Universität des Saarlandes

Fachrichtung Mathematik

Preprint Nr. 398

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Michael Bildhauer, Martin Fuchs and Jan Müller

Saarbrücken 2018

Fachrichtung Mathematik
Universität des Saarlandes

Preprint No. 398
submitted: February 27, 2018

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Michael Bildhauer
Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
bibi@math.uni-sb.de

Martin Fuchs
Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Jan Müller
Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
jmueller@math.uni-sb.de

Edited by
FR Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

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M. Bildhauer M. Fuchs J. Müller

AMS classification: 76A05, 76M30, 49Q20, 49N60
Keywords: non-Newtonian fluids, potentials with linear growth, functions of bounded deformation, existence and regularity of solutions

Abstract

We consider the slow flow of an incompressible fluid assuming in addition that the flow is also stationary. Our main assumption concerns the dissipative potential which is of linear growth with respect to the symmetric gradient of the velocity field. Thus our model can be seen as an approximation of the perfectly plastic case introduced by von Mises, and we will establish various results on existence and regularity of a solution.

1 Introduction

On a bounded region $\Omega \subset \mathbb{R}^d$ we investigate the following set of equations ($\varepsilon(u) := \frac{1}{2}(\nabla u + \nabla u^T)$ denoting the symmetric gradient)

$$(1.1) \qquad \begin{cases} -\operatorname{div}\left[DF\big(\varepsilon(u)\big)\right] + \nabla\pi = f, \\ \operatorname{div} u = 0 \ \text{ on } \Omega \end{cases}$$

together with the homogeneous boundary condition

$$(1.2) \qquad u\big|_{\partial\Omega} = 0,$$

Michael Bildhauer (bibi@math.uni-sb.de),
Martin Fuchs (fuchs@math.uni-sb.de),
✉Jan Müller (corresponding author) (jmueller@math.uni-sb.de),
Saarland University (Department of Mathematics), P.O. Box 15 11 50, 66041 Saarbrücken, Germany.

which can be regarded as the governing laws of a stationary flow of an incompressible fluid through the domain Ω assuming in addition that the velocity field $u : \Omega \to \mathbb{R}^d$ is "small", which means that we neglect the convective term $(\nabla u)u$ and just look at a Stokes-type problem with non-slip boundary condition (1.2) and a given system of volume forces $f : \Omega \to \mathbb{R}^d$, whereas the quantity π occurring in (1.1) denotes the a priori unknown pressure function $\pi : \Omega \to \mathbb{R}^d$. The mathematical and physical background leading to the equations (1.1) and (1.2) is explained in the fundamental monographs of Ladyzhenskaya [1] and Galdi [2], [3].

Our main concern is the discussion of the situation, when the dissipative potential

$$F : \mathbb{S}^d \to [0,\infty) \quad (\mathbb{S}^d \text{ being the space of symmetric } (d \times d)\text{-matrices})$$

approximates the case of a perfectly plastic fluid first introduced by von Mises [4] with further investigations e.g. in [5] and [6]. For a perfectly plastic fluid it holds

$$F(\varepsilon) = \nu_1 |\varepsilon|, \quad \varepsilon \in \mathbb{S}^d, \tag{1.3}$$

ν_1 denoting a positive constant, and since in case (1.3) the potential F is neither strictly convex nor differentiable, it is not obvious how to solve (1.1) together with (1.2). We therefore replace (1.3) with a density F of the form

$$F(\varepsilon) = h\big(|\varepsilon|\big) \tag{1.4}$$

for a smooth and strictly convex function $h : [0,\infty) \to [0,\infty)$, which in addition is strictly increasing and of linear growth in the sense that

$$\nu_1 t - \nu_2 \le h(t) \le \nu_3 t + \nu_4, \quad t \ge 0 \tag{1.5}$$

holds with constants $\nu_1, \nu_3 > 0$, $\nu_2, \nu_4 \ge 0$. A typical example is given by $h(t) := \Phi_\mu(t)$ with

$$\begin{aligned} \Phi_\mu(t) &:= \int_0^t \int_0^s (1+r)^{-\mu} \,\mathrm{d}r \,\mathrm{d}s \\ &= \begin{cases} \dfrac{1}{\mu-1} t + \dfrac{1}{\mu-1}\dfrac{1}{\mu-2}\big(t+1\big)^{-\mu+2} - \dfrac{1}{\mu-1}\dfrac{1}{\mu-2}, \ \mu \neq 2, \\ t - \ln(1+t), \ \mu = 2, \end{cases} \end{aligned} \tag{1.6}$$

where $\mu \in (1,\infty)$ is fixed. Note that

$$\lim_{\mu\to\infty} (\mu-1)\Phi_\mu\big(|\varepsilon|\big) = |\varepsilon|, \quad \varepsilon \in \mathbb{S}^d, \tag{1.7}$$

i.e. we have an approximation of the perfectly plastic case. For completeness we would like to mention that if we formally let $\mu = 1$ in the definition of Φ_μ, then we recover the Prandtl-Eyring fluid model for which – up to negligible terms – it holds $F(\varepsilon) = |\varepsilon| \ln\big(1+|\varepsilon|\big)$. In this case our problem (1.1), (1.2) admits a unique solution in the Orlicz-Sobolev space generated by the N-function $t \to t\ln(1+t)$. Moreover, various regularity results are available. We refer the reader to [7], chapter 4 of [8], [9] and the references quoted therein. If we consider values $\mu \in (-\infty, 1)$ in the definition (1.6) of Φ_μ, then we obtain a dissipative potential of p-growth by letting $F(\varepsilon) := \Phi_\mu\big(|\varepsilon|\big)$, i.e.

$$F(\varepsilon) \approx |\varepsilon|^p, \quad p := 2 - \mu \in (1, \infty),$$

and problem (1.1), (1.2) is well-posed in the standard Sobolev space $W^{1,p}(\Omega, \mathbb{R}^d)$, compare e.g. [1], [2], [3] and chapter 3 of [8]. For regularity results in the setting of power law models, the reader should consult the papers [10], [11], [12] and even more general potentials F of superlinear growth are the subject of the investigations in [13, 14, 15, 16, 17, 18].

Let us come back to the linear growth case (1.5) to which none of the above mentioned references applies. The appropriate weak form of (1.1) is

$$\int_\Omega DF\big(\varepsilon(u)\big) : \varepsilon(\varphi)\,\mathrm{d}x = \int_\Omega f \cdot \varphi \,\mathrm{d}x \quad \text{for all } \varphi \in C_0^\infty(\Omega, \mathbb{R}^d) \text{ with } \operatorname{div}\varphi = 0 \tag{1.8}$$

and the function u has to be found in a suitable space of solenoidal fields together with an appropriate version of (1.2). From the theory of perfect plasticity it is known that a suitable version of a safe load condition (cf. [8], [19]) is necessary for proving an existence result. Here we require

$$\Big|\int_\Omega f \cdot \varphi \,\mathrm{d}x\Big| \le \widetilde{\nu}_1 \int_\Omega |\varepsilon(\varphi)|\,\mathrm{d}x \quad \text{for all } \varphi \in C_0^\infty(\Omega, \mathbb{R}^d) \text{ with } \operatorname{div}\varphi = 0, \tag{1.9}$$

with a constant $\widetilde{\nu}_1$ satisfying (for ν_1 from (1.5))

$$\widetilde{\nu}_1 \in (0, \nu_1). \tag{1.10}$$

Of course it would be desirable to produce a solution of (1.8), (1.2) in the space $\overset{\circ}{W}{}^{1,1}(\Omega, \mathbb{R}^d)$ or its "symmetric analog" $\mathrm{LD}_0(\Omega, \mathbb{R}^d)$ (see Section 2 for the definitions of the function spaces), but, unfortunately, we could only prove the following weaker result:

MAIN THEOREM. *Let $d = 2$, let F satisfy (1.4) and (1.5) and suppose that the safe load condition (1.9) holds together with (1.10). Additionally, assume that F is μ-elliptic with*

$$\mu \in \big(1, {}^3\!/_2\big), \tag{1.11}$$

e.g. $F(\varepsilon) = \Phi_\mu\big(|\varepsilon|\big)$ (see (1.6)). Then there exists a solution

$$u \in \mathrm{LD}(\Omega, \mathbb{R}^d) \cap \mathrm{BD}_{\mathrm{div}}(\Omega, \mathbb{R}^d) \tag{1.12}$$

of equation (1.8). This solution additionally satisfies

$$u \in W^{1,r}_{\mathrm{loc}}(\Omega, \mathbb{R}^d) \cap W^{2,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^d) \tag{1.13}$$

for any $r < \infty$ and all $s < 2$.

The notion of μ-ellipticity is explained in Section 2, where we will also discuss the question of uniqueness to some extent. It turns out (compare Theorem 2.1 and its consequences) that u is unique up to rigid motions.

