Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 190

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Saarbrücken 2007

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 190
submitted: February 16, 2007

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Michael Didas

Let $\Omega \subset \mathbb{C}^n$ be a bounded convex or strictly pseudoconvex open subset. Given a separable Hilbert space K and a weak* closed subspace $\mathcal{T} \subset B(K)$, we show that the space $H^\infty(\Omega, \mathcal{T})$ of all bounded holomorphic $\mathcal{T}$-valued functions on Ω possesses the tensor product representation $H^\infty(\Omega, \mathcal{T}) = H^\infty(\Omega)\overline{\otimes}\mathcal{T}$ with respect to the normal spatial tensor product. As a consequence we deduce that $H^\infty(\Omega)$ has property S_σ. This implies that, if $S \in B(H)^n$ is a subnormal tuple of class $\mathbb{A}$ on a strictly pseudoconvex or bounded symmetric domain and $T \in B(K)^m$ is a commuting tuple satisfying $\operatorname{AlgLat}(T) = \mathcal{A}_T$ (where $\mathcal{A}_T$ denotes the unital dual operator algebra generated by T), then the tensor product tuple $(S \otimes 1, 1 \otimes T)$ is reflexive.

1 Property S_σ and the reflexivity of tensor products

Given a complex Hilbert space H and an arbitrary family $\mathcal{S} \subset B(H)$ of bounded linear operators, we define $\mathcal{W}_\mathcal{S}$ to be the smallest WOT-closed subalgebra of $B(H)$ containing $\mathcal{S}$ and the identity 1_H. As usual, we write $\operatorname{Lat}(\mathcal{S})$ for the set of all closed subspaces of H that are invariant under each member of $\mathcal{S}$ and we define $\operatorname{AlgLat}(\mathcal{S})$ to be the set of all operators $C \in B(H)$ with $\operatorname{Lat}(C) \supset \operatorname{Lat}(\mathcal{S})$. Obviously $\operatorname{AlgLat}(\mathcal{S})$ is a WOT-closed unital subalgebra of $B(H)$ containing $\mathcal{S}$ (and hence $\mathcal{W}_\mathcal{S}$). The family $\mathcal{S}$ is called *reflexive* if the identity

$$\operatorname{AlgLat}(\mathcal{S}) = \mathcal{W}_\mathcal{S}$$

holds. For many concrete examples of reflexive systems $\mathcal{S}$, the algebra $\mathcal{W}_\mathcal{S}$ coincides with the unital dual operator algebra

$$\mathcal{A}_\mathcal{S} = \overline{\operatorname{alg}(\mathcal{S} \cup \{1_H\})}^{w^*} \subset B(H)$$

generated by $\mathcal{S}$, e.g. if $\mathcal{S} = \{S\}$ consists of a single von Neumann operator $S \in B(H)$, and hence in particular if S is subnormal (Conway and Dudziak [1], Corollary 3.2), or if $\mathcal{S} = (S_1, \ldots, S_n) \in B(H)^n$ is a von Neumann n-tuple of class $\mathbb{A} \cap \mathbb{A}_{1,\aleph_0}$ on a strictly pseudoconvex domain (see [2], Corollary 4.4.4). In what follows, such a family $\mathcal{S} \subset B(H)$ which is reflexive and satisfies the identity $\mathcal{A}_\mathcal{S} = \mathcal{W}_\mathcal{S}$ will be called *strongly reflexive*, for short. Observe that a family $\mathcal{S}$ of operators is strongly reflexive if and only if the equality

AlgLat$(\mathcal{S}) = \mathcal{A}_\mathcal{S}$ holds or, equivalently, if AlgLat$(\mathcal{A}_\mathcal{S}) = \mathcal{A}_\mathcal{S}$.[1] Clearly, the family $\mathcal{S}$ is strongly reflexive if and only if so is the dual operator algebra $\mathcal{A}_\mathcal{S}$.

