Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint

# New Facts around the Choquet Integral

Heinz König

Preprint No. 62

Saarbrücken 2002

Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

# New Facts around the Choquet Integral

Heinz König

Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken
Germany
E-Mail: hkoenig@math.uni-sb.de

submitted: June 13, 2002

Preprint No. 62

Saarbrücken 2002

Edited by
FR 6.1 – Mathematik
Im Stadtwald
D–66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

Séminaire d'Analyse Fonctionelle
Université Pierre et Marie Curie (Paris VI)
11 Avril 2002

# NEW FACTS AROUND THE CHOQUET INTEGRAL

HEINZ KÖNIG

In section 48 of his famous Theory of Capacities [2] Gustave Choquet introduced a certain class of functionals with the flavour of an *integral*, but invented for an important issue in capacities and not at all for the sake of measure and integration. Yet the concept showed basic qualities in that other respect too: It was in the initial spirit of Lebesgue [11] to construct the integral via decomposition into horizontal strips rather than into vertical ones, which had fallen into oblivion in the course of the 20th century, and was simpler and much more comprehensive than the usual constructions. Thus in subsequent decades the concept developed into a universal one in measure and integration, called the *Choquet integral.* One could even wonder why the Choquet integral did not become the foundation for all of integration theory.

But the fact that this did not happen had an immediate reason: The basic hardship with the Choquet integral is that it is a priori obscure whether and when it is *additive*, which one best even subdivides into *subadditive* and *superadditive.* To this issue Choquet contributed in his final section 54 a spectacular, because much more abstract idea: On certain lattice cones *all submodular and positive-homogeneous real-valued functionals must be subadditive*, and the same for *super* in place of *sub*. It is this assertion which forms the theme of the present note (in the sequel the two cases will be united via an obvious *sub/super* shorthand notation). The treatment of Choquet was kind of an outline, and his proof limited to a rather narrow special case. While the Choquet integral has been explored in subsequent decades, the abstract assertion remained unsettled up to now.

In recent years the present author became motivated because he needed an assertion of this kind for the further development of his extended Daniell-Stone and Riesz representation theorems [7]. The present note is a summary of his recent paper [10]. It presents a counterexample which shows that the initial context for the abstract assertion has to be modified, and then in new context a comprehensive theorem which fulfils all needs turned up so far. There are two proofs for the basic step within the so-called *finite* situation. One of them is a distributional version of the initial proof due to Choquet, while the other one furnishes, via a remarkable fact on convex functions, an essential fortification in the finite situation.

**1. The Choquet integral.** The Choquet integral as evolved in the second half of the 20th century exists in different versions. The present version is from the author's textbook [7] section 11. It features *two* classes of admissible functions. The reason is that the two variants are in perfect accord with the two extension theories in measure and integration, the inner and the outer one, developed in [7].

Let $X$ be a nonvoid set and $\mathfrak{S}$ be a lattice of subsets with $\varnothing \in \mathfrak{S}$ in $X$. We define $\mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ to consist of the functions $f \in [0,\infty]^X$ such that $[f \geqq t]/[f > t] \in \mathfrak{S}$

for all $0 < t < \infty$, called *upper/lower measurable* $\mathfrak{S}$. We fix an increasing set function $\varphi : \mathfrak{S} \to [0, \infty]$ with $\varphi(\varnothing) = 0$ and define the *Choquet integral*

$$\rlap{\,-}\int f d\varphi := \int_{0\leftarrow}^{\rightarrow\infty} \varphi([f \geqq t])dt \in [0, \infty] \quad \text{for } f \in \mathrm{UM}(\mathfrak{S}),$$

$$\rlap{\,-}\int f d\varphi := \int_{0\leftarrow}^{\rightarrow\infty} \varphi([f > t])dt \in [0, \infty] \quad \text{for } f \in \mathrm{LM}(\mathfrak{S}),$$

both times as an improper Riemann integral of a decreasing function with values in $[0, \infty]$. It is well-defined since in case $f \in \mathrm{UM}(\mathfrak{S}) \cap \mathrm{LM}(\mathfrak{S})$ the two second members are equal. Thus for $A \in \mathfrak{S}$ we have $\chi_A \in \mathrm{UM}(\mathfrak{S}) \cap \mathrm{LM}(\mathfrak{S})$ with $\rlap{\,-}\int \chi_A d\varphi = \varphi(A)$. If $\mathfrak{S}$ is a $\sigma$ algebra then $\mathrm{UM}(\mathfrak{S}) = \mathrm{LM}(\mathfrak{S})$ consists of the $f \in [0, \infty]^X$ *measurable* $\mathfrak{S}$ in the usual sense, and in case of a measure $\varphi$ then $\rlap{\,-}\int f d\varphi$ is the usual integral $\int f d\varphi$.