REMARK 1.1. *By definition (compare [20]) the space* $\mathrm{BD}_{\mathrm{div}}(\Omega, \mathbb{R}^d)$ *consists of all functions of bounded deformation* $w \in \mathrm{BD}(\Omega, \mathbb{R}^d)$ *such that*

$$\int_\Omega u \cdot \nabla\varphi \, \mathrm{d}x = 0 \quad \textit{holds for all } \varphi \in W^{1,n}(\Omega), \tag{1.14}$$

which means that u has vanishing divergence (in the weak sense) up to the boundary of Ω *(and* ***not only locally*** *inside* Ω*, as the notion* $\mathrm{BD}_{\mathrm{div}}$ *might suggest!). Note that if* $\partial\Omega$ *is Lipschitz continuous, Gauss's theorem in* BD *(see e.g. Theorem 1.4 in [21]) together with the identity (1.14) implies that the normal component of the trace of u vanishes on* $\partial\Omega$*. This latter statement is a rather weak interpretation of (1.2), however, it is the natural extension of this condition to the* BD*-context. It is not to be expected that the solution u satisfies (1.2) in the trace sense of* $\mathrm{BD}(\Omega, \mathbb{R}^d)$*.*

REMARK 1.2. *A simple example taken from a slightly different setting (see [22]) suggests that even in the 1D-case we can not allow values* $\mu > 2$ *in the condition of* μ*-ellipticity. In this situation the underlying variational problem (1.8) admits only generalized solutions in the space* BV*, i.e. the derivatives do not belong to* L^1 *and (1.8) does not make any sense. However it might be interesting to discuss a reasonable concept of a "very weak solution" to (1.8).*

REMARK 1.3. *The Main Theorem can easily be extended to the case* $d = 3$ *provided we replace (1.11) by the inequality* $1 < \mu < 9/7 = 3d/3d-2$*; under this hypothesis we find a solution satisfying (1.12) and with the additional property that* $u \in W^{1,r}_{\mathrm{loc}}(\Omega, \mathbb{R}^d)$ *for some* $r > 1$*.*

REMARK 1.4. *Another extension of the Main Theorem concerns densities of* $(1, p)$*-growth, which means that on the r.h.s. of (1.5) we replace "t" with "t^p" for some* $p > 1$*. We will comment on this anisotropic case in the final section.*

Our paper is organized as follows: in Section 2 we first collect the necessary background material on function spaces and discuss some related inequalities. Furthermore, an appropriate variational formulation of equation (1.8) is given together with a precise formulation of our results including the Main Theorem. In Section 3 we prove the existence of a minimizer to the variational problem introduced in Theorem 2.1 of Section 2 in a very general setting, Section 4 contains the proof of the Main Theorem under suitable restrictions as e.g. (1.11). Finally, in Section 5 we have a brief look at anisotropic densities of $(1, p)$-growth. We wish to note that most of our arguments are based on suitable uniform estimates (at least locally) of a Stokes-type regularization.

2 Notation and Results

Throughout this section $\Omega \subset \mathbb{R}^d$ denotes a bounded Lipschitz domain.

2.1 Some Facts about Function Spaces

As usual we let for $m \geq 1$

$$C_0^\infty(\Omega, \mathbb{R}^m) := \text{ all smooth functions with compact support in } \Omega,$$

and define

$$C_{0,\mathrm{div}}^\infty(\Omega, \mathbb{R}^m) := \Big\{ w \in C_0^\infty(\Omega, \mathbb{R}^m) : \operatorname{div} w = 0 \text{ on } \Omega \Big\}.$$

The Lebesgue and Sobolev classes $(1 \leq p \leq \infty,\ k \in \mathbb{N})$

$$L^p(\Omega, \mathbb{R}^m), \quad W^{k,p}(\Omega, \mathbb{R}^m), \quad \overset{\circ}{W}{}^{k,p}(\Omega, \mathbb{R}^m)$$

together with the local spaces

$$L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^m), \quad W^{k,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$$

are introduced in the usual way, we refer e.g. to [23]. Further we let

$$\overset{(\circ)}{W}{}^{k,p}_{\mathrm{div}}(\Omega, \mathbb{R}^m) := \Big\{ w \in \overset{(\circ)}{W}{}^{k,p}(\Omega, \mathbb{R}^m) \ : \ \operatorname{div} w = 0 \text{ on } \Omega \Big\},$$

and recall the L^p-variant of Korn's inequality

Lemma 2.1. *Let $1 < p < \infty$. Then there is a constant $C = C(d, p, \Omega)$ such that*

$$\|u\|_{W^{1,p}} \leq C \Big[\|u\|_{L^p} + \|\varepsilon(u)\|_{L^p} \Big] \tag{2.1}$$

for any $u \in W^{1,p}(\Omega, \mathbb{R}^d)$. If u is in the subspace $\overset{\circ}{W}{}^{1,p}(\Omega, \mathbb{R}^d)$, then it even holds

$$\|u\|_{W^{1,p}} \leq C \|\varepsilon(u)\|_{L^p}. \tag{2.2}$$

Proof. We refer to [24] (or [25], [26]). The case $p = 2$ is classical and can be found for instance in [27, 28], [29], [30] or [31]. □

We emphasize that Korn's inequality fails to hold in case $p = 1$. For this reason it is often more appropriate to replace the space $\overset{(\circ)}{W}{}^{1,1}(\Omega, \mathbb{R}^d)$ with the class

$$\mathrm{LD}(\Omega, \mathbb{R}^d) := \Big\{ w \in L^1(\Omega, \mathbb{R}^d) \ : \ \varepsilon_{ij}(w) := \frac{1}{2}(\partial_i u^j + \partial_j u^i) \in L^1(\Omega),\ i, j = 1, ..., d \Big\}$$

consisting of all functions $w : \Omega \to \mathbb{R}^d$ whose symmetric derivative is a tensor of class $L^1(\Omega, \mathbb{S}^d)$. The following facts can be found e.g. in [19]:

i) $\mathrm{LD}(\Omega, \mathbb{R}^d)$ is complete with respect to the norm

$$\|w\|_{\mathrm{LD}} := \|w\|_{L^1} + \|\varepsilon(w)\|_{L^1}.$$

ii) The trace theorem for $W^{1,1}(\Omega,\mathbb{R}^d)$ $\big(\subsetneq \mathrm{LD}(\Omega,\mathbb{R}^d)\big)$ extends to the space $\mathrm{LD}(\Omega,\mathbb{R}^d)$.

On account of ii) we define

$$\mathrm{LD}_0(\Omega,\mathbb{R}^d) := \Big\{ w \in \mathrm{LD}(\Omega,\mathbb{R}^d) \; : \; w\big|_{\partial\Omega} = 0 \Big\},$$

moreover, we introduce the subspaces

$$\begin{aligned} \mathrm{LD}_{\mathrm{div}}(\Omega,\mathbb{R}^d) &:= \Big\{ w \in \mathrm{LD}(\Omega,\mathbb{R}^d) \; : \; \operatorname{div} w = 0 \text{ on } \Omega \Big\}, \\ \mathrm{LD}_{0,\mathrm{div}}(\Omega,\mathbb{R}^d) &:= \Big\{ w \in \mathrm{LD}_{\mathrm{div}}(\Omega,\mathbb{R}^d) \; : \; w\big|_{\partial\Omega} = 0 \Big\} \\ &= \mathrm{LD}_0(\Omega,\mathbb{R}^d) \cap \mathrm{LD}_{\mathrm{div}}(\Omega,\mathbb{R}^d). \end{aligned}$$

The class $\mathrm{LD}(\Omega,\mathbb{R}^d)$ together with its subspaces plays some kind of intermediate role: if one considers variational problems of linear growth for functions $w : \Omega \to \mathbb{R}^d$, where the energy density depends on $\varepsilon(w)$, then – due to the lack of Korn's inequality – the Sobolev class $W^{1,1}(\Omega,\mathbb{R}^d)$ is inappropriate since we have no coercivity in this space. As a matter of fact, $\mathrm{LD}(\Omega,\mathbb{R}^d)$ is the correct class in which we can hope for such a behavior. On the other hand, due to the lack of (weak) compactness of bounded sequences in $\mathrm{LD}(\Omega,\mathbb{R}^d)$, coercivity of an energy functional is not very helpful: one has to replace $\mathrm{LD}(\Omega,\mathbb{R}^d)$ with the space

$$\mathrm{BD}(\Omega,\mathbb{R}^d) := \left\{ u \in L^1(\Omega,\mathbb{R}^d) \; : \; \varepsilon_{ij}(u),\; i,j = 1,\ldots,d,\; \begin{array}{c} \text{is a (signed) Radon measure} \\ \text{on } \Omega \text{ with finite total mass} \end{array} \right\}$$

introduced in [32], [33] with further contributions in [34] and [21]. As usual we write $\varepsilon(u)$ for the $\mathbb{S}^d$-valued Radon measure generated by the distributions $\frac{1}{2}(\partial_i u^j + \partial_j u^i)$ and denote by $\int_\Omega |\varepsilon(u)|$ its total variation on Ω, i.e. the quantity $|\varepsilon(u)|(\Omega)$ which is finite by definition. We have the obvious inclusions

$$W^{1,1}(\Omega,\mathbb{R}^d) \subsetneq \mathrm{LD}(\Omega,\mathbb{R}^d) \subsetneq \mathrm{BD}(\Omega,\mathbb{R}^d).$$

Next we collect some properties of the space $\mathrm{BD}(\Omega,\mathbb{R}^d)$.

Lemma 2.2. *i) If $u_n \in \mathrm{BD}(\Omega,\mathbb{R}^d)$ are such that*

$$\sup_{n\in\mathbb{N}} \Big\{ \|u_n\|_{L^1} + \int_\Omega |\varepsilon(u_n)| \Big\} < \infty,$$

then there exists a function $u \in \mathrm{BD}(\Omega,\mathbb{R}^d)$ such that (at least for a subsequence)

$$\begin{aligned} &u_n \longrightarrow u \text{ in } L^p(\Omega,\mathbb{R}^d) \text{ for all } p < {}^d\!/_{d-1}, \\ &u_n \rightharpoonup u \text{ in } L^{d/d-1}(\Omega,\mathbb{R}^d). \end{aligned}$$

ii) If $u_n,\ u \in L^1(\Omega,\mathbb{R}^d)$ satisfy

$$\lim_{n\to\infty} \|u - u_n\|_{L^1} = 0,$$

then

$$\int_\Omega |\varepsilon(u)| \le \liminf_{n\to\infty} \int_\Omega |\varepsilon(u_n)|,$$

in particular we obtain $u \in \mathrm{BD}(\Omega,\mathbb{R}^d)$ if the r.h.s. is finite.

