0^s (1+r)^{-\mu}\,\mathrm{d}r\,\mathrm{d}s + \int_0^t \int_0^s (1-\eta(r))\left\{(1+r)^{p-2} - (1+r)^{-\mu}\right\}\mathrm{d}r\,\mathrm{d}s\,.$$

For simplicity let us assume $p \le 2$. Then $\{\ldots\} \le 1$, hence(compare (1.5) - (1.8))

$$\begin{aligned} g(t) &\le c_{61}\left\{t + \int_0^t \int_0^s (1-\eta(r))\,\mathrm{d}r\,\mathrm{d}s\right\} \le c_{61}\left\{t + \int_0^t \int_0^t (1-\eta(r))\,\mathrm{d}r\,\mathrm{d}s\right\} \\ &\le c_{61}\left\{t + t\int_0^\infty (1-\eta(r))\,\mathrm{d}r\right\} \le c_{62}\, t\left\{1 + \sum_{k=1}^\infty \varepsilon_k\right\}\ , \end{aligned}$$

and we obtain (5.10) from the requirement that $\sum_{k=1}^\infty \varepsilon_k < \infty$. In the case $p > 2$ a slight modification is necessary still leading to (5.10). In addition, we can choose different functions η to get $\lim_{t\to\infty} g(t)/t^q > 0$ for a given number $q \in (1,p)$.

REMARK 5.2. *As a matter of fact our previous considerations extend to densities*

$$G(\nabla u) = \int_0^{|\nabla u|} \int_0^s \left[\eta(r)(\varepsilon_1 + r)^{-\mu} + (1-\eta(r))\,(\varepsilon_2 + r)^{p-2}\right]\mathrm{d}r\,\mathrm{d}s$$

with positive numbers $\varepsilon_1, \varepsilon_2$ and with weight-function η as in (5.8).

Example 5.2. *We let*

$$G(\xi) := g(|\xi|) := \int_0^{|\xi|} \int_0^s (\varepsilon + r)^{\rho(r)-2}\,\mathrm{d}r\,\mathrm{d}s,\ \ \xi \in \mathbb{R}^2\,, \tag{5.11}$$

with $\varepsilon > 0$ and for a continuous and decreasing function

$$\rho : [0,\infty) \to [-\mu+2, p], \quad \rho(0) = p, \quad \lim_{r\to\infty} \rho(r) = 2-\mu \tag{5.12}$$

with exponents $p, \mu > 1$. Note that (5.11), (5.12) can be seen as an approximation of the density $G(\nabla u) = |\nabla u|^{p(|\nabla u|)}$ where $p(|\nabla u|)$ decreases from p to 1 as $|\nabla u|$ ranges from 0 to ∞, introduced by Blomgren, Chan and Mulet [26] for $p = 2$.

Proposition 5.5. *The density G from (5.11) with ρ defined in (5.12) satisfies the ellipticity condition (1.12), moreover, the balancing inequality (5.3) holds.*

Proof. W.l.o.g. we let $\varepsilon = 1$ and observe for any $t \ge 0$

$$(1+t)^{-\mu} \le (1+t)^{\rho(t)-2} = g''(t) \le (1+t)^{p-2},$$

moreover, it holds

$$\frac{1}{t}g'(t) = \frac{1}{t}\int_0^t \big(1+s\big)^{\rho(s)-2}\,\mathrm{d}x \le \frac{1}{t}\int_0^t (1+s)^{p-2}\ \mathrm{d}s \le c_{63}(1+t)^{p-2},$$
$$\frac{1}{t}\ g'(t) \ge \frac{1}{t}\int_0^t (1+s)^{-\mu}\,\mathrm{d}s \ge c_{64}(1+t)^{-\mu},$$

we refer to the proof of Proposition 5.4. Thus (1.12) follows from Proposition 5.1. Next we discuss (5.4) for g by referring to Proposition 5.3: we have

$$g(2t) = \int_0^{2t} g'(s)\,\mathrm{d}s = \int_0^t 2g'(2s)\,\mathrm{d}s$$

$$\overset{(5.11)}{=} 2\int_0^t\int_0^{2s} (1+r)^{\rho(r)-2}\,\mathrm{d}r\,\mathrm{d}s = 4\int_0^t\int_0^s (1+2r)^{\rho(2r)-2}\,\mathrm{d}r\,\mathrm{d}s\,.$$

In case $p \le 2$ we get by the properties of ρ

$$(1+2r)^{\rho(2r)-2} \le (1+r)^{\rho(2r)-2}, \quad (1+r)^{\rho(2r)-2} \le (1+r)^{\rho(r)-2}$$

(note: $\rho(2r) \le \rho(r)$), hence

$$g(2t) \le 4\int_0^t\int_0^s (1+r)^{\rho(r)-2}\,\mathrm{d}r\,\mathrm{d}s = 4g(t)\,.$$

If the value of $p > 1$ is arbitrary, we write

$$(1+2r)^{\rho(2r)-2} = (1+r)^{\rho(2r)-2}\left(\frac{1+2r}{1+r}\right)^{\rho(2r)-2}$$

and use the fact that

$$\lim_{r\to\infty}\left(\frac{1+2r}{1+r}\right)^{\rho(2r)-2} = 2^{-\mu},$$

thus $(1+2r)^{\rho(2r)-2} \le c_{65}(1+r)^{\rho(2r)-2}$, and by recalling $\rho(2r) \le \rho(r)$ we obtain as before inequality (5.6) with a suitable constant. It remains to check (5.5): we have

$$g'(t) = \int_0^t (1+s)^{\rho(s)-2}\,\mathrm{d}s \ge \int_{t/2}^t (1+s)^{\rho(s)-2}\,\mathrm{d}s \ge \int_{t/2}^t (1+s)^{\rho(t)-2}\,\mathrm{d}s,$$

since ρ decreases. Writing

$$\int_{t/2}^t (1+s)^{\rho(t)-2}\,\mathrm{d}s = (1+t)^{\rho(t)-2}\int_{t/2}^t \left\{\frac{1+s}{1+t}\right\}^{\rho(t)-2}\,\mathrm{d}s$$

and observing that

$$\left\{\frac{1+s}{1+t}\right\}^{\rho(t)-2} \ge c_{66} > 0 \ \text{ on } [t/2, t]\,,$$

we see that $g'(t) \ge c_{66}\ \frac{t}{2}(1+t)^{\rho(t)-2}$ and (5.5) is established. □

References

[1] L. I. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259 – 268, 1992.