The prototype of the Choquet integral defined in [2] was for the lattice $\mathfrak{S} = \mathrm{Comp}(X)$ of the compact subsets in a locally compact Hausdorff topological space $X$ and under the assumption $\varphi < \infty$, but restricted to the function class

$$\mathrm{CK}(X, [0, \infty[) \subset \mathrm{USCK}(X, [0, \infty[) \subset \mathrm{UM}(\mathrm{Comp}(X)) \cap [0, \infty[^X,$$

with these classes defined to consist of the continuous and of the upper semicontinuous functions $X \to [0, \infty[$ with compact support. Therefore the set functions $\varphi$ had sometimes to be restricted to the downward $\tau$ continuous ones, that is to the *capacities* in the sense of [2].

We return to the full Choquet integral. For the sequel we need a few terms on nonvoid function systems $S \subset [0, \infty]^X$ and functionals $I : S \to [0, \infty]$. The list will be continued in section 5 below.

D1) $I$ is called *(sub/super)additive* iff $I(u + v) \leqq/\geqq I(u) + I(v)$ for all $u, v \in S$ with $u + v \in S$. Thus $S$ need not be stable under addition.

D2) Assume that $S$ is stable under the pointwise lattice operations $\max \min = \vee\wedge$. Then $I$ is called *(sub/super)modular* iff $I(u \vee v) + I(u \wedge v) \leqq/\geqq I(u) + I(v)$ for all $u, v \in S$.

One then notes the properties which follow.

1.1 Properties. i) $\mathrm{UM}(\mathfrak{S})$ *and* $\mathrm{LM}(\mathfrak{S})$ *are positive-homogeneous* (under multiplication with real numbers $0 < t < \infty$) *with* $0$ *and stable under* $\vee\wedge$.

ii) *If* $\mathfrak{S}$ *is stable under countable intersections then* $\mathrm{UM}(\mathfrak{S})$ *is stable under addition and* $\mathrm{UM}(\mathfrak{S}) \supset \mathrm{LM}(\mathfrak{S})$. *If* $\mathfrak{S}$ *is stable under countable unions then* $\mathrm{LM}(\mathfrak{S})$ *is stable under addition and* $\mathrm{LM}(\mathfrak{S}) \supset \mathrm{UM}(\mathfrak{S})$.

iii) *The Choquet integral* $I : I(f) = \rlap{\,-}\int f d\varphi$ *on* $\mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ *is positive-homogeneous with* $I(0) = 0$ *and increasing under the pointwise order* $\leqq$.

The basic question is when the Choquet integral $I : I(f) = \rlap{\,-}\int f d\varphi$ on $\mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ is *(sub/super)additive*, and also when it is *(sub/super)modular*, in relation to the respective behaviour of the set function $\varphi$. For set functions on *lattices* the adequate notion is *(sub/super)modular*, defined to mean that

$$\varphi(A \cup B) + \varphi(A \cap B) \leqq/\geqq \varphi(A) + \varphi(B) \quad \text{for all } A, B \in \mathfrak{S}.$$

One notes the simple observations

(A) $\varphi$ (sub/super)modular $\Longleftarrow I$ (sub/super)additive,
(M) $\varphi$ (sub/super)modular $\Longleftrightarrow I$ (sub/super)modular,

under the reservation that for the prototype in [2] the two implications $\Longleftarrow$ require that $\varphi$ be a *capacity*. The decisive question is whether $\Longrightarrow$ holds true in (A). It will be dealt with in the subsequent sections.

**2. The work of Choquet 1953/54.** Choquet in [2] noted that for his prototype the implication $\Longrightarrow$ in (A) is valid, and hence that the three properties involved are equivalent for *capacities* $\varphi$. But what is more and deserves to be called *spectacular*, he had the idea that the implication

$$I \text{ (sub/super)modular } \Longrightarrow I \text{ (sub/super)additive}$$

must be valid for a much wider class of functionals (he also knew that this cannot be true for the converse implication). His precise formulation 54.1 was as follows.

2.1 CHOQUET'S VISION. *Let $E$ be an ordered vector space with order $\sqsubseteq$ and positive cone $E^+$, and assume that $E$ (or at least $E^+$) is a lattice under $\sqsubseteq$ with lattice operations $\sqcup\sqcap$. Let $I : E^+ \to \mathbb{R}$ be positive-homogeneous. If $I$ is* (sub/super)modular *under $\sqcup\sqcap$ then it must be* (sub/super)additive.

From this vision 2.1 applied to $E = \mathrm{CK}(X, \mathbb{R})$ on the locally compact Hausdorff $X$ with pointwise order $\leqq$ and lattice operations $\vee\wedge$, and to the restricted Choquet integral $I : I(f) = \fint f d\varphi$ on $E^+ = \mathrm{CK}(X, [0, \infty[)$ with an arbitrary $\varphi$, and combined with $\Longrightarrow$ in (M), it follows indeed that Choquet's prototype fulfils the desired implication $\Longrightarrow$ in (A).