Proof. We refer to the above mentioned references and also to [8], Theorem A.3.1, where also the "density" of the class $C_0^\infty(\Omega,\mathbb{R}^d)$ in $\mathrm{BD}(\Omega,\mathbb{R}^d)$ is established, i.e. for $u \in \mathrm{BD}(\Omega,\mathbb{R}^d)$ there exists a sequence $u_n \in C_0^\infty(\Omega,\mathbb{R}^d)$ such that

$$\begin{cases} u_n \longrightarrow u \text{ in } L^1(\Omega,\mathbb{R}^d), \\ \displaystyle\int_\Omega |\varepsilon(u_n)|\,\mathrm{d}x \longrightarrow \int_\Omega |\varepsilon(u)| \text{ as } n \to \infty. \end{cases}$$

□

According to, e.g. [34], there is a trace operator $\gamma : \mathrm{BD}(\Omega,\mathbb{R}^d) \to L^1(\partial\Omega,\mathbb{R}^d)$ such that

$$\gamma(u) = u\big|_{\partial\Omega}$$

for $u \in \mathrm{BD}(\Omega,\mathbb{R}^d) \cap C^0\big(\overline{\Omega},\mathbb{R}^d\big)$. Moreover, we note (see [34] again) that $\varepsilon(u) = 0$ for $u \in \mathrm{BD}(\Omega,\mathbb{R}^d)$ implies that u is a rigid motion.

During the investigation of equation (1.8) in a variational setting we have to consider the space

$$\mathrm{BD}_{\mathrm{div}}(\Omega,\mathbb{R}^d) := \Big\{u \in \mathrm{BD}(\Omega,\mathbb{R}^d) \,:\, \int_\Omega u \cdot \nabla\varphi\,\mathrm{d}x = 0 \text{ for all } \varphi \in W^{1,n}(\Omega)\Big\}.$$

The reader should note that this space is defined in such a way that the extension $\widehat{u}$ of any $u \in \mathrm{BD}_{\mathrm{div}}(\Omega,\mathbb{R}^d)$ through zero outside of Ω, i.e.

$$\widehat{u}(x) := \begin{cases} u(x), \ x \in \Omega, \\ (0,...,0), \ x \in \mathbb{R}^d - \Omega. \end{cases} \tag{2.3}$$

gives an element of $\mathrm{BD}_{\mathrm{div}}(\mathbb{R}^d,\mathbb{R}^d)$. With respect to our desired boundary condition (1.2) it would be natural to work in the space

$$\overset{\circ}{\mathrm{BD}}_{\mathrm{div}}(\Omega,\mathbb{R}^d) := \mathrm{BD}_{\mathrm{div}}(\Omega,\mathbb{R}^d) \cap \ker\gamma.$$

However, the boundary operator γ is not weakly continuous and therefore a minimizing sequence u_n with $\gamma(u_n) = 0$ might produce a "solution" u for which $\gamma(u) \neq 0$. Since we have no control on the limiting behavior of the boundary values during the minimization procedure, which will be applied later on (see Section 2.2), we decided to work in the class $\mathrm{BD}_{\mathrm{div}}(\Omega,\mathbb{R}^d)$ giving at least some weak interpretation of (1.2) (cf. Remark 1.1). The reader should note that we have the obvious inclusion

$$\overset{\circ}{W}{}^{1,p}_{\mathrm{div}}(\Omega,\mathbb{R}^d) \subset \mathrm{BD}_{\mathrm{div}}(\Omega,\mathbb{R}^d). \tag{2.4}$$

Let us now state some further auxiliary results:

Lemma 2.3. *There is a constant $C_p = C(d,\Omega)$ such that*

$$\|u\|_{L^1} \le C_p \int_\Omega |\varepsilon(u)| \tag{2.5}$$

holds for all $u \in \mathrm{BD}(\Omega,\mathbb{R}^d)$ with $\gamma(u) = 0$. If $d = 2$ and if Ω is convex, then we may choose $C_p = \frac{1}{\sqrt{2}}\,\mathrm{diam}(\Omega)$.

REMARK 2.1. *In fact we may choose* $\|u\|_{L^{d/d-1}}$ *on the left-hand side of (2.5). Lemma 2.3 is proved in [21], Corollary 1.11; for the case* $d = 2$ *we refer to [35], Theorem 1.2.*

Lemma 2.4. *Let* $u \in \mathrm{BD}_{\mathrm{div}}(\Omega, \mathbb{R}^d)$ *and define* $\widehat{u}$ *according to (2.3). Then it holds*

$$\int_{\partial\Omega} |\varepsilon(\widehat{u})| = \int_{\partial\Omega} |\gamma(u) \odot \mathfrak{n}| \,\mathrm{d}\mathcal{H}^{d-1}. \tag{2.6}$$

Here we have abbreviated

$$\gamma(u) \odot \mathfrak{n} := \frac{1}{2}\big(\gamma(u)^i \mathfrak{n}^j + \gamma(u)^j \mathfrak{n}^i\big)_{1 \le i,j \le d} \in \mathbb{S}^d,$$

$\mathfrak{n}$ *denoting the outward unit normal to* $\partial\Omega$ *and* $\mathcal{H}^{d-1}$ *is Hausdorff's measure of dimension* $d - 1$.

Proof. We apply Theorem 1.5 ii) from [21] to the function $\widehat{u}$ and observe that the exterior trace of $\widehat{u}$ vanishes whereas the interior trace equals $\gamma(u)$. □

The next result on the density of smooth functions with vanishing divergence in $\mathrm{BD}_{\mathrm{div}}(\Omega, \mathbb{R}^d)$ can be found in [20], Theorem 4.1. However, for our purposes, we need to add another convergence property of the approximating sequence concerning the quantity $\int_\Omega \sqrt{1 + |\varepsilon(w)|^2}$ which, for $w \in \mathrm{BD}(\Omega, \mathbb{R}^d)$, is defined by

$$\int_\Omega \sqrt{1 + |\varepsilon(w)|^2} := \int_\Omega \sqrt{1 + |\varepsilon^a(w)|^2}\,\mathrm{d}x + \int_\Omega |\varepsilon^s(w)|,$$

where

$$\varepsilon(w) = \varepsilon^a(w) \cdot \mathcal{L}^d + \varepsilon^s(w) \tag{2.7}$$

is the Lebesgue decomposition of the measure $\varepsilon(w)$ (cf. [36] for the details of this construction).

Lemma 2.5. *If in addition to our previous assumptions the domain* Ω *is star-shaped, then – for every* $u \in \mathrm{BD}_{\mathrm{div}}(\Omega, \mathbb{R}^d)$ *– there exists a sequence* $u_k \in C^\infty_{0,\mathrm{div}}(\Omega, \mathbb{R}^d)$ *such that*

$$\begin{cases} \text{i)} & u_k \longrightarrow u \text{ in } L^{d/d-1}(\Omega, \mathbb{R}^d), \\ \text{ii)} & \displaystyle\int_\Omega |\varepsilon(u_k)|\,\mathrm{d}x \longrightarrow \int_\Omega |\varepsilon(u)| + \int_{\partial\Omega} |\gamma(u) \odot \mathfrak{n}|\,\mathrm{d}\mathcal{H}^{d-1}, \\ \text{iii)} & \displaystyle\int_\Omega \sqrt{1 + |\varepsilon(u_k)|^2}\,\mathrm{d}x \longrightarrow \int_\Omega \sqrt{1 + |\varepsilon(u)|^2} + \int_{\partial\Omega} |\gamma(u) \odot \mathfrak{n}|\,\mathrm{d}\mathcal{H}^{d-1} \end{cases} \tag{2.8}$$

as $k \to \infty$.

REMARK 2.2. *The reader should note that Lemma 2.5 is an essential improvement of Lemma A.3.1 in [8] and more adequate to the setting of fluid mechanics. We further remark that iii) of (2.8) has not been proved in [20].*

Proof. Define $\widehat{u}$ according to (2.3). Then, for all $\varphi \in C^\infty(\mathbb{R}^d)$, it holds

$$\int_{\mathbb{R}^d} \widehat{u} \cdot \nabla\varphi \,\mathrm{d}x = 0. \tag{2.9}$$

Let $t_k > 1$ be a sequence such that $t_k \longrightarrow 1$ as $k \to \infty$ and set

$$\rho_k := \frac{1}{2}\operatorname{dist}\big(\Omega, \partial(t_k\Omega)\big) \xrightarrow{k\to\infty} 0.$$

Next we consider the mollification of $\widehat{u}$ with radius ρ_k,

$$\widehat{u}_k := (\widehat{u})_{\rho_k} \in C_0^\infty(t_k\Omega).$$

Then, by the same computation as on p. 14 of [20] we find that

$$\operatorname{div}\widehat{u}_k(x) = 0 \quad \text{for all } x \in \mathbb{R}^d.$$

Moreover, an application of Lemma 2.2 from [36] yields

$$\begin{aligned}\int_{\mathbb{R}^d} |\varepsilon(\widehat{u}_k)|\,\mathrm{d}x = \int_{\mathbb{R}^d} \big|\big(\varepsilon(\widehat{u})\big)_{\rho_k}\big|\,\mathrm{d}x &\xrightarrow{k\to\infty} \int_{\mathbb{R}^d} |\varepsilon(\widehat{u})| \\ &= \int_\Omega |\varepsilon(u)| + \int_{\partial\Omega} |\gamma(u)\odot\mathfrak{n}|\,\mathrm{d}\mathcal{H}^{d-1}\end{aligned}$$

as well as

$$\begin{aligned}\int_{\mathbb{R}^d} \sqrt{1+|\varepsilon(\widehat{u}_k)|^2}\,\mathrm{d}x &\xrightarrow{k\to\infty} \int_{\mathbb{R}^d} \sqrt{1+|\varepsilon(\widehat{u})|^2} \\ &= \int_\Omega \sqrt{1+|\varepsilon(u)|^2} + \underbrace{\int_{\partial\Omega} \sqrt{1+|\varepsilon(u)|^2}}_{= \int_{\partial\Omega} |\gamma(u)\odot\mathfrak{n}|\,\mathrm{d}\mathcal{H}^{d-1}.}\end{aligned} \tag{2.10}$$

