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The Virtual Element Method on Anisotropic Polygonal Discretizations*

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Abstract. In recent years, the numerical treatment of boundary value problems with the help of polygonal and polyhedral discretization techniques has received a lot of attention within several disciplines. Due to the general element shapes an enormous flexibility is gained and can be exploited, for instance, in adaptive mesh refinement strategies. The Virtual Element Method (VEM) is one of the new promising approaches applicable on general meshes. Although polygonal element shapes may be highly adapted, the analysis relies on isotropic elements which must not be very stretched. But, such anisotropic element shapes have a high potential in the discretization of interior and boundary layers. Recent results on anisotropic polygonal meshes are reviewed and the Virtual Element Method is applied on layer adapted meshes containing isotropic and anisotropic polygonal elements.

1 Introduction

In the numerical treatment of boundary value problems the flexibility in the discretization of the computational domain has gained more and more importance during the last years. Therefore, approximation strategies applicable on general polygonal and polyhedral meshes attracted a lot of interest. These approaches include, e.g., the BEM-based FEM [12], where BEM stands for Boundary Element Method, the Virtual Element Method (VEM) [5], Mimetic Finite Differences, polygonal Discontinuous Galerkin methods, and Hybrid High-Order schemes (see [4] and the papers therein cited). Since polygonal elements may contain an arbitrary number of nodes on their boundary, the notion of "hanging nodes" is naturally included in the previously mentioned approaches. Consequently, the application in adaptive mesh refinement strategies becomes very attractive. For this reason, a posteriori error estimates have been developed and employed for the BEM-based FEM as well as for MFD and VEM in recent publications, see [1,7,10,11,17,19,20].

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For classical Finite Element Methods (FEM), it is widely recognized that anisotropic mesh refinements have significant potential for improving the efficiency of the solution process when dealing with sharp layers in the solution. Pioneering works for anisotropic triangular and tetrahedral meshes have been done by Apel [3] as well as by Formaggia and Perotto [13,14]. Furthermore, a posteriori error estimates for driving adaptivity with anisotropic elements have been studied by Kunert [15]. The anisotropic refinement of classical elements, however, results in certain restrictions due to the limited element shapes and the necessity to remove or handle hanging nodes in the discretization. Polygonal elements, in contrast, are much more flexible and adapt to anisotropic element shapes easily. This new topic has been addressed in [18], where anisotropic interpolation error estimates have been proved and utilized to generate highly adapted meshes.

The aim of this paper is to investigate the Virtual Element Method on such anisotropically adapted meshes. In Sect. 2, some results of [18] are highlighted. After a short review of the VEM in Sect. 3, we give a numerical experiment in Sect. 4 demonstrating the applicability of the method on anisotropic meshes.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and let $\mathcal{K}_h$ be a decomposition of Ω into disjoint polygonal elements K such that $\overline{\Omega} = \bigcup_{K\in\mathcal{K}_h} \overline{K}$. Each element K consists of an arbitrary (uniformly bounded) number of vertices which correspond to the nodes in the polygonal mesh $\mathcal{K}_h$. The edges e of K are always located between two nodes. Several nodes may lie on a straight line and thus the notion of "hanging nodes" in classical finite element discretizations is naturally included in polygonal meshes. In order to prove convergence estimates for the later discussed approach, the meshes have to fulfil certain regularity assumptions. A classical choice for them is, cf. [5,18], that all elements $K \in \mathcal{K}_h$ with diameter h_K fulfil:

- K is a star-shaped polygon with respect to a circle of radius ρ_K and with a uniformly bounded aspect ratio h_K/ρ_K.
- For the element K and all its edges $e \subset \partial K$ the ratio $h_K/|e|$ is uniformly bounded, where $|e|$ denotes the edge length.

These regularity assumptions obviously do not allow stretched anisotropic elements, since their aspect ratio degenerates. Therefore, the regularity for meshes with anisotropic elements has to be adapted. The geometric information of the polygonal element K is encoded in the symmetric and positive definite covariance matrix

$$M_{\mathrm{Cov}}(K) = \frac{1}{|K|}\int_K (\boldsymbol{x}-\bar{\boldsymbol{x}}_K)(\boldsymbol{x}-\bar{\boldsymbol{x}}_K)^\top \, d\boldsymbol{x} \in \mathbb{R}^{d\times d}, \quad \bar{\boldsymbol{x}}_K = \frac{1}{|K|}\int_K \boldsymbol{x}\, d\boldsymbol{x}.$$