However, Choquet did not prove his vision 2.1 in its full extent. His proof was restricted to the case $E = \mathbb{R}^n$ with pointwise order $\leqq$ and lattice operations $\vee\wedge$, and to the positive-homogeneous functions $I : E^+ = [0, \infty[^n \to \mathbb{R}$ which are continuous on $[0, \infty[^n$ and $\mathrm{C}^2$ on $]0, \infty[^n$. The explanation is that the entire context was at the end and outside the mainstream of the memoir [2]. Nevertheless Choquet's proof was so well-founded that after half a century it was capable to furnish a proof of the basic step for the present new main theorem. Therefore we include a sketch of the proof.

2.2 PROPOSITION. *Assume that $I : ]0, \infty[^n \to \mathbb{R}$ is positive-homogeneous and $\mathrm{C}^2$. If $I$ is (sub/super)modular then it is (sub/super)additive.*

Sketch of proof. Assume that $I : ]0, \infty[^n \to \mathbb{R}$ is $\mathrm{C}^2$. Let $X_1, \cdots, X_n : ]0, \infty[^n \to ]0, \infty[$ denote the coordinate functions and $D_1, \cdots, D_n$ the partial derivations. One verifies three facts.

0) For $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$ one has the identity

$$\sum_{k,l=1}^{n} z_k z_l (D_k D_l I) = \sum_{k=1}^{n} \frac{z_k^2}{X_k} \Big( \sum_{l=1}^{n} X_l (D_k D_l I) \Big) - \frac{1}{2} \sum_{k,l=1}^{n} X_k X_l \Big( \frac{z_k}{X_k} - \frac{z_l}{X_l} \Big)^2 (D_k D_l I).$$

1) If $I$ is positive-homogeneous then $\sum_{l=1}^{n} X_l (D_k D_l I) = 0$ for $1 \leqq k \leqq n$.

2) If $I$ is (sub/super)modular then $D_k D_l I \leqq / \geqq 0$ for $1 \leqq k \neq l \leqq n$.

For $I$ positive-homogeneous and (sub/super)modular these facts combine with Taylor's formula to furnish that $I$ is convex/concave, that is (sub/super)additive. □

**3. The need for reconsideration.** We have seen that Choquet's vision 2.1 furnishes the decisive implication $\Longrightarrow$ in (A) for his prototype. But we have to note that it does *not* furnish this implication for the full Choquet integral. An obvious reason is that the functions $f \in \mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ and the functional $I : I(f) = ⨍ f d\varphi$ can attain the value $\infty$, and another one that the domains $\mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ need not be stable under addition. But there is a deeper reason: Let for example $X = [0,1]$ and $\mathfrak{S} = \mathrm{Comp}(X)$. Then $\mathrm{USC}(X, [0,\infty[) = \mathrm{UM}(\mathfrak{S}) \cap [0,\infty[^X$ is a convex cone in the vector space $E = \mathrm{B}(X,\mathbb{R})$ of bounded functions, *pointed* and *salient* in the usual sense and hence the positive cone $E^+ \subset E$ in the so-called *intrinsic order* $\sqsubseteq$ which it produces. But one proves that $E^+$ is *not a lattice* under $\sqsubseteq$. Thus 2.1 cannot be applied.

To be sure, the implication $\Longrightarrow$ in (A) holds true for the full Choquet integral, even though this does not follow from Choquet's vision 2.1. In the second half of the 20th century the implication has been proved a number of times for the different versions of the Choquet integral. As far as the author is aware, the first proof is due to Topsøe [13] in 1974. For other proofs see [1] [6] [12] [3]. The implication for the present version is in [7] 11.11.

On the other hand Choquet's vision 2.1 remained open in [2], and in fact remained open all the time so far. Now in March 2002 the present author observed that the statement is not true as it stands.

3.1 Example. Let $E = \mathbb{R}^2$ be equipped with the lexicographical order $\sqsubseteq$, that is $u = (u_1, u_2)$ and $v = (v_1, v_2)$ fulfil $u \sqsubseteq v$ iff either $u_1 < v_1$ or $u_1 = v_1$ and $u_2 \leqq v_2$. The order $\sqsubseteq$ is compatible with the standard vector space structure, and $E^+$ consists of the halfspace $\{x : x_1 > 0\}$ and the halfline $\{x : x_1 = 0 \text{ and } x_2 \geqq 0\}$. Moreover $\sqsubseteq$ is *total* and hence a lattice order with trivial lattice operations $\sqcup\sqcap$. Thus in particular all positive-homogeneous functions $I : E^+ \to \mathbb{R}$ are modular $\sqcup\sqcap$. But of course most of them are *not additive*. A simple example is $I : I(x) = x_2^+$, since $u = (u_1, u_2)$ and $v = (v_1, v_2)$ with $u_1, v_1 > 0$ and $u_2 < 0 < v_2$ are in $E^+$ and fulfil $I(u+v) = (u_2 + v_2)^+ < v_2 = v_2^+ = I(v) = I(u) + I(v)$. $\square$

The same idea works for any real vector space $E$ of dimension $> 1$, in that one defines a compatible and total order $\sqsubseteq$ on $E$ via the choice of a basis $B$ of $E$ and of a well-order of $B$.