Now, for $x \in \Omega$, we let $u_k(x) := \widehat{u}_k(t_k x) \in C^\infty_{0,\mathrm{div}}(\Omega, \mathbb{R}^d)$ and claim that this sequence is the desired approximation. Indeed, we have

$$\int_\Omega |u(x) - u_k(x)|\,\mathrm{d}x \le \underbrace{\int_\Omega |u(x) - \widehat{u}(t_k x)|\,\mathrm{d}x}_{\longrightarrow 0 \text{ as } k\to\infty} + \underbrace{\int_\Omega |\widehat{u}(t_k x) - \widehat{u}_k(t_k x)|\,\mathrm{d}x}_{= t_k^{-d}\int_{t_k\Omega} |\widehat{u}(y) - \widehat{u}_k(y)|\,\mathrm{d}y \xrightarrow{k\to\infty} 0} \xrightarrow{k\to\infty} 0.$$

Furthermore,

$$\int_\Omega |\varepsilon(u_k)|\,\mathrm{d}x = t_k^{1-d}\int_{\mathbb{R}^d} \big|\varepsilon\big(\widehat{u}_k(y)\big)\big|\,\mathrm{d}y \xrightarrow{k\to\infty} \int_{\mathbb{R}^d} |\varepsilon(\widehat{u})|$$

and finally

$$\int_\Omega \sqrt{1+|\varepsilon(u_k)|^2}\,\mathrm{d}x = t_k^{1-d}\int_{t_k\Omega} \sqrt{\frac{1}{t_k^2} + |\varepsilon(\widehat{u}_k)|^2}\,\mathrm{d}y \xrightarrow{k\to\infty} \int_{\mathbb{R}^d} \sqrt{1+|\varepsilon(\widehat{u})|^2},$$

since $\sqrt{\frac{1}{t_k^2} + x^2}$ converges uniformly to $\sqrt{1+x^2}$ on $\mathbb{R}$ as $k \to \infty$ and due to (2.10). $\square$

2.2 Formulation of the Variational Problems

Coming back to (1.1) and (1.8) we will consider an energy density $F : \mathbb{S}^d \to [0,\infty)$ being strictly convex and of linear growth in the sense that

$$\nu_1|\varepsilon| - \nu_2 \le F(\varepsilon) \le \nu_3|\varepsilon| + \nu_4 \tag{2.11}$$

holds for all $\varepsilon \in \mathbb{S}^d$ with constants $\nu_1, \nu_3 > 0$, $\nu_2, \nu_4 \ge 0$. Suppose further that we are given volume forces

$$f \in L^\infty(\Omega, \mathbb{R}^d) \tag{2.12}$$

(clearly (2.12) can be weakened to some degree) satisfying the "safe load condition"

$$\left| \int_\Omega f \cdot \varphi \,\mathrm{d}x \right| \le \nu_1^* \int_\Omega |\varepsilon(\varphi)| \,\mathrm{d}x \tag{2.13}$$

for any $\varphi \in C^\infty_{0,\mathrm{div}}(\Omega, \mathbb{R}^d)$ with a constant ν_1^* such that

$$0 < \nu_1^* < \nu_1 \tag{2.14}$$

for ν_1 from (2.11). Note that, on account of Lemma 2.3, condition (2.14) particularly holds if

$$C_p \|f\|_{L^\infty(\Omega)} < \nu_1, \tag{2.15}$$

which, in the case $d = 2$ and for convex Ω, can be replaced with

$$\frac{1}{\sqrt{2}} \operatorname{diam}(\Omega) \|f\|_{L^\infty(\Omega)} < \nu_1. \tag{2.16}$$

Under the above assumptions we now introduce the following energy functionals ($\delta > 0$):

$$\begin{aligned}
J[u,\Omega] &:= \int_\Omega F\big(\varepsilon(u)\big) \,\mathrm{d}x - \int_\Omega f \cdot u \,\mathrm{d}x, \quad u \in \mathrm{LD}(\Omega, \mathbb{R}^d), \\
J_\delta[u,\Omega] &:= \int_\Omega F\big(\varepsilon(u)\big) \,\mathrm{d}x + \frac{\delta}{2} \int_\Omega |\varepsilon(u)|^2 \,\mathrm{d}x - \int_\Omega f \cdot u \,\mathrm{d}x, \quad u \in W^{1,2}(\Omega, \mathbb{R}^d), \\
\widehat{J}[u,\Omega] &:= \int_\Omega F\big(\varepsilon^a(u)\big) \,\mathrm{d}x + \int_\Omega F^\infty\left(\frac{\varepsilon^s(u)}{|\varepsilon^s(u)|}\right) \mathrm{d}|\varepsilon^s(u)| \\
&\quad + \int_{\partial\Omega} F^\infty\big(\gamma(u) \odot \mathfrak{n}\big) \,\mathrm{d}\mathcal{H}^{d-1} - \int_\Omega f \cdot u \,\mathrm{d}x, \ u \in \mathrm{BD}(\Omega, \mathbb{R}^d),
\end{aligned} \tag{2.17}$$

where in the definition of $\widehat{J}$ the quantities $\varepsilon^a(u)$ and $\varepsilon^s(u)$ are defined in (2.7), and F^∞ is the recession function of F, i.e.

$$F^\infty(\varepsilon) := \lim_{t\to\infty} \frac{F(t\varepsilon)}{t}.$$

Finally, $\frac{\varepsilon^s(u)}{|\varepsilon^s(u)|}$ denotes the Radon-Nikodym derivative.

Lemma 2.6. *The variational problem*

$$J_\delta[\,\cdot\,,\Omega] \longrightarrow \min \ \ \textit{in}\ \overset{\circ}{W}{}^{1,2}_{\mathrm{div}}(\Omega,\mathbb{R}^d) \tag{2.18}$$

admits a unique solution u_δ for any $\delta > 0$. We have

$$\sup_{1>\delta>0} \delta \int_\Omega |\varepsilon(u_\delta)|^2 \,\mathrm{d}x < \infty, \tag{2.19}$$

$$\sup_{1>\delta>0} \Big(\|u_\delta\|_{L^1} + \int_\Omega |\varepsilon(u_\delta)| \,\mathrm{d}x \Big) < \infty \tag{2.20}$$

as well as

$$u_\delta \in W^{2,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^d) \qquad \textit{(not necessarily uniformly with respect to } \delta!\textit{)}. \tag{2.21}$$

The proof of Lemma 2.6 together with the proof of the following theorem will be presented in Section 3.

THEOREM 2.1. *Let (2.11)–(2.14) hold and assume that Ω is star-shaped. Then, for a suitable sequence $\delta \to 0$, the functions u_δ from Lemma 2.6 converge in $L^1(\Omega,\mathbb{R}^d)$ towards a function $u \in \mathrm{BD}_{\mathrm{div}}(\Omega,\mathbb{R}^d)$ which is a solution of*

$$\widehat{J}[\,\cdot\,,\Omega] \longrightarrow \min \ \ \textit{in}\ \mathrm{BD}_{\mathrm{div}}(\Omega,\mathbb{R}^d). \tag{2.22}$$

Corollary 2.1. *Under the assumptions and with the notation of Theorem 2.1 assume that, in addition, it holds*

$$u \in \mathrm{LD}(\Omega,\mathbb{R}^d). \tag{2.23}$$

a) If the density F belongs to the class $C^1(\mathbb{S}^d)$, then we have for all $\varphi \in C^\infty_{0,\mathrm{div}}(\Omega,\mathbb{R}^d)$

$$\int_\Omega DF\big(\varepsilon(u)\big) : \varepsilon(\varphi)\,\mathrm{d}x = \int_\Omega f\cdot\varphi\,\mathrm{d}x, \tag{2.24}$$

hence equation (1.1) holds in the weak sense.

b) Suppose that $v \in \mathrm{BD}_{\mathrm{div}}(\Omega,\mathbb{R}^d)$ is a solution of (2.22) satisfying (2.23). Then $u = v + r$ on Ω for some rigid motion r.