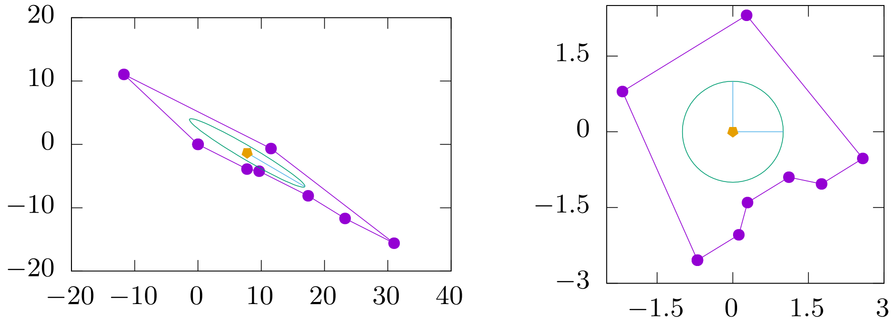

Fig. 1. Anisotropic polygonal element (left) and transformed reference configuration (right) with ellipse given by the eigenvectors of M_{Cov} scaled by the square root of the corresponding eigenvalues

This matrix admits an eigenvalue decomposition $M_{\mathrm{Cov}}(K) = U_K \Lambda_K U_K^\top$ with $U^\top = U^{-1}$ and $\Lambda_K = \mathrm{diag}(\lambda_{K,1}, \lambda_{K,2})$. Exploiting this information, we define the mapping

$$\boldsymbol{x} \mapsto F_K(\boldsymbol{x}) = A_K \boldsymbol{x} \quad \text{with} \quad A_K = \Lambda_K^{-1/2} U_K^\top,$$

which transforms the element K into a reference configuration $F_K(K)$. An example is given in Fig. 1. Thus, we call $\mathcal{K}_h$ a regular anisotropic mesh if:

- The reference configuration $F_K(K)$ for all $K \in \mathcal{K}_h$ is a regular polygonal element according to the previous assumptions.
- Neighbouring elements in $\mathcal{K}_h$ behave similarly in their anisotropy, i.e., their characteristic directions are scaled and oriented in a comparable way, see [18] for details.

Under these assumptions on the mesh, an anisotropic error estimate for the Clément interpolation operator on polygonal discretizations can be derived, see [18, Sec. 4.1]. For $v \in H^1(\Omega)$, we denote its interpolation by $\mathfrak{I}_C v \in V_h$, where the discrete space V_h is given in the next section and the expansion coefficients are defined as usual as averages over neighbouring elements of the nodes. The Clément interpolation fulfils

$$\|v - \mathfrak{I}_C v\|_{L_2(\Omega)}^2 \leq c \sum_{K \in \mathcal{K}_h} \lambda_{K,1}\, \boldsymbol{u}_{K,1}^\top\, G_K^*(v)\, \boldsymbol{u}_{K,1} + \lambda_{K,2}\, \boldsymbol{u}_{K,2}^\top\, G_K^*(v)\, \boldsymbol{u}_{K,2},$$

where $G_K^*(v) = \left(\int_K \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}\, d\boldsymbol{x} \right)_{i,j=1}^2 \in \mathbb{R}^{2\times 2}$ for $\boldsymbol{x} = (x_1, x_2)^\top$ and $\boldsymbol{u}_{K,i}$ are the eigenvectors of $M_{\mathrm{Cov}}(K)$ corresponding to the eigenvalues $\lambda_{K,i}$, $i = 1, 2$. This result generalizes the work of Formaggia and Perotto [14].

The generation of such general meshes, however, cannot be performed with standard tools. For the mesh refinement we use a bisection of the polygonal element through its barycentre into two new elements. The direction of the bisection might be determined by $\boldsymbol{u}_{K,2}$ that yields an isotropic refinement or by the eigenvector corresponding to the smallest eigenvalue of $G_K^*(v)$. The