Thus there is quite some reason for reconsideration. This does not mean to impose questions upon the wonderful overall implication

$$I \text{ (sub/super)modular } \Longrightarrow I \text{ (sub/super)additive},$$

which will henceforth be called the *fundamental implication.* In the sequel the author wants to summarize what he observed since 1998. Section 4 will be devoted to the special case $E = \mathbb{R}^n$ with pointwise order $\leqq$ and lattice operations $\vee\wedge$, henceforth called the *finite* situation, and section 6 to the *full* situation, as we shall see with pointwise order and lattice operations as well. Then section 7 will outline the application to the Daniell-Stone and Riesz representation theorems mentioned in the introduction.

**4. The finite situation.** It is important to start with the *open* cone $]0,\infty[^n$. The basic step is to prove the above result 2.2 of Choquet [2] under the assumption that $I$ instead of $\mathrm{C}^2$ is *continuous.*

4.1 Proposition. *Assume that* $I : ]0, \infty[^n \to \mathbb{R}$ *is positive-homogeneous and continuous. Then I fulfils the fundamental implication.*

There are two proofs. The first proof extends 2.2 via distribution theory. This has not been done before, perhaps because one had tried to extend the *result* of 2.2 via regularization, which does not work. By contrast the author [10] 2.1 followed Choquet's *proof* of 2.2, in that he took the above 0)1)2) in the distributional sense.

The second proof of 4.1 is with bare hands [10] section 2. It leads in two variants 1)2) to the two sharper results which assume $I$ instead of *continuous* to fulfil the respective two conditions

1) for each pair $u, v \in ]0, \infty[^n$ the function $t \mapsto I\big((1-t)u + tv\big)$ is continuous on $0 < t < 1$; and

2) for each pair $u, v \in ]0, \infty[^n$ the function $t \mapsto I\big((1-t)u + tv\big)$ is bounded (above/below) on some nondegenerate subinterval of $\{t \in \mathbb{R} : (1-t)u + tv > 0\}$.

The result under 2) requires the somewhat mysterious fact on convex functions which follows, obtained in [10] 2.4. The special case $\varphi =$ const is the classical result [5] theorem 111 that a midpoint convex function on an interval in $\mathbb{R}$ is convex when it is bounded above on some nondegenerate subinterval.

4.2 Theorem. *Let* $K \subset E$ *be a nonvoid convex subset of the real vector space* $E$ *and* $f : K \to \mathbb{R}$. *Assume that*

i) *there exists an affine function* $\varphi : K \to ]0, \infty[$ *such that*

$$f\Big(\frac{\sqrt{\varphi(v)}u + \sqrt{\varphi(u)}v}{\sqrt{\varphi(v)} + \sqrt{\varphi(u)}}\Big) \leqq \frac{\sqrt{\varphi(v)}f(u) + \sqrt{\varphi(u)}f(v)}{\sqrt{\varphi(v)} + \sqrt{\varphi(u)}} \quad \text{for } u, v \in K;$$

ii) *for each pair* $u, v \in K$ *the function* $t \mapsto f\big((1-t)u + tv\big)$ *is bounded above on some nondegenerate subinterval of* $\{t \in \mathbb{R} : (1-t)u + tv \in K\}$.

*Then* $f$ *is convex.*

An important specialization of 4.1 is the case that $I$ is *increasing* under $\leqq$ (also called *isotone*), because it is the unique one which will reach the full situation, but on the other hand will cover all applications known so far.

4.3 Specialization. *Assume that* $I : ]0, \infty[^n \to \mathbb{R}$ *is positive-homogeneous and increasing. Then* $I$ *is* $\geqq 0$ *and continuous. Thus* $I$ *fulfils the fundamental implication.*

Proof. 1) For $x \in ]0, \infty[^n$ we have $I(x) \leqq I(2x) = 2I(x)$ and hence $I(x) \geqq 0$. 2) If $a \in ]0, \infty[^n$ and $0 < \varepsilon < 1$ then $\{x : (1-\varepsilon)a \leqq x \leqq (1+\varepsilon)a\}$ is a neighbourhood of $a$ on which $(1-\varepsilon)I(a) = I\big((1-\varepsilon)a\big) \leqq I(x) \leqq I\big((1+\varepsilon)a\big) = (1+\varepsilon)I(a)$, that is $|I(x) - I(a)| \leqq \varepsilon I(a)$. □