REMARK 2.3. *Corollary 2.1 states that property (2.23) is the key tool for obtaining a solution of (1.1) as well as uniqueness up to rigid motions.*

Proof of Corollary 2.1. a) The convexity of F together with the linear growth condition (2.11) implies on account of Lemma 2.2, p. 156, in [37] that DF is bounded. From $\widehat{J}[u,\Omega] \le \widehat{J}[u+t\varphi,\Omega]$, $t\in\mathbb{R}$, $\varphi \in C^\infty_{0,\mathrm{div}}(\Omega,\mathbb{R}^d)$ we obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \widehat{J}[u+t\varphi,\Omega],$$

and since $\varepsilon^s(u) = \varepsilon^s(u + t\varphi)$, assertion (2.24) is immediate due to $|DF| \in L^\infty(\mathbb{S}^d)$.

b) Obviously, $\widehat{J}[\,\cdot\,, \Omega]$ is convex and even strictly convex w.r.t. $\varepsilon^a(\,\cdot\,)$, which follows from the strict convexity of F. So, if we assume $\varepsilon(u) \neq \varepsilon(v)$ on a subset of Ω with positive measure, then we obtain the contradiction

$$\widehat{J}\Big[\tfrac{1}{2}(u+v), \Omega\Big] < \frac{1}{2}\widehat{J}[u, \Omega] + \frac{1}{2}\widehat{J}[v, \Omega] = \inf_{w \in \mathrm{BD}_{\mathrm{div}}(\Omega)} \widehat{J}[w, \Omega].$$

□

2.3 Formulation of the Main Result

Here we give a precise version of the Main Theorem. As a matter of fact, we need to impose conditions on the data such that (2.23) from Corollary 2.1 can be established. This will be possible for a particular class of densities F: consider a function $h : [0, \infty) \to [0, \infty)$ such that

$$h \in C^2\big([0, \infty)\big), \quad h(0) = h'(0) = 0, \tag{2.25}$$

$$c_1 \frac{1}{(1+t)^\mu} \leq \min\left\{h''(t), \frac{h'(t)}{t}\right\}, \tag{2.26}$$

$$\max\left\{h''(t), \frac{h'(t)}{t}\right\} \leq \frac{c_2}{1+t} \tag{2.27}$$

for any $t \geq 0$, and with an exponent

$$\mu \in (1, \infty) \tag{2.28}$$

and positive constants c_1, c_2. Note that the functions Φ_μ from (1.6) satisfy the requirements (2.25)–(2.27) provided (2.28) holds. We then let

$$F : \mathbb{S}^d \to [0, \infty), \quad F(\varepsilon) := h\big(|\varepsilon|\big) \tag{2.29}$$

and observe

Lemma 2.7. *The density F from (2.29) is μ-elliptic in the sense that*

$$c_1 \frac{1}{(1+|\varepsilon|)^\mu} |\sigma|^2 \leq D^2F(\varepsilon)(\sigma, \sigma) \leq c_2 \frac{|\sigma|^2}{1+|\varepsilon|} \tag{2.30}$$

holds for all $\varepsilon, \sigma \in \mathbb{S}^d$ with c_1, c_2 from (2.26) and (2.27), respectively. Moreover, F satisfies the growth estimate (2.11) with $\nu_1 = \nu_2 := 2^{-\mu}c_1$, $\nu_3 := c_2$ and $\nu_4 := 0$ (with c_1, c_2 from (2.26) and (2.27), respectively).

Proof. Inequality (2.30) is an immediate consequence of (2.26) and (2.27) in combination with the estimate

$$\min\left\{h''\big(|\varepsilon|\big), \frac{h'\big(|\varepsilon|\big)}{|\varepsilon|}\right\}|\sigma|^2 \le D^2F(\varepsilon)(\sigma,\sigma) \le \max\left\{h''\big(|\varepsilon|\big), \frac{h'\big(|\varepsilon|\big)}{|\varepsilon|}\right\}|\sigma|^2,$$

which follows from the formula

$$D^2F(\varepsilon)(\sigma,\sigma) = \frac{1}{|\varepsilon|}h'\big(|\varepsilon|\big)\left[|\sigma|^2 - \frac{(\varepsilon:\sigma)^2}{|\varepsilon|^2}\right] + h''\big(|\varepsilon|\big)\frac{(\varepsilon:\sigma)^2}{|\varepsilon|^2}.$$

Let us discuss (2.11): from (2.27) it follows that

$$h'(t) \le c_2\frac{t}{1+t} \le c_2,$$

thus (recall $h(0) = 0$ by (2.25)) $h(t) \le c_2 t$, which means that we can choose $\nu_3 = c_2$, $\nu_4 = 0$. Since $h(0) = h'(0) = 0$, it holds for $x \ge 0$

$$h(x) = \int_0^1 (1-t)h''(tx)\,\mathrm{d}t\; x^2,$$

hence, by (2.26),

$$h(x) \ge c_1\int_0^1 \frac{1-t}{(1+tx)^\mu}\,\mathrm{d}t\; x^2.$$

Let us assume that $x \ge 1$. Then

$$\int_0^1 \frac{1-t}{(1+tx)^\mu}\,\mathrm{d}t\; x^2 \ge \int_0^{1/x} \frac{1-t}{(1+tx)^\mu}\,\mathrm{d}t\; x^2 \ge \int_0^{1/x} \frac{1-t}{2^\mu}\,\mathrm{d}t\; x^2 = \frac{x-\frac{1}{2}}{2^\mu}$$

and in conclusion we may choose (c_1 from (2.26))

$$\nu_1 = \nu_2 := 2^{-\mu}c_1. \tag{2.31}$$

□

After these preparations we can give a precise statement of the Main Theorem from Section 1.

THEOREM 2.2. *Let $d = 2$ and consider a star-shaped domain Ω. Let h satisfy the conditions (2.25)–(2.28) for some exponent $\mu \in \big(1, 3/2\big)$ and define F according to (2.29). Suppose that the volume forces f satisfy (2.12)–(2.14) with ν_1 from formula (2.31). Then the minimizer $u \in \mathrm{BD}_{\mathrm{div}}(\Omega,\mathbb{R}^2)$ of $\widehat{J}[\,\cdot\,,\Omega]$ constructed in Theorem 2.1 satisfies*

$$u \in \mathrm{LD}_{\mathrm{div}}(\Omega,\mathbb{R}^2), \tag{2.32}$$

and it holds

$$\int_\Omega DF\big(\varepsilon(u)\big) : \varepsilon(\varphi)\,\mathrm{d}x = \int_\Omega f\cdot\varphi\,\mathrm{d}x \tag{2.33}$$

for any $\varphi \in C^\infty_{0,\mathrm{div}}(\Omega,\mathbb{R}^2)$. Moreover, u is an element of the space

$$W^{1,r}_{\mathrm{loc}}(\Omega,\mathbb{R}^2)\cap W^{2,s}_{\mathrm{loc}}(\Omega,\mathbb{R}^2)$$

for arbitrary values $r<\infty$ and $s<2$.

REMARK 2.4. *An inspection of the proof of Theorem 2.2 (see Section 4) will show that (2.32) and (2.33) extend to the case $d=3$ provided we impose the bound $\mu\in\big(1,{}^{3d}/_{3d-2}\big)$ and keep all other assumptions of the theorem. Moreover, the solution u is in some space $W^{1,r}_{\mathrm{loc}}(\Omega,\mathbb{R}^d)$ with $r>1$. We further note that, for Lipschitz continuous volume forces $f\in W^{1,\infty}(\Omega,\mathbb{R}^d)$, the assertion of Theorem 2.2 holds even under the assumption $\mu<2$ (which is optimal in this context, cf. [22]).*

3 Proof of Theorem 2.1

We start with the *proof of Lemma 2.6.* With δ being fixed, we deduce from (2.11)–(2.14), recalling definition (2.17),

$$J_\delta[w,\Omega]\geq \frac{\delta}{2}\int_\Omega|\varepsilon(w)|^2\,\mathrm{d}x+(\nu_1-\nu_1^*)\int_\Omega|\varepsilon(w)|\,\mathrm{d}x-\nu_2\mathcal{L}^d(\Omega), \tag{3.1}$$

which holds for all $w\in \overset{\circ}{W}{}^{1,2}_{\mathrm{div}}(\Omega,\mathbb{R}^d)$. Here we have used the fact that (2.13) extends to $\varphi\in \overset{\circ}{W}{}^{1,2}_{\mathrm{div}}(\Omega,\mathbb{R}^d)$ by a standard approximation argument. By (3.1), the infimum of $J_\delta[\,\cdot\,,\Omega]$ in the space $\overset{\circ}{W}{}^{1,2}_{\mathrm{div}}(\Omega,\mathbb{R}^d)$ is finite, and we may consider a minimizing sequence $(u_n)_{n=1}^\infty$. If we combine (3.1) with inequality (2.2) (choosing $p=2$) it is immediate that

$$\sup_{n\in\mathbb{N}}\|u_n\|_{W^{1,2}}<\infty. \tag{3.2}$$

From (3.2) we deduce the existence of $u_\delta\in \overset{\circ}{W}{}^{1,2}(\Omega,\mathbb{R}^d)$ such that (at least for a subsequence)

$$u_n \rightharpoondown u_\delta \text{ in } \overset{\circ}{W}{}^{1,2}(\Omega,\mathbb{R}^d). \tag{3.3}$$

Clearly, u_δ is in the space $\overset{\circ}{W}{}^{1,2}_{\mathrm{div}}(\Omega,\mathbb{R}^d)$, and standard results on lower-semicontinuity (compare [38]) together with (3.3) imply

$$J_\delta[u_\delta,\Omega]=\inf\Big\{J_\delta[w,\Omega]\,:\,w\in \overset{\circ}{W}{}^{1,2}_{\mathrm{div}}(\Omega,\mathbb{R}^d)\Big\}. \tag{3.4}$$

If $\widetilde{u}_\delta$ is another solution of (3.4), then, by the strict convexity of the energy density with respect to the quantity "$\varepsilon(w)$", we must have $\varepsilon(u_\delta - \widetilde{u}_\delta) = 0$ on Ω, hence $u_\delta - \widetilde{u}_\delta$ is a rigid motion implying $u = \widetilde{u}_\delta$ on account of $u_\delta\big|_{\partial\Omega} = 0 = \widetilde{u}_\delta\big|_{\partial\Omega}$. Next we observe that

$$J_\delta[u_\delta, \Omega] \leq J_\delta[0, \Omega] = F(0)\mathcal{L}^d(\Omega)$$

and deduce from this inequality together with (3.1) that

$$\sup_{1>\delta>0} \Big\{ \delta \int_\Omega |\varepsilon(u_\delta)|^2 \,\mathrm{d}x + \int_\Omega |\varepsilon(u_\delta)| \,\mathrm{d}x \Big\} < \infty. \tag{3.5}$$