[2] E. Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser, Basel, 1984.

[3] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Clarendon Press, Oxford, 2000.

[4] A. Tikhonov and V. Arsenin. *Solutions of ill-posed problems. Translated from the Russian. Preface by translation editor Fritz John.* Scripta Series in Mathematics. V. H. Winston & Sons, Washington, D.C.: John Wiley & Sons, New York-Toronto, Ont.-London, 1977.

[5] J. Weickert. *Anisotropic Diffusion in Image Processing*. ECMI Series. Teubner, Stuttgart, 1998.

[6] R. A. Adams. *Sobolev spaces*, volume 65 of *Pure and Applied Mathematics*. Academic Press, New-York-London, 1975.

[7] J. Malỳ and W.P. Ziemer. *Fine Regularity of Solutions of Elliptic Partial Differential Equations*. Mathematical surveys and monographs. American Mathematical Society, 1997.

[8] M. Bildhauer and M. Fuchs. A variational approach to the denoising of images based on different variants of the TV-regularization. *Appl. Math. Optim.*, 66(3):331 – 361, 2012.

[9] M. Bildhauer and M. Fuchs. On some perturbations of the total variation image inpainting method. part I: regularity theory. *J. Math. Sciences*, 202(2):154 – 169, 2014.

[10] M. Bildhauer and M. Fuchs. On some perturbations of the total variation image inpainting method. part II: relaxation and dual variational formulation. *J. Math. Sciences*, 205(2):121 – 140, 2015.

[11] M. Bildhauer, M. Fuchs, and C. Tietz. $C^{1,\alpha}$-interior regularity for minimizers of a class of variational problems with linear growth related to image inpainting. *Algebra i Analiz*, 27(3):51–65, 2015.

[12] M. Bildhauer, M. Fuchs, and J. Weickert. Denoising and inpainting of images using tv-type energies: Theoretical and computational aspects. *J. Math. Sciences*, 219(6):899–910, 2016.

[13] M. Fuchs, J. Müller, and C. Tietz. Signal recovery via TV-type energies. To appear in St.Petersburg Math.J.

[14] P. Marcellini. Regularity of minimizers of integrals of the calculus of variations with non-standard growth conditions. *Arch. Rat. Mech. Anal.*, 105:267–284, 1989.

[15] P. Marcellini. Regularity and existence of solutions of elliptic equations with p,q-growth conditions. *J. Diff. Equ.*, 105:296–333, 1993.

[16] P. Marcellini. Regularity for elliptic equations with general growth conditions. *J. Diff. Equ.*, 90(1):1–30, 1991.

[17] P. Marcellini. Regularity for some scalar variational problems under general growth conditions. *Journal of Optimization Theory and Applications*, 90(1):161–181, 1996.

[18] L. Esposito, F. Leonetti, and G. Mingione. Regularity results for minimizers of irregular integrals with (p,q) growth. *Forum Math.*, 14(2):245–272, 1998.

[19] L. Esposito, F. Leonetti, and G. Mingione. Regularity for minimizers of convex integrals with $L \log L$-growth. *Nonlinear Diff. Equ. Appl.*, 7:107–125, 2000.

[20] M.Bildhauer and M. Fuchs. Partial regularity for variational integrals with (s,μ,q)-growth. *Calc. Var.*, 13:637–560, 2001.

[21] M.Bildhauer and M. Fuchs. Twodimensional anisotropic variational problems. *Calc. Var.*, 16:177–186, 2003.

[22] M.Bildhauer and M. Fuchs. Elliptic variational problems with nonstandard growth. *International Mathematical Series, Vol. 1, Nonlinear problems in mathematical physics and related topics I, in honor of Prof. O.A. Ladyzhenskaya. By T. Rozhkovskaya, Novosibirsk, Russia, 2002 (in Russian), 49–62. By Kluwer/Plenum Publishers, 2002 (in English), 53–66.*

[23] M. Bildhauer. *Convex Variational Problems: Linear, nearly Linear and Anisotropic Growth Conditions*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2003.

[24] M. Giaquinta. *Multiple integrals in the calculus of variations and nonlinear elliptic systems*. Annals of Mathematics Studies. Princeton University Press, Princeton, New Jersey, 1983.

[25] M. Fuchs and C. Tietz. Existence of generalized minimizers and of dual solutions for a class of variational problems with linear growth related to image recovery. *J. Math. Sciences*, 210(4):458 – 475, 2015.

[26] P. Blomgren, T.F. Chan, and P. Mulet. Extensions to total variation denoising. *F.T. Luk (Ed.), Advanced signal processing: algorithms, architectures, and implementations VII*, SPIE Vol. 3162, 1997.

✉ Martin Fuchs (fuchs@math.uni-sb.de), Jan Müller (jmueller@math.uni-sb.de)
Saarland University (Department of Mathematics), P.O. Box 15 11 50, 66041 Saarbrücken, Germany