So much for the open cone $]0, \infty[^n$. The four results obtained thereon, the initial 4.1 with the fortified versions based on 1)2) and the more special 4.3, can all be transferred to the positive cone $E^+ = [0, \infty[^n$ of $E = \mathbb{R}^n$. This is obvious for 4.1 and requires a little inductive proof for the other three results. However, the transferred 4.1 is kind of a dead end, because it involves an unnatural proper restriction: For $I : [0, \infty[^n \to \mathbb{R}$ positive-homogeneous the properties (sub/super)additive mean *convex/concave*, and it is well-known that these functions need not be continuous at the boundaries of their domains. An example is the function

$$I : [0, \infty[^2 \to [0, \infty[ \quad \text{with } I(x) = x_1 \text{ for } x_2 > 0 \text{ and } I(x) = 0 \text{ for } x_2 = 0,$$

which is positive-homogeneous, supermodular and superadditive, and moreover increasing. Thus we shall ignore the transferred 4.1 but restate the transferred versions of the other three results, which do not share that defect. The statement based on 2) will be our most comprehensive result in the *finite* situation, while the transferred 4.3 will be the basis for the treatment of the *full* situation in section 6. We add that the subsequent assertions can also be formulated with the value $\infty$ admitted in adequate manner. We avoid this at present, but will do it in the full situation.

4.4 THEOREM. *Assume that* $I : [0, \infty[^n \to \mathbb{R}$ *is positive-homogeneous. The further assumptions are*

1) *for each pair* $u, v \in [0, \infty[^n$ *the function* $t \mapsto I\big((1-t)u + tv\big)$ *is continuous on* $0 < t < 1$*;*

2) *for each pair* $u, v \in [0, \infty[^n$ *the function* $t \mapsto I\big((1-t)u + tv\big)$ *is bounded (above/below) on some nondegenerate subinterval of* $\{t \in \mathbb{R} : (1-t)u + tv \geqq 0\}$.

Of course 1) $\Longrightarrow$ 2). *Each of these assumptions implies that I fulfils the fundamental implication.*

4.5 THEOREM. *Assume that* $I : [0, \infty[^n \to [0, \infty[$ *is positive-homogeneous and increasing. Then I fulfils the fundamental implication.*

**5. Return to the Choquet integral.** Our treatment of the full situation will be under the strict requirement that it comprises the (sub/super)additivity theorem for the full Choquet integral, that is the implication $\Longrightarrow$ in (A) above. We see from 1.1 that in this case we have positive-homogeneous domains $S \subset [0, \infty]^X$ with $0 \in S$ which are stable under the pointwise lattice operations $\vee\wedge$, and positive-homogeneous functionals $I : S \to [0, \infty]$ with $I(0) = 0$ which are *increasing* in the pointwise order $\leqq$.

We need some further properties of the Choquet integral. We first continue the former list of terms on nonvoid function systems $S \subset [0, \infty]^X$ and functionals $I : S \to [0, \infty]$.

D3) $S$ is called *Stonean* iff $f \in S \Longrightarrow f \wedge t, (f - t)^+ \in S$ for $0 < t < \infty$; note that $f = f \wedge t + (f - t)^+$. In this case $I$ is called *Stonean* iff

$$I(f) = I(f \wedge t) + I((f - t)^+) \quad \text{for all } f \in S \text{ and } 0 < t < \infty.$$

Moreover an *increasing* $I$ is called *truncable* iff

$$I(f) = \sup\{I((f - a)^+ \wedge (b - a)) : 0 < a < b < \infty\} \quad \text{for all } f \in S.$$

We note that this relation holds true when $f$ on its $[f > 0]$ fulfils $\alpha \leqq f \leqq \beta$ for some constants $0 < \alpha < \beta < \infty$, because then $f \leqq (\alpha/\alpha - a)((f - a)^+ \wedge (b - a))$ for $0 < a < \alpha < \beta < b < \infty$. Thus to be truncable is a mild continuity condition on $I$.

D4) Assume that $0 \in S$ and that $I$ is increasing with $I(0) = 0$. Then we define the *envelopes* $I^\star, I_\star : [0, \infty]^X \to [0, \infty]$ to be

$$\begin{aligned} I^\star(f) &= \inf\{I(u) : u \in S \text{ with } u \geqq f\} \quad \text{with } \inf \varnothing := \infty, \text{ and} \\ I_\star(f) &= \sup\{I(u) : u \in S \text{ with } u \leqq f\}. \end{aligned}$$

Thus $I_\star \leqq I^\star$ and $I^\star|S = I_\star|S = I$. Moreover $I^\star$ and $I_\star$ are increasing. When $S$ is stable under $\vee\wedge$ then to be *submodular* $\vee\wedge$ carries over from $I$ to $I^\star$, and to be *supermodular* $\vee\wedge$ carries over from $I$ to $I_\star$.

One then notes the properties which follow. The subsequent representation theorem is in essence due to Greco [4]. Let $\mathfrak{S}$ be a lattice of subsets with $\varnothing \in \mathfrak{S}$ in $X$.