If two commuting Hilbert-space multi-operators $S \in B(H)^n$ and $T \in B(K)^m$ are (strongly) reflexive, then it is natural to ask for the (strong) reflexivity of the tensor product tuple

$$(S \otimes 1, 1 \otimes T) \in B(H \otimes K)^{n+m}.$$

We will focus on the strong-reflexivity version of this question, namely if $\mathcal{A}_{(S\otimes 1, 1\otimes T)}$ is strongly reflexive whenever so are $\mathcal{A}_S$ and $\mathcal{A}_T$. Answering this question turns out to be equivalent to solving a reflexivity problem for tensor products of dual algebras. To point this out, we have to recall that the normal spatial tensor product of two arbitrary weak* closed subspaces $\mathcal{S} \subset B(H)$ and $\mathcal{T} \subset B(K)$ is defined by

$$\mathcal{S}\overline{\otimes}\mathcal{T} = \overline{\mathcal{S} \otimes \mathcal{T}}^{w^*} \subset B(H \otimes K),$$

where $\mathcal{S} \otimes \mathcal{T} = LH\{x \otimes y : x \in \mathcal{S}, y \in \mathcal{T}\}$ stands for the algebraic tensor product of $\mathcal{S}$ and $\mathcal{T}$. It is a simple observation that the dual algebra generated by any tensor product tuple splits with respect to the normal spatial tensor product.

1.1 Lemma. *For arbitrary commuting tuples $S \in B(H)^n$ and $T \in B(K)^m$, we have $\mathcal{A}_{(S\otimes 1, 1\otimes T)} = \mathcal{A}_S \overline{\otimes} \mathcal{A}_T$.*

Proof. Since the set on the right-hand side is a unital dual operator algebra containing $S \otimes 1$ and $1 \otimes T$, the inclusion "$\subset$" follows by the minimality of the algebra on the left. To prove the non-trivial inclusion "$\supset$" first note that all elementary tensors $A \otimes B$ with $A \in \mathbb{C}[S]$ and $B \in \mathbb{C}[T]$ are clearly contained in the set on the left-hand side. The weak* continuity of the mapping $B(H) \to B(H \otimes K)$, $A \mapsto A \otimes B$, for each fixed $B \in B(K)$ therefore implies that the set of all elementary tensors $A \otimes B$ with $A \in \mathcal{A}_S$ and $B \in \mathbb{C}[T]$ is contained in $\mathcal{A}_{(S\otimes 1, 1\otimes T)}$. By the same argument with the roles of the first and second factor exchanged we may also replace the condition $B \in \mathbb{C}[T]$ by $B \in \mathcal{A}_T$. Passing to the linear hull, we deduce that the algebraic tensor product $\mathcal{A}_S \otimes \mathcal{A}_T$ is contained in $\mathcal{A}_{(S\otimes 1, 1\otimes T)}$. This observation finishes the proof. □

The reflexivity problem for tensor products of dual operator algebras has been studied intensively in a series of papers by Jon Kraus (see e.g. [6] and [7]).

[1] It should be remarked that the notion of reflexivity is not uniformly defined in the literature. Operator algebraists often use the identity AlgLat$(\mathcal{A}_\mathcal{S}) = \mathcal{A}_\mathcal{S}$ as the definition for the reflexivity of the operator algebra $\mathcal{A}_\mathcal{S}$ (see e.g. [7]) while our definition, which is commonly used in operator theory, says that the family $\mathcal{A}_\mathcal{S}$ is reflexive if the weaker condition AlgLat$(\mathcal{A}_\mathcal{S}) = \mathcal{W}_{\mathcal{A}_\mathcal{S}} = \mathcal{W}_\mathcal{S}$ holds (being equivalent to the reflexivity of the family $\mathcal{S}$ itself). Therefore we introduced the new notion of strong reflexivity hoping to avoid a misunderstanding.

Kraus showed that the tensor product of two strongly reflexive dual operator algebras remains strongly reflexive if one of the factors satisfies a certain (natural) splitting property (called S_σ).

Towards a precise formulation of property S_σ we have to recall the definition of Tomiyama's slice maps. Let $C^1(H)$ denote the space of all trace class operators on the Hilbert space H. Recall that we may identify $B(H)$ with the dual space of $C^1(H)$ via the bilinear form $C^1(H) \times B(H) \to \mathbb{C}$, $(C, A) \mapsto \text{trace}(CA)$. The right slice map R_C associated with a given element $C \in C^1(H)$ now can be defined as the adjoint of the continuous linear map

$$(R_C)_* : C^1(K) \longrightarrow C^1(H \otimes K), \qquad D \mapsto C \otimes D.$$