Recalling (2.5), it is immediate how to deduce the claims (2.19) and (2.20) from (3.5). Finally, assertion (2.21) follows from an application of the well-known difference quotient technique; we refer to Theorem 3.1 in [39] for an application of this procedure in a similar setting. □

Now we come to the *proof of Theorem 2.1.* We observe that

$$u_\delta \in \overset{\circ}{W}{}^{1,2}_{\mathrm{div}}(\Omega, \mathbb{R}^d) \cap \mathrm{BD}_{\mathrm{div}}(\Omega, \mathbb{R}^d)$$

together with (recall (2.20))

$$\sup_{1>\delta>0} \Big\{ \int_\Omega |u_\delta| \,\mathrm{d}x + \int_\Omega |\varepsilon(u_\delta)| \,\mathrm{d}x \Big\} < \infty. \tag{3.6}$$

From (3.6) together with Lemma 2.2 i) we deduce the existence of $u \in \mathrm{BD}(\Omega, \mathbb{R}^d)$ with the property

$$u_\delta \longrightarrow u \quad \text{in } L^1(\Omega, \mathbb{R}^d) \tag{3.7}$$

at least for a suitable sequence $\delta \to 0$ and, by (3.6), (3.7) implies $u \in \mathrm{BD}_{\mathrm{div}}(\Omega, \mathbb{R}^d)$. Next we show that u in fact solves the variational problem (2.22). To this purpose, consider an arbitrary $w \in \mathrm{BD}_{\mathrm{div}}(\Omega, \mathbb{R}^d)$. Then, by Lemma 2.5, we find a sequence w_k in $C^\infty_{0,\mathrm{div}}(\Omega, \mathbb{R}^d)$ such that (see (2.8))

$$w_k \longrightarrow w \quad \text{in } L^1(\Omega, \mathbb{R}^d), \tag{3.8}$$

$$\int_\Omega |\varepsilon(w_k)| \,\mathrm{d}x \longrightarrow \int_\Omega |\varepsilon(w)| + \int_{\partial\Omega} |w \odot \mathfrak{n}| \,\mathrm{d}\mathcal{H}^{d-1}, \tag{3.9}$$

$$\int_\Omega \sqrt{1 + |\varepsilon(w_k)|^2} \,\mathrm{d}x \longrightarrow \int_\Omega \sqrt{1 + |\varepsilon(w)|^2} + \int_{\partial\Omega} |w \odot \mathfrak{n}| \,\mathrm{d}\mathcal{H}^{d-1} \tag{3.10}$$

as $k \to \infty$. By (2.6) of Lemma 2.4 it holds ($\widehat{w}$, $\widehat{w}_k$ meaning extension by zero outside Ω)

$$\int_{\partial\Omega} |w \odot \mathfrak{n}| \,\mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\Omega} |\varepsilon(\widehat{w})|,$$

and since

$$\int_{\mathbb{R}^d} |\varepsilon(\widehat{w})| = \int_\Omega |\varepsilon(\widehat{w})| + \int_{\partial\Omega} |\varepsilon(\widehat{w})|,$$
$$\int_\Omega |\varepsilon(\widehat{w})| = \int_\Omega |\varepsilon(w)|,$$

we deduce from (3.9)

$$\int_{\mathbb{R}^d} |\varepsilon(\widehat{w}_k)| \,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^d} |\varepsilon(\widehat{w})|. \tag{3.11}$$

In the same way we obtain

$$\int_{\mathbb{R}^d} \sqrt{1+|\varepsilon(\widehat{w}_k)|^2} \,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^d} \sqrt{1+|\varepsilon(\widehat{w})|}. \tag{3.12}$$

Now we observe that for the functional $\widehat{J}$ from (2.17) it holds

$$\widehat{J}[w,\Omega] = \int_{\mathbb{R}^d} F\big(\varepsilon^a(\widehat{w})\big) \,\mathrm{d}x + \int_{\mathbb{R}^d} F^\infty\left(\frac{\varepsilon^s(\widehat{w})}{|\varepsilon^s(w)|}\right) \mathrm{d}|\varepsilon^s(w)| - \int_\Omega f\cdot w \,\mathrm{d}x,$$

which together with (3.8) and (3.12), using Reshetnyak's continuity theorem (cf. Proposition 2.2 in [40]), implies

$$\widehat{J}[w,\Omega] = \lim_{k\to\infty} \widehat{J}[w_k,\Omega]. \tag{3.13}$$

Obviously,

$$\widehat{J}[w_k,\Omega] = J[w_k,\Omega] \tag{3.14}$$

and the $J_\delta[\,\cdot\,,\Omega]$-minimality of u_δ implies

$$J_\delta[u_\delta,\Omega] \le J_\delta[w_k,\Omega]. \tag{3.15}$$

From (3.7) and the lower semicontinuity of $\widehat{J}[\,\cdot\,,\Omega]$ we further obtain

$$\widehat{J}[u,\Omega] \le \liminf_{\delta\to 0} \widehat{J}[u_\delta,\Omega],$$

hence, on account of $\widehat{J}[u_\delta,\Omega] = J[u_\delta,\Omega]$, we infer the following chain of inequalities:

$$\begin{aligned}\widehat{J}[u,\Omega] &\le \liminf_{\delta\to 0} J[u_\delta,\Omega] \le \liminf_{\delta\to 0} J_\delta[u_\delta,\Omega] \overset{(3.15)}{\le} \liminf_{\delta\to 0} J_\delta[w_k,\Omega]\\ &= \liminf_{\delta\to 0} J[w_k,\Omega] = J[w_k,\Omega] \underset{(3.14)}{=} \widehat{J}[w_k,\Omega].\end{aligned}$$

Finally, (3.13) implies

$$\widehat{J}[u,\Omega] \le \widehat{J}[w,\Omega],$$

and the proof of Theorem 2.1 is complete. □

4 Proof of Theorem 2.2

Let the assumptions of Theorem 2.2 hold, i.e. in particular $d = 2$ in the course of this section. We may assume that Ω has a smooth boundary since, otherwise, we can restrict our considerations to a compact subset $\omega \Subset \Omega$. By c we denote a generic positive constant being independent of the parameter $\delta \in (0,1)$. Our starting point is the following Euler-Lagrange equation, which is a direct consequence of the $J_\delta[\,\cdot\,,\Omega]$-minimality of u_δ from Lemma 2.6 (we abbreviate $F_\delta(\varepsilon) := \frac{\delta}{2}|\varepsilon|^2 + F(\varepsilon)$, $\varepsilon \in \mathbb{S}^d$):

$$\int_\Omega DF_\delta\big(\varepsilon(u_\delta)\big) : \varepsilon(\varphi)\,\mathrm{d}x = \int_\Omega f \cdot \varphi\,\mathrm{d}x \quad \forall \varphi \in C^\infty_{0,\mathrm{div}}(\Omega, \mathbb{R}^2). \tag{4.1}$$

In the following, we write

$$\sigma_\delta := DF_\delta\big(\varepsilon(u_\delta)\big)$$

and observe that, due to (2.21), it holds $\sigma_\delta \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$. Then (4.1) is equivalent to the identity

$$\int_\Omega \big(\operatorname{div}\sigma_\delta - f\big) \cdot \varphi\,\mathrm{d}x = 0 \quad \forall \varphi \in C^\infty_{0,\mathrm{div}}(\Omega, \mathbb{R}^2), \tag{4.2}$$

which, by Lemma 1.1 on p.101 in [2], implies the existence of a pressure function $p_\delta \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega \big(\operatorname{div}\sigma_\delta - f\big) \cdot \varphi\,\mathrm{d}x = \int_\Omega \nabla p_\delta \cdot \varphi\,\mathrm{d}x \quad \forall \varphi \in C^\infty_0(\Omega, \mathbb{R}^2). \tag{4.3}$$

W.l.o.g., we may assume the normalization

$$\int_\Omega p_\delta\,\mathrm{d}x = 0. \tag{4.4}$$

Let us further define the tensor $\tau_\delta := \sigma_\delta - p_\delta \mathbb{1}$ ($\mathbb{1} \in \mathbb{S}^2$ being the 2×2-identity matrix). Then (4.3) yields

$$\int_\Omega \tau_\delta : \varepsilon(\varphi)\,\mathrm{d}x = -\int_\Omega f \cdot \varphi\,\mathrm{d}x \quad \forall \varphi \in C^\infty_0(\Omega, \mathbb{R}^2),$$

which means

$$\operatorname{div}\tau_\delta = f \quad \text{in the weak sense.} \tag{4.5}$$

Now fix a Ball $B_{2R}(x_0) \Subset \Omega$ inside the domain Ω, and choose a cut-off function $\eta \in C^\infty_0(\Omega)$ such that

$$\begin{cases} \mathrm{spt}(\eta) \subset B_{2R}(x_0), \\ \eta \equiv 1 \text{ on } B_R(x_0) \text{ and } |\nabla\eta| \le cR^{-1}. \end{cases} \tag{4.6}$$

Note that, by approximation, equation (4.1) is valid even for $\varphi \in \overset{\circ}{W}{}^{1,2}(\Omega, \mathbb{R}^2)$. In particular, we may choose $\varphi = \partial_\alpha(\eta^{2k}\partial_\alpha u_\delta)$ $(\alpha \in \{1, 2\})$, where the power $k \in \mathbb{N}$ will be specified later. This leads us to

$$\int_\Omega \partial_\alpha \tau_\delta : \varepsilon(\eta^{2k}\partial_\alpha u_\delta)\,\mathrm{d}x = \int_\Omega f \cdot \partial_\alpha(\eta^{2k}\partial_\alpha u_\delta)\,\mathrm{d}x. \tag{4.7}$$