5.1 PROPERTIES. i) $\mathrm{UM}(\mathfrak{S})$ *and* $\mathrm{LM}(\mathfrak{S})$ *are Stonean.*

ii) *For an increasing* $\varphi : \mathfrak{S} \to [0, \infty]$ *with* $\varphi(\varnothing) = 0$ *the Choquet integral* $I : I(f) = \fint f d\varphi$ *on* $\mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ *is Stonean and truncable.*

5.2 THEOREM. *Assume that* $S \subset \mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ *is positive-homogeneous with* $0 \in S$ *and Stonean, and that* $I : S \to [0, \infty]$ *with* $I(0) = 0$ *is increasing. Then*

*there exist increasing set functions* $\varphi : \mathfrak{S} \to [0, \infty]$ *with* $\varphi(\varnothing) = 0$
*which represent* $I : I(f) = \fint f d\varphi$ *for all* $f \in S$

*iff* $I$ *is Stonean and truncable. In this case an increasing* $\varphi : \mathfrak{S} \to [0, \infty]$ *with* $\varphi(\varnothing) = 0$ *represents* $I$ *iff* $I_\star(\chi_A) \leqq \varphi(A) \leqq I^\star(\chi_A)$ *for all* $A \in \mathfrak{S}$.

**6. The full situation.** We observe that the above representation theorem 5.2 allows to formulate the (sub/super)additivity theorem for the Choquet integral *in exclusive terms of the functional* $I$ *without reference to the set function* $\varphi$.

6.1 REMARK. *On a nonvoid set* $X$ *the following are equivalent.* i) *For each lattice* $\mathfrak{S}$ *with* $\varnothing \in \mathfrak{S}$ *in* $X$ *and each increasing* $\varphi : \mathfrak{S} \to [0, \infty]$ *with* $\varphi(\varnothing) = 0$ *one has the implication* $\Longrightarrow$ *in* (A).

ii) *For each positive-homogeneous* $S \subset [0, \infty]^X$ *with* $0 \in S$ *which is stable under* $\vee\wedge$ *and Stonean, and for each positive-homogeneous* $I : S \to [0, \infty]$ *with* $I(0) = 0$ *which is increasing, Stonean and truncable, one has the fundamental implication.*

Proof. One obtains ii)$\Longrightarrow$i) as an immediate consequence of 1.1 and 5.1 combined with $\Longrightarrow$ in (M). To see i)$\Longrightarrow$ii) one applies 5.2 to $I$ and $\mathfrak{S} = \mathfrak{P}(X)$, and takes

$$\begin{aligned} \varphi : \varphi(A) &= I^\star(\chi_A) \text{ for } A \subset X \text{ when } I \text{ is submodular } \vee\wedge, \\ \varphi(A) &= I_\star(\chi_A) \text{ for } A \subset X \text{ when } I \text{ is supermodular } \vee\wedge. \end{aligned}$$

Then $\varphi : \mathfrak{P}(X) \to [0, \infty]$ represents $I$, and is (sub/super)modular in view of D4). Thus i) asserts that $f \mapsto \fint f d\varphi$ on $[0, \infty]^X$ is (sub/super)additive, and hence that $I$ is (sub/super) additive in the sense of the above definition D1). □

The new formulation 6.1.ii) of the implication $\Longrightarrow$ in (A) looks in fact like the *theorem on the fundamental implication* we are in search of. However, there are several additional conditions: Besides the almost familiar condition that $I$ be *increasing* these are the conditions that $I$ be *Stonean* and *truncable* (with the prerequisite one that $S$ be *Stonean*). The condition to be *truncable* can be dismissed as a mild continuity assumption. However, the condition that $I$ be *Stonean* is a critical one, because it expresses that $I$ be *additive* in a certain partial sense, and thus collides with the conclusion. In fact, there are situations where the prospective theorem will be invoked *in order to conclude that* $I$ *is Stonean.* A case in point will be described below. Therefore it is imperative that in a comprehensive version of the theorem like the desired one the assumption that $I$ be *Stonean* does not occur.

Now the fundamental fact is that the above reformulation 6.1.ii) holds true *without the assumption that* $I$ *be Stonean.* This is the main and final result of the present work. It is much more comprehensive than 6.1.ii).

6.2 THEOREM. *Assume that the positive-homogeneous* $S \subset [0, \infty]^X$ *with* $0 \in S$ *is stable under* $\vee\wedge$ *and Stonean, and that the positive-homogeneous* $I : S \to [0, \infty]$ *with* $I(0) = 0$ *is increasing and truncable. Then* $I$ *fulfils the fundamental implication.*

The proof starts from the result 4.5 in the finite situation (enriched with $\infty$) and proceeds via certain approximations which make essential use of the assumption that $I$ be *increasing*. An important intermediate step is the specialization $S = [0,\infty]^X$.