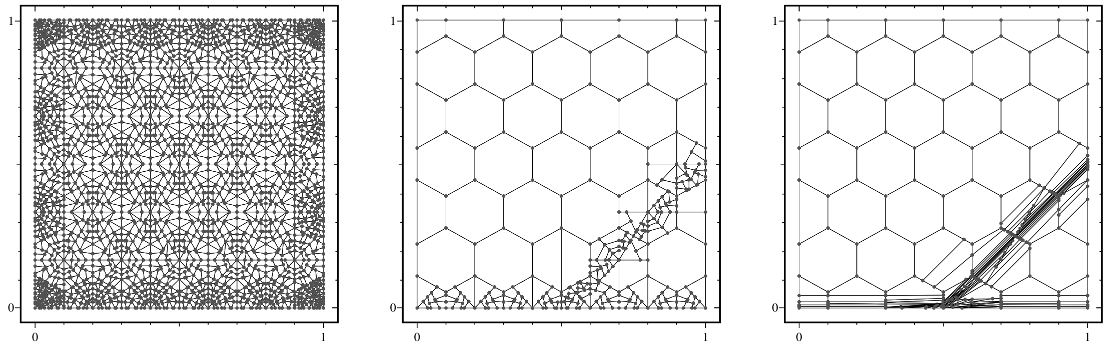

Fig. 2. Uniform (left) and adaptive meshes with isotropic (middle) and anisotropic elements (right) after 6 refinement steps starting from a hexagonal mesh.

later strategy results in an anisotropic refinement, where the characteristics of the function v are incorporated. In Fig. 2, three meshes are visualized which are obtained after 6 refinement steps using a uniform and adaptive strategy with isotropic and anisotropic bisection, starting from a hexagonal mesh. These refinement procedures are exploited in the later numerical experiment.

3 Virtual Element Method

It remains to discuss the numerical approximation of boundary value problems on polygonal meshes. We restrict ourselves for simplicity to the Poisson problem. For a given source function $f \in L^2(\Omega)$, we consider the following formulation: find $u \in V = H_0^1(\Omega)$ such that:

$$a(u,v) = \int_\Omega fv, \quad \forall v \in V, \tag{1}$$

where $a(\cdot,\cdot) = (\nabla\cdot,\nabla\cdot)_{0,\Omega}$. Problem (1) is well-posed. For future use, it is convenient to split the (continuous) bilinear form $a(\cdot,\cdot)$ defined in (1) into a sum of local contributions:

$$a(u,v) = \sum_{K\in\mathcal{K}_h} a^K(u,v) \quad \forall u,\, v \in V, \quad \text{where } a^K(\cdot,\cdot) = (\nabla\cdot,\nabla\cdot)_{0,K}.$$

In order to construct the lowest order VEM approximation of (1), we need the following ingredients:

- Finite dimensional subspaces $V_h(K)$ of $V(K) = V \cap H^1(K)$ $\forall K \in \mathcal{K}_h$;
- Local symmetric bilinear forms $a_h^K : V_h(K) \times V_h(K) \to \mathbb{R}$ $\forall K \in \mathcal{K}_h$ so that
$$a_h(u_h,v_h) = \sum_{K\in\mathcal{K}_h} a_h^K(u_h,v_h) \quad \forall u_h,\, v_h \in V_h; \tag{2}$$
- A duality pairing $\langle f_h,\cdot\rangle_h$, where $f_h \in V_h'$ and V_h' is the dual space of V_h.

The above ingredients must be built in such a way that the discrete version of (1): find $u_h \in V_h$ such that:

$$a_h(u_h, v_h) = \langle f_h, v_h \rangle_h, \quad \forall v_h \in V_h, \tag{3}$$

is well-posed and optimal a priori energy error estimates hold, cf. [5].

We begin by introducing the local space $V_h(K)$ for $K \in \mathcal{K}_h$

$$V_h(K) = \left\{ v_h \in H^1(K) \mid \Delta v_h = 0,\, v_h|_{\partial K} \in \mathbb{B}_1(\partial K) \right\}, \tag{4}$$

where $\mathbb{B}_1(\partial K) = \left\{ v_h \in \mathcal{C}^0(\partial K) \mid v_h|_e \in \mathbb{P}_1(e),\, \forall e \in \partial K \right\}$. The global space is then obtained by gluing continuously the local spaces:

$$V_h = \left\{ v_h \in H_0^1(\Omega) \cap \mathcal{C}^0(\overline{\Omega}) \mid v_h|_K \in V_h(K),\, \forall K \in \mathcal{K}_h \right\}. \tag{5}$$

Here, we only revised the space yielding first order approximations. But, the approach can be extended to k-th order approximation spaces V_h^k for $k > 1$, cf. [5,6]. We endow the space (4) with the values of v_h at the vertices of K. Reasoning as in [5], it is easy to see that this is a unisolvent set of degrees of freedom. Owing the definition (4) of the VE local space and the choice of the degrees of freedom, it is possible to compute the $H^1(K)$ projector $\Pi_1^\nabla : V_h(K) \to \mathbb{P}_1(K)$

$$\begin{cases} a^K(\Pi_1^\nabla v_h - v_h, q) = 0, & \forall q \in \mathbb{P}_1(K), \\ \int_{\partial K} (\Pi_1^\nabla v_h - v_h) = 0, & \end{cases} \qquad \forall v_h \in V_h(K), \tag{6}$$

see [5] for details. We observe that the last condition in (6) is needed in order to fix the constant part of the energy projector.