From now on, we basically follow the ideas in [41] (where some adjustments due to the presence of the exterior force f become necessary), i.e. our goal is to establish

$$\phi_\delta := \big(1 + |\varepsilon(u_\delta)|\big)^{1-\mu/2} \in W^{1,2}_{\mathrm{loc}}(\Omega) \text{ uniformly with respect to } \delta. \tag{4.8}$$

It is then straightforward from Korn's inequality in $\mathbb{R}^2$ (see Lemma 2.1) that

$$u_\delta \in W^{1,r}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap W^{2,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \text{ uniformly}$$

for r, s as declared in Theorem 2.2, and $u \in \mathrm{LD}_{\mathrm{div}}(\Omega, \mathbb{R}^2)$ follows since $u_\delta \longrightarrow u$ in $L^1(\Omega, \mathbb{R}^2)$ and $\mathrm{BD}_{\mathrm{div}}(\Omega, \mathbb{R}^2) \cap W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \subset \mathrm{LD}_{\mathrm{div}}(\Omega, \mathbb{R}^2)$. We want to apply summation convention with respect to indices that appear twice and we will omit the index δ in the following calculations. Expanding equation (4.7), using $\operatorname{div} u_\delta \equiv 0$, we obtain

$$\begin{aligned}\int_\Omega \eta^{2k}\partial_\alpha\sigma : \varepsilon(\partial_\alpha u)\,\mathrm{d}x = &-2\int_\Omega \partial_\alpha\tau_{ij}\varepsilon_{i\alpha}(u)\partial_j\eta^{2k}\,\mathrm{d}x\\ &+\int_\Omega \partial_\alpha\tau_{ij}\partial_i u^\alpha\partial_j\eta^{2k}\,\mathrm{d}x\\ &+\int_\Omega f\cdot\partial_\alpha(\eta^{2k}\partial_\alpha u_\delta)\,\mathrm{d}x =: -2I_1 + I_2 + I_3.\end{aligned} \tag{4.9}$$

Ad I_1:

$$\begin{aligned} I_1 &= \int_\Omega (\partial_\alpha\sigma_{ij} - \delta_{ij}\partial_\alpha p)\varepsilon_{i\alpha}(u)\partial_j\eta^{2k}\,\mathrm{d}x\\ &\underset{(4.3)}{=} \int_\Omega \partial_\alpha\sigma_{ij}\varepsilon_{i\alpha}(u)\partial_j\eta^{2k} - \partial_\alpha\sigma_{j\alpha}\varepsilon_{i\alpha}(u)\partial_i\eta^{2k}\,\mathrm{d}x + \underbrace{\int_\Omega f_\alpha\varepsilon_{i\alpha}(u)\partial_j\eta^{2k}\,\mathrm{d}x}_{=:\,T}.\end{aligned}$$

The term T is bounded by our assumption (2.12) and (2.20). Interchanging the indices j and α, we may further write

$$\begin{aligned}|I_1| &\le \left|\int_\Omega \partial_\alpha\sigma_{ij}\varepsilon_{i\alpha}(u)\partial_j\eta^{2k} - \partial_\alpha\sigma_{j\alpha}\varepsilon_{lj}(u)\partial_l\eta^{2k}\,\mathrm{d}x\right| + c\\ &= \left|\int_\Omega \partial_\alpha\sigma_{ij}\Big[\varepsilon_{i\alpha}(u)\partial_j\eta^{2k} - \delta_{i\alpha}\varepsilon_{lj}(u)\partial_l\eta^{2k}\Big]\,\mathrm{d}x\right| + c,\end{aligned}$$

and with the same calculation that is carried out on p. 328 of [41] we establish

$$|I_1| \le c\left(\int_\Omega \eta^{2k}\partial_\alpha\sigma : \varepsilon(\partial_\alpha u)\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega \delta|\varepsilon(u)|^2 + \frac{|\varepsilon(u)|^2}{1+|\varepsilon(u)|}\,\mathrm{d}x\right)^{1/2} + c, \tag{4.10}$$

where we have used the growth estimate (2.30). Now, (2.12) and (2.20) together with an application of Young's inequality yield ($\rho > 0$ can be chosen arbitrarily small)

$$|I_1| \le \rho \int_\Omega \eta^{2k} \partial_\alpha \sigma : \varepsilon(\partial_\alpha u)\,\mathrm{d}x + c. \tag{4.11}$$

<u>*Ad* I_2:</u> at the beginning, an integration by parts yields

$$\begin{aligned} I_2 = \int_\Omega \partial_\alpha \tau_{ij} \partial_i u^\alpha \partial_j \eta^{2k}\,\mathrm{d}x &= -\int_\Omega \tau_{ij} \partial_i \partial_\alpha u^\alpha \partial_j \eta^{2k}\,\mathrm{d}x - \int_\Omega \tau_{ij} \partial_i u^\alpha \partial_\alpha \partial_j \eta^{2k}\,\mathrm{d}x \\ &\underset{\operatorname{div} u_\delta = 0}{=} -\int_\Omega \tau_{ij} \partial_i u^\alpha \partial_\alpha \partial_j \eta^{2k}\,\mathrm{d}x, \end{aligned}$$

and from this, by another integration by parts and using (4.5), we obtain

$$I_2 = \int_\Omega \tau_{ij} u^\alpha \partial_\alpha \partial_i \partial_j \eta^{2k}\,\mathrm{d}x + \underbrace{\int_\Omega f_j u^\alpha \partial_\alpha \partial_j \eta^{2k}\,\mathrm{d}x}_{=:\,T'}.$$

Clearly, the term T' is bounded due to (2.12) and (2.20), and an application of Hölder's inequality to the first summand gives

$$|I_2| \le c\left[1 + \left(\int_{B_{2R}(x_0)} |\tau|^2\,\mathrm{d}x\right)^{1/2} \left(\int_{B_{2R}(x_0)} |u|^2\,\mathrm{d}x\right)^{1/2}\right]. \tag{4.12}$$

By Korn's inequality (Lemma 2.1), u_δ is bounded in $L^2(\Omega, \mathbb{R}^2)$ and, owing to (4.4), Lemma 1.1 on p. 180 in [2] (see also Corollary 2.2 in [42]) implies

$$\int_{B_{2R}(x_0)} |\tau|^2\,\mathrm{d}x \le c \int_{B_{2R}(x_0)} |f|^2 + |\sigma|^2\,\mathrm{d}x \overset{(2.12),\,(2.15)}{<} \infty,$$

so that

$$|I_2| \le c. \tag{4.13}$$

Reintroducing the index δ, we observe that from the definition of ϕ_δ (see (4.8)) and the growth estimate (2.30) it follows that

$$\begin{aligned} \int_\Omega \eta^{2k} |\nabla \phi_\delta|^2\,\mathrm{d}x &\le c \int_\Omega \eta^{2k} \frac{|\nabla \varepsilon(u_\delta)|^2}{\big(1 + |\varepsilon(u_\delta)|\big)^\mu}\,\mathrm{d}x \\ &\le c \int_\Omega \eta^{2k} D^2 F_\delta\big(\varepsilon(u_\delta)\big)\big(\partial_\alpha \varepsilon(u_\delta), \partial_\alpha \varepsilon(u_\delta)\big)\,\mathrm{d}x \\ &\le c \int_\Omega \eta^{2k} \partial_\alpha \sigma_\delta : \varepsilon(\partial_\alpha u_\delta)\,\mathrm{d}x. \end{aligned} \tag{4.14}$$

Combining this with the estimates (4.9), (4.11) and (4.13), we deduce

$$\int_\Omega \eta^{2k} \frac{|\nabla\varepsilon(u_\delta)|^2}{\big(1+|\varepsilon(u_\delta)|\big)^\mu} \,\mathrm{d}x \le c(1+I_3). \tag{4.15}$$

<u>*Ad* I_3:</u> by (2.12) we have

$$|I_3| \le \int_\Omega \eta^{2k}|\nabla^2 u_\delta| \,\mathrm{d}x + 2k\int_\Omega \eta^{2k-1}|\nabla\eta||\nabla u_\delta| \,\mathrm{d}x =: S_1 + 2kS_2.$$

For the term S_1 we note that, due to the identity

$$\partial_j\partial_k w^i = \partial_j\big(\varepsilon(w)_{ik}\big) + \partial_k\big(\varepsilon(w)_{ij}\big) - \partial_i\big(\varepsilon(w)_{jk}\big) \quad \text{for all } w \in W^{2,1}(\Omega,\mathbb{R}^2),$$

we have

$$|\nabla^2 u_\delta| \le c|\nabla\varepsilon(u_\delta)|,$$

hence

$$|S_1| \le c\int_\Omega \eta^{2k}|\nabla\varepsilon(u_\delta)| \,\mathrm{d}x$$

and an application of Young's inequality therefore yields ($\rho > 0$ can be chosen arbitrarily small)

$$|S_1| \le \rho\int_\Omega \eta^{2k} \frac{|\nabla\varepsilon(u_\delta)|^2}{\big(1+|\varepsilon(u_\delta)|\big)^\mu} \,\mathrm{d}x + c\int_\Omega \eta^{2k}\big(1+|\varepsilon(u_\delta)|\big)^\mu \,\mathrm{d}x. \tag{4.16}$$

To S_2, we apply Young's inequality:

$$\begin{aligned}|S_2| &\le \varepsilon\int_\Omega \eta^{(2k-1)\mu}|\nabla u_\delta|^\mu \,\mathrm{d}x + c \overset{\mu\le 2}{\le} c\left(1+\int_\Omega \big|\nabla\big(\eta^{2k-1}u_\delta\big)\big|^\mu \,\mathrm{d}x\right)\\ &\le c\left(1+\int_\Omega \big|\varepsilon\big(\eta^{2k-1}u_\delta\big)\big|^\mu \,\mathrm{d}x\right) \le c\left(1+\int_\Omega \eta^{(2k-1)\mu}|\varepsilon(u_\delta)|^\mu \,\mathrm{d}x\right),\end{aligned}$$

where we have used Korn's inequality in $L^\mu(\Omega,\mathbb{R}^2)$ (see Lemma 2.1). Thus, if we choose $k \ge \frac{\mu}{2(\mu-1)}$, we obtain

$$|S_2| \le c\left(1+\int_\Omega \eta^{2k}\big(1+|\varepsilon(u_\delta)|\big)^\mu \,\mathrm{d}x\right). \tag{4.17}$$