6.3 SPECIALIZATION. *Assume that the positive-homogeneous* $I : [0,\infty]^X \to [0,\infty]$ *with* $I(0) = 0$ *is increasing and truncable. Then* $I$ *fulfils the fundamental implication.*

The above specialization is the first result 1998 of the author in the present context [8] theorem 1.1. It is, aside from the theories of measure and integration developed in [7], the basic pillar which carries the comprehensive Daniell-Stone and Riesz type representation theorems [8] 5.3 = [9] 6.3 and [8] 5.8 = [9] 6.6. These theorems are the other important application of the new (sub/super)additivity theorem. They will be described in the final section 7 below. We note that the applications of [8] 1.1 which served to obtain these theorems were in the proof of [8] 3.10 and had in fact the aim to prove that the functionals under consideration were *Stonean*. For the details we have to refer to that paper.

We conclude the section with one more example, in order to show what can happen when the functional $I$ is not increasing.

6.4 EXAMPLE. Let $X \subset \mathbb{R}$ be an interval with $\sup X = \infty$. Define $P \subset [0,\infty[^X$ to consist of the functions $f : X \to [0,\infty[$ which are constant *near* $\infty$, that means on some upward unbounded subinterval of $X$, and $Q \subset [0,\infty[^X$ to consist of the functions $f : X \to [0,\infty[$ which are strictly decreasing near $\infty$. Then $P \cap Q = \varnothing$, and $S := P \cup Q \subset [0,\infty[^X$ is a convex cone with $0 \in S$ which is stable under $\vee\wedge$. Define $I : S \to [0,\infty[$ to be $I(f) = 0$ for $f \in P$ and $I(f) = \lim\limits_{t\to\infty} f(t)$ for $f \in Q$. Thus $I$ is positive-homogeneous with $I(0) = 0$, but of course not increasing. One verifies that $I$ is modular $\vee\wedge$. But $I$ is not additive, since for $u \in P$ with $u = c > 0$ near $\infty$ and $v \in Q$ one has $u + v \in Q$ with $I(u+v) = c + I(v) > I(v) = I(u) + I(v)$. We note that $I$ has certain continuity properties. Thus $S$ is Stonean, and $I$ is *truncable* in the sense that

$$I\big((f-a)^+ \wedge (b-a)\big) \uparrow I(f) \text{ under } a \downarrow 0 \text{ and } b \uparrow \infty \quad \text{for all } f \in S.$$

Also for each pair $u, v \in S$ the function $t \mapsto I\big((1-t)u + tv\big)$ is continuous on $0 < t < 1$ (but need not be continuous on $0 \leqq t \leqq 1$).

**7. The Daniell-Stone-Riesz representation theorems.** The representation theorems of the present section are quite different from the former representation theorem 5.2: The aim is to represent particular classes of functionals in terms of certain classes of distinguished set functions, like in the classical Riesz representation theorem, but in a much more extended frame. The basis are the extension theories in measure and integration developed in the author's textbook [7] and in subsequent articles like [8], and summarized in [9]. We recall that there are parallel inner and outer extension theories, and also parallel sequential and nonsequential versions (as usual labelled as $\sigma$ and $\tau$ versions). We also recall the most basic notions: For an increasing set function $\varphi : \mathfrak{S} \to [0,\infty]$ on a set system $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ and for $\bullet = \sigma\tau$ one forms the *envelopes*

$$\varphi_\bullet, \varphi^\bullet : \mathfrak{P}(X) \to [0,\infty] \text{ with the } \textit{satellites } \varphi^B_\bullet : \mathfrak{P}(X) \to [0,\infty[ \text{ with } B \in \mathfrak{S},$$

and on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ one defines the *inner* and *outer* $\bullet$ *premeasures* $\varphi$. Likewise for an increasing functional $I : S \to [0,\infty]$ on a function class $S \subset [0,\infty]^X$ with $0 \in S$ and $I(0) = 0$ and for $\bullet = \sigma\tau$ one forms the *envelopes*

$$I_\bullet, I^\bullet : [0,\infty]^X \to [0,\infty] \text{ with the } \textit{satellites } I_\bullet^v : [0,\infty]^X \to [0,\infty] \text{ with } v \in S;$$

the notions of *inner* and *outer* $\bullet$ *preintegrals* $I$ will appear below.

For the sequel we assume a positive-homogeneous function class

$S \subset [0,\infty[^X$ in the inner situation,
$S \subset [0,\infty]^X$ in the outer situation,

with $0 \in S$ which is stable under $\vee\wedge$ and *Stonean*, and a functional

$I : S \to [0,\infty[$ in the inner situation,
$I : S \to [0,\infty]$ in the outer situation,

with $I(0) = 0$ which is increasing. These are the assumptions made in [8] [9], while those in [7] were much narrower. We form the set systems

$\mathrm{um}(S) = \{[f \geqq t] : f \in S \text{ and } 0 < t < \infty\}$ for the inner situation,
$\mathrm{lm}(S) = \{[f > t] : f \in S \text{ and } 0 < t < \infty\}$ for the outer situation,

which are lattices with $\varnothing$. Then we define

the *inner sources* of $I$ to be those increasing set functions $\varphi : \mathrm{um}(S) \to [0,\infty[$,
the *outer sources* of $I$ to be those increasing set functions $\varphi : \mathrm{lm}(S) \to [0,\infty]$,

which have $\varphi(\varnothing) = 0$ and which represent $I : I(f) = ⨍ f d\varphi$ for all $f \in S$. The representation theorem 5.2 tells us that such inner/outer sources of $I$ exist iff $I$ is *Stonean* and *truncable*. In this case their characterization is $I_\star(\chi_A) \leqq \varphi(A) \leqq I^\star(\chi_A)$ for all $A \in \mathrm{um}(S)/\mathrm{lm}(S)$, so that as a rule one must expect a lot of inner and outer sources of $I$.