Next, we introduce the discrete right-hand side $f_h \in V_h'$ and the associated duality pairing, i.e. $\langle f_h, v_h \rangle_h = \sum_{K \in \mathcal{K}_h} \int_K \Pi_0^0 f \overline{v}_h$, where Π_0^0 is the L^2 projection on constants and $\overline{v}_h = \frac{1}{|\partial K|} \int_{\partial K} v_h$. Finally, we consider the discrete bilinear form and we require that the local bilinear forms $a_h^K : V_h(K) \times V_h(K) \to \mathbb{R}$ satisfy, for all $K \in \mathcal{K}_h$, the following two assumptions

(**A1**) **consistency**: $a^K(q, v_h) = a_h^K(q, v_h) \quad \forall q \in \mathbb{P}_1(K),\, \forall v_h \in V_h(K)$;

(**A2**) **stability**: there exist two positive constants $0 < \alpha_* < \alpha^* < +\infty$ (possibly depending on the shape regularity of K), such that $\forall v_h \in V_h(K)$

$$\alpha_* |v_h|_{1,K}^2 \leq a_h^K(v_h, v_h) \leq \alpha^* |v_h|_{1,K}^2.$$

Assumption (**A1**) guarantees that the method is exact whenever the solution of (1) is a polynomial of degree one, whereas assumption (**A2**) guarantees the well-posedness of problem (3). Let now Id_h be the identity operator on the space $V_h(K)$, we set for every $u_h, v_h \in V_h(K)$

$$a_h^K(u_h, v_h) = a^K(\Pi_1^\nabla u_h, \Pi_1^\nabla v_h) + S_h^K((\mathrm{Id}_h - \Pi_1^\nabla)u_h, (\mathrm{Id}_h - \Pi_1^\nabla)v_h), \tag{7}$$

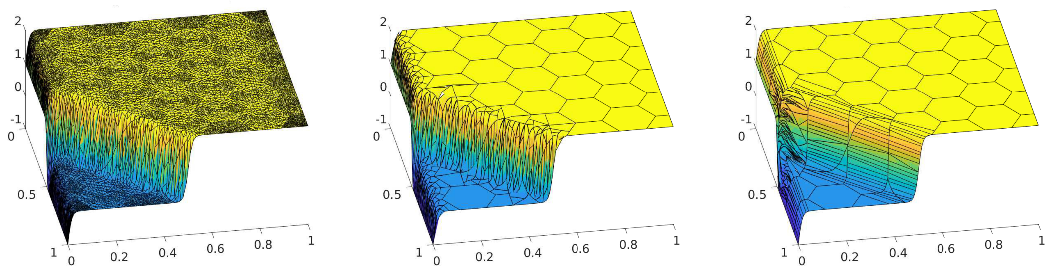

Fig. 3. Computed solution on uniform (left) and adaptive meshes with isotropic (middle) and anisotropic elements (right) after 8 refinement steps, with $k = 1$

where Π_1^∇ is defined in (6) and the local bilinear form $S_h^K(\cdot,\cdot)$

$$S_h^K(u_h, v_h) = \sum_{i=1}^{\dim(V_h(K)\cap V_h)} \mathrm{dof}_i(u_h)\mathrm{dof}_i(v_h) \tag{8}$$

satisfies $c_*|v_h|^2_{1,K} \le S_h^K(v_h,v_h) \le c^*|v_h|^2_{1,K}$ for all $v_h \in \ker(\Pi_1^\nabla)$, where c_* and c^* might depend on the shape regularity of the polygon, and the local discrete bilinear form (7) satisfies (**A1**) and (**A2**).