Combining (4.16) and (4.17) with (4.15) and (4.14) yields

$$\int_\Omega \eta^{2k}|\nabla\phi_\delta|^2 \,\mathrm{d}x \le c\Big(1+\underbrace{\int_\Omega \eta^{2k}\big(1+|\varepsilon(u_\delta)|\big)^\mu \,\mathrm{d}x}_{=:\,S_3}\Big). \tag{4.18}$$

Let us define

$$\psi_\delta := \big(1 + |\varepsilon(u_\delta)|\big)^{\mu/2}.$$

Following ideas from [43] and using the Sobolev-Poincaré inequality in $\mathbb{R}^2$, we infer

$$|S_3| \le \int_\Omega |\eta^k \psi_\delta|^2 \,\mathrm{d}x \le c \left(\int_\Omega |\nabla(\eta^k \psi_\delta)| \,\mathrm{d}x \right)^2 \le c(\eta) + c \left(\int_\Omega \eta^k |\nabla \psi_\delta| \,\mathrm{d}x \right)^2 .$$

Observing $\psi_\delta = \phi_\delta^{\mu/(2-\mu)}$, we further obtain

$$\begin{aligned}\int_\Omega \eta^k |\nabla \psi_\delta| \,\mathrm{d}x &\le c \int_\Omega \eta^k |\nabla \phi_\delta| \phi_\delta^{\mu/(2-\mu)-1} \,\mathrm{d}x \\ &\le c \left(\int_\Omega \eta^{2k} |\nabla \phi_\delta|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_\Omega \eta^{2k} \phi_\delta^{\frac{4\mu-4}{2-\mu}} \,\mathrm{d}x \right)^{1/2}\end{aligned}$$

and therefore

$$|S_3| \le c(\eta) + c \left(\int_\Omega \eta^{2k} |\nabla \phi_\delta|^2 \,\mathrm{d}x \right) \left(\int_\Omega \eta^{2k} \phi_\delta^{\frac{4\mu-4}{2-\mu}} \,\mathrm{d}x \right).$$

Noting that $\frac{4\mu-4}{2-\mu} < 1$ due to our assumption $1 < \mu < 3/2$ as well as that $\phi_\delta \in L^1(\Omega)$ uniformly by (2.20), we may choose $R < R_0$ small enough such that the second summand in the above estimate of $|S_3|$ can be absorbed in the left-hand side of (4.18) and we finally obtain

$$\int_{B_R(x_0)} |\nabla \phi_\delta|^2 \,\mathrm{d}x \le \int_\Omega \eta^{2k} |\nabla \phi_\delta|^2 \,\mathrm{d}x \le c(R_0),$$

which implies (4.8) and thereby finishes the proof of Theorem 2.2. □

REMARK 4.1. *We would like to note that, for $d = 3$, the critical estimate (4.12) can be treated in the same way as the corresponding quantity in inequality (3.19) on p.329 of [41].*

5 Some extensions to the case of dissipative potentials of $(1, p)$-growth

We end our considerations with a short discussion of the anisotropic growth case, which can be treated by combining our foregoing arguments with the results from [44]. As in the preceding section, let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $F(\varepsilon) = h\big(|\varepsilon|\big)$ hold for a strictly convex scalar function $h : [0, \infty) \to [0, \infty)$, satisfying (2.25) and (2.26) with $\mu > 1$, as well as the anisotropic estimate

$$(2.22^*) \qquad \max\left\{ h''(t), \frac{h'(t)}{t} \right\} \le c_2 (1 + t)^{p-2}$$

for some $p > 1$. For the function F this means that

$$\nu_1|\varepsilon| - \nu_2 \le F(\varepsilon) \le \nu_3|\varepsilon|^p + \nu_4 \tag{5.1}$$

holds for all $\varepsilon \in \mathbb{S}^2$, with constants $\nu_1, ..., \nu_4$ as in (2.11). An example of such a density can e.g. be constructed via the formula

$$F(\varepsilon) := \int_0^{|\varepsilon|} \int_0^s \big(\varepsilon + r\big)^{\varphi(r)-2} \,\mathrm{d}r \,\mathrm{d}s,$$

where $\phi : [0, \infty) \to [2 - \mu, p]$ is a continuous decreasing function with $\varphi(0) = p$ and $\lim\limits_{r\to\infty} \varphi(r) = 2 - \mu$; we refer to [44] for further examples and a more detailed discussion of the growth condition (5.1).

Assume further (2.12)-(2.14) for the volume forces f with ν_1 from (2.31). Still, our variational problem

$$J[\,\cdot\,, \Omega] \to \min$$

is well defined on the space $\mathrm{LD}(\Omega, \mathbb{R}^2)$, and we can show:

THEOREM 5.1. *Let either*

$$1 < \mu < {}^3\!/\!_2 \quad \textit{and} \quad 1 < p < \mu, \tag{I}$$

or

$$\text{(II)} \qquad \begin{cases} 1 < \mu < {}^3\!/\!_2 \quad \textit{and} \quad p \in (1, 2] \quad \textit{together with the “balancing condition”} \\ \textit{(cf. the discussion after Proposition 5.2 in [44])} \\ \qquad\qquad \big|D^2F(\varepsilon)\big||\varepsilon|^2 \le c_3\big(F(\varepsilon) + 1\big), \quad \varepsilon \in \mathbb{S}^2 \end{cases}$$

hold. Then there exists $u \in \mathrm{LD}_{\mathrm{div}}(\Omega, \mathbb{R}^2)$ such that

$$\int_\Omega DF\big(\varepsilon(u)\big) : \varepsilon(\varphi) \,\mathrm{d}x = \int_\Omega f \cdot \varphi \,\mathrm{d}x \quad \textit{for all } \varphi \in C^\infty_{0,\mathrm{div}}(\Omega, \mathbb{R}^2). \tag{5.2}$$

REMARK 5.1. *i) Since for $p > 1$ the problem “$J[\,\cdot\,, \Omega] \to \min$ in $\mathrm{LD}(\Omega, \mathbb{R}^2)$” does not admit an evident relaxation in the space $\mathrm{BD}_{\mathrm{div}}(\Omega, \mathbb{R}^2)$, it is not clear whether u from Theorem 5.1 can be regarded as a minimizer in the sense of Theorem 2.1.*

ii) One could probably admit values $p > 2$ in the condition (II) above. However, this would require to adapt Lemma 2.6 to a suitable p-regularized version of our problem.

Sketch of the proof. Again we take the sequence $u_\delta \in \overset{\circ}{W}{}^{1,2}(\Omega, \mathbb{R}^2)$ from Lemma 2.6 and claim that its $L^1(\Omega, \mathbb{R}^2)$-limit $u \in \mathrm{BD}_{\mathrm{div}}(\Omega, \mathbb{R}^2)$ is the desired function. Repeating the calculations from the proof of Theorem 2.2, we see that estimate (4.10) for the quantity I_1 (defined in (4.9)) now has to be replaced with

$$|I_1| \le c\left(\int_\Omega \eta^{2k}\partial_\alpha\sigma : \varepsilon(\partial_\alpha u) \,\mathrm{d}x\right)^{1/2}\left(\int_{B_{2R}(x_0)} \delta|\varepsilon(u)|^2 + \big(1 + |\varepsilon(u)|\big)^p \,\mathrm{d}x\right)^{1/2} + c, \tag{5.3}$$

hence

$$|I_1| \le \varepsilon \int_\Omega \eta^{2k} \partial_\alpha \sigma : \varepsilon(\partial_\alpha u)\,\mathrm{d}x + c\left(1 + \int_{B_{2R}(x_0)} \big(1+|\varepsilon(u)|\big)^p\,\mathrm{d}x\right). \tag{5.4}$$

For ϕ_δ as defined in (4.8) we thus obtain

$$\int_\Omega \eta^{2k}|\nabla\phi_\delta|^2\,\mathrm{d}x \le c\bigg(1 + \int_{B_{2R}(x_0)} \eta^{2k}\big(1+|\varepsilon(u_\delta)|\big)^\mu\,\mathrm{d}x + \int_{B_{2R}(x_0)} \big(1+|\varepsilon(u)|\big)^p\,\mathrm{d}x\bigg),$$

which corresponds to inequality (4.3) in [44]. If now $p < \mu$, then the p-term can be absorbed in the integral $\int_{B_{2R}(x_0)} \eta^{2k}\big(1+|\varepsilon(u_\delta)|\big)^\mu\,\mathrm{d}x$ and we may argue as in the proof of Theorem 2.2 to infer

$$u_\delta \in W^{1,r}_{\mathrm{loc}}(\Omega,\mathbb{R}^2) \cap W^{2,s}_{\mathrm{loc}}(\Omega,\mathbb{R}^2) \text{ uniformly.}$$

The assertion of Theorem 5.1 now follows from

$$u_\delta \longrightarrow u \text{ in } L^1(\Omega,\mathbb{R}^2).$$

For the balanced case (II), we refer to Remark 3.1 in [44]. □

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Michael Bildhauer (bibi@math.uni-sb.de)
Martin Fuchs (fuchs@math.uni-sb.de)
Jan Müller (jmueller@math.uni-sb.de)

Saarland University
Department of Mathematics

P.O. Box 15 11 50
66041 Saarbrücken
Germany