After this we define for $\bullet = \sigma\tau$ the functional $I$ to be an *inner/outer* $\bullet$ *preintegral* iff it admits at least one inner/outer source which is an inner/outer $\bullet$ premeasure. Then the fundamental results quoted above are the theorems on the inner and outer $\bullet$ preintegrals which follow.

7.1 Inner Theorem ($\bullet = \sigma\tau$). *The functional $I$ is an inner $\bullet$ preintegral iff*

1) *$I$ is supermodular and Stonean and downward $\bullet$ continuous at $\varnothing$,*
2) *$I(v) \leqq I(u) + I_\bullet^v(v-u)$ for all $u \leqq v$ in $S$.*

*In this case $\varphi := I^\star(\chi_\cdot)|\mathrm{um}(S)$ is the unique inner source of $I$ which is an inner $\bullet$ premeasure. It fulfils $I_\bullet(f) = ⨍ f d\varphi_\bullet$ for all $f \in [0,\infty]^X$.*

7.2 Outer Theorem ($\bullet = \sigma\tau$). *The functional $I$ is an outer $\bullet$ preintegral iff*

1) *$I$ is submodular and Stonean and upward $\bullet$ continuous,*
2) *$I(v) \geqq I(u) + I^\bullet(v-u)$ for all $u \leqq v$ in $S$ with $u < \infty$,*
3) *moreover for $\bullet = \tau$* (while this is automatic for $\bullet = \sigma$)

$$I^\bullet(f) = \sup\{I^\bullet(f \wedge u) : u \in [I < \infty]\} \quad \textit{for all } f \in [I^\bullet < \infty].$$

*In this case $\varphi := I_\star(\chi_\cdot)|\mathrm{lm}(S)$ is the unique outer source of $I$ which is an outer $\bullet$ premeasure. It fulfils $I^\bullet(f) = ⨍ f d\varphi^\bullet$ for all $f \in [0,\infty]^X$.*

We refer to the cited papers for the collection of more or less familiar special cases. Thus the classical Riesz representation theorem and its extension to arbitrary Hausdorff topological spaces are immediate consequences of the inner $\tau$ theorem, whereas the conventional Daniell-Stone theorem falls under the outer $\sigma$ theorem.

## References


[1] R.C.Bassanezi and G.H.Greco, Sull'additività dell'integrale. Rend.Sem.Mat.Univ. Padova 72(1984), 249-275.

[2] G.Choquet, Theory of Capacities. Ann.Inst.Fourier (Grenoble) 5(1953/54), 131-295.

[3] D.Denneberg, Non-additive Measure and Integration. Kluwer 1994.

[4] G.H.Greco, Sulla rappresentazione di funzionali mediante integrali. Rend.Sem.Mat.Univ. Padova 66(1982), 21-42.

[5] G.H.Hardy, J.E.Littlewood and G.Polya, Inequalities. Cambridge Univ.Press 2nd ed. 1952.

[6] J.Kindler, A Mazur-Orlicz theorem for submodular set functions. Journ.Math.Analysis Appl. 120(1986), 533-564.

[7] H.König, Measure and Integration: An Advanced Course in Basic Procedures and Applications. Springer 1997.

[8] H.König, Measure and Integration: Integral representations of isotone functionals. Annales Univ. Saraviensis Ser.Math. 9(1998), 123-153.

[9] H.König, Measure and Integration: An attempt at unified systematization. Rend.Istit.Mat. Univ.Trieste, to appear. Preprint under http://www.math.uni-sb.de/EX/koenig/.

[10] H.König, The (sub/super)additivity assertion of Choquet. Preprint 2002.

[11] H.Lebesgue, Sur une généralization de l'intégrale définie. C.R.Acad.Sci.Paris 132(1901), 1025-1027 and 332(2001), 85-90.

[12] D.Schmeidler, Integral representation without additivity. Proc.Amer.Math.Soc. 97(1986), 255-261.

[13] F.Topsøe, On construction of measures. In: Proc.Conf.Topology and Measure I, Zinnowitz 1974. Ernst-Moritz-Arndt Univ. Greifswald 1978, Part 2, pp.343-381.



Universität des Saarlandes, Fakultät für Mathematik und Informatik, D-66041 Saarbrücken, Germany

*E-mail address*: hkoenig@math.uni-sb.de