4 Numerical Experiment

In this last section we report some preliminary results obtained solving the Poisson problem with non-homogeneous Dirichlet boundary condition

$$-\Delta u = f \quad \text{in } \Omega = (0,1)^2, \qquad u = g \quad \text{on } \partial\Omega,$$

by the Virtual Element Method on uniformly and locally refined meshes with isotropic as well as anisotropic elements. The computations are done with first and higher order approximation spaces V_h^k. In the considered test problem, the exact solution is

$$u(\boldsymbol{x}) = \tanh(60x_2) - \tanh(60(x_1 - x_2) - 30), \quad \boldsymbol{x} = (x_1, x_2)^\top \in \mathbb{R}^2,$$

chosen because it has a strong boundary layer on the bottom of the domain and a strong internal layer connecting the bottom and the right part of the boundary. The Dirichlet data and the forcing function are chosen appropriately as $g = u\big|_{\partial\Omega}$ and

$$f(\boldsymbol{x}) = 14400\frac{\tanh(30 - 60(x_1 - x_2))}{\cosh^2(30 - 60(x_1 - x_2))} + 7200\frac{\tanh(60x_2)}{\cosh^2(60x_2)}.$$

In Fig. 2, we display the meshes and in Fig. 3 the projections of the VEM solution with order $k = 1$ on piecewise linear functions over each element. We can easily observe that the solution is properly described on all the meshes,

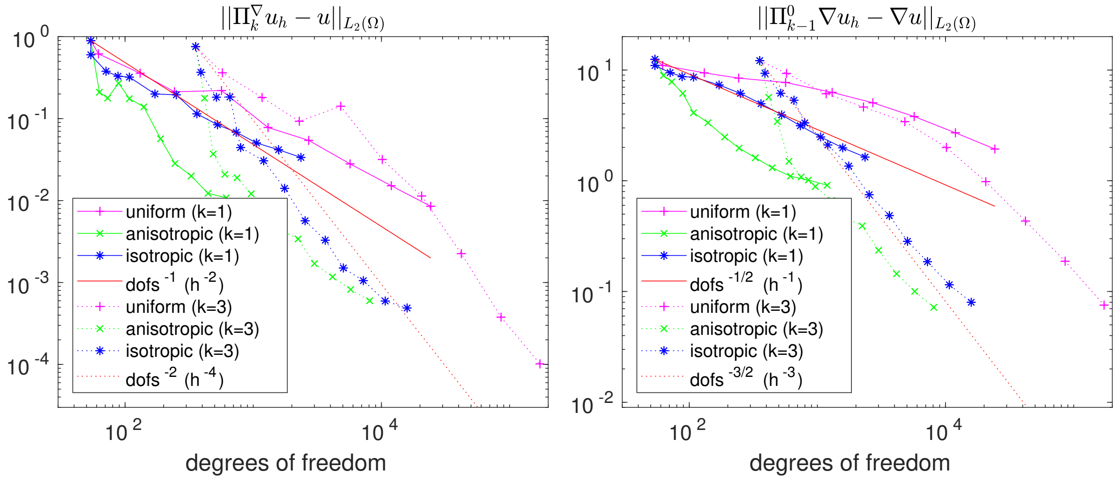

Fig. 4. Errors (L_2 left and H^1 right) on uniform and adaptive meshes with isotropic and anisotropic elements in logarithmic scale

although the number of degrees of freedom (dofs) on them is very different. Similar approximations can be obtained considering advection dominated advection diffusion problems. For these problems the VEM solution requires special stabilization for preventing spurious oscillations [8]. The anisotropic VEM elements generated by the approach presented in [18] have a quite large aspect ratio. For these elements the VEM construction could require the implementation of suitable polynomial basis functions in order to prevent problems due to the ill conditioning of the VEM projectors introduced in Sect. 3 as described, e.g., in [2,9,16].

In Fig. 4, we report the convergence histories in the approximate L_2 error norm defined by the Π_k^∇-projection of the VEM solution and the H^1 error semi-norm defined considering the L_2 projection on the space of polynomials of order $k-1$ of the gradient of the solution, cf. [6]. In the convergence graphs, we consider $k=1$ and $k=3$, where the continuous lines indicate the theoretical rates of convergence. The errors are given with respect to the number of degrees of freedom and, in the legend, we recall the corresponding rates with respect to the mesh-size h on quasi uniform meshes as well. Both, for the L_2 and H^1 errors as well as for $k=1$ and $k=3$, the convergence histories fit very well the expected theoretical behaviours. The anisotropic meshes always perform superior providing a smaller error with respect to the uniform or the isotropic adaptive mesh.

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