The mapping R_C obtained in this way is the unique weak* continuous linear operator

$$R_C : B(H \otimes K) \to B(K) \quad \text{satisfying} \quad R_C(A \otimes B) = \langle C, A \rangle B$$

for every $A \in B(H)$ and $B \in B(K)$, where $\langle C, A \rangle = \text{trace}(CA)$. Given $D \in C^1(K)$, the assignment $L_D(A \otimes B) = \langle D, B \rangle A$, where $A \otimes B \in B(H \otimes K)$, can in a completely analoguous manner be extended to a weak* continuous linear map $L_D : B(H \otimes K) \to B(H)$, called the left slice map induced by D. For further properties of slice maps, see Kraus [6] and the references therein. Let us, for later use, just mention the intertwining property explicitly, which can be easily verified by the reader and – in the context of right slice maps – says that

$$R_C((1 \otimes V)X(1 \otimes W)) = V R_C(X) W,$$

whenever $V, W \in B(K)$ and $X \in B(H \otimes K)$. Analogously, for the left slice maps we have $L_D((V \otimes 1)X(W \otimes 1)) = V L_D(X) W$ for $V, W \in B(H)$ and $X \in B(H \otimes K)$ (see the formulas (1.3) and (1.4) in Kraus [6]).

In order to define property S_σ, we associate with each pair of weak* closed subspaces $\mathcal{S} \subset B(H)$ and $\mathcal{T} \subset B(K)$ the so-called Fubini product

$$F(\mathcal{S}, \mathcal{T}) = \left\{ A \in B(H \otimes K) \;\middle|\; \begin{array}{c} R_C(A) \in \mathcal{T} \quad \text{and} \quad L_D(A) \in \mathcal{S} \\ \text{whenever } C \in C^1(H) \text{ and } D \in C^1(K) \end{array} \right\}$$

which is easily seen to be a weak* closed subspace of $B(H \otimes K)$ containing $\mathcal{S} \overline{\otimes} \mathcal{T}$. Now following Kraus [7] we say that a weak* closed subspace $\mathcal{S} \subset B(H)$ satisfies property S_σ if the subspace tensor product formula

$$F(\mathcal{S}, \mathcal{T}) = \mathcal{S} \overline{\otimes} \mathcal{T}$$

holds whenever $\mathcal{T} \subset B(K)$ is a weak* closed subspace of $B(K)$ for any Hilbert space K. As shown by Kraus in [6], it suffices to consider the case where K is separable and infinite dimensional.

For later reference we remark that the Fubini product can be expressed using right slice maps only. Theorem 1.9 in [6] guarantees that $B(K)$ has property

S_σ and, consequently, we have $F(\mathcal{S},\mathcal{T}) \subset F(\mathcal{S}, B(K)) = \mathcal{S}\overline{\otimes}B(K)$. Using this and the fact that $L_D(\mathcal{S}\overline{\otimes}B(K)) \subset \mathcal{S}$ we obtain the desired representation

$$F(\mathcal{S},\mathcal{T}) = \{A \in \mathcal{S}\overline{\otimes}B(K) : R_C(A) \in \mathcal{T} \text{ for all } C \in C^1(H)\}.$$

Let us now turn back to the reflexivity problem for tensor product tuples. In Section 3 of [6], Kraus settles a link between property S_σ and the reflexivity of tensor products which, in our context, reads as follows.

1.2 Proposition. (Kraus) *Let $S \in B(H)^n$ and $T \in B(K)^m$ be commuting tuples of bounded linear Hilbert-space operators which are strongly reflexive in the sense that $AlgLat(S) = \mathcal{A}_S$ and $AlgLat(T) = \mathcal{A}_T$. If $\mathcal{A}_S$ has property S_σ, then the tensor product tuple $(S \otimes 1, 1 \otimes T) \in B(H \otimes K)^{n+m}$ satisfies*

$$AlgLat(S \otimes 1, 1 \otimes T) = F(\mathcal{A}_S, \mathcal{A}_T) = \mathcal{A}_{(S\otimes 1, 1\otimes T)}.$$

Proof. Let $M \in \mathrm{Lat}(T)$ and let $P \in B(K)$ denote the orthogonal projection with range M. Since $H \otimes M$ is $(S \otimes 1, 1 \otimes T)$-invariant, an operator $A \in \mathrm{AlgLat}(S \otimes 1, 1 \otimes T)$ clearly satisfies $(1 \otimes P)A(1 \otimes P) = A(1 \otimes P)$. Using the intertwining property of the right slice-map R_C, we deduce that

$$PR_C(A)P = R_C((1 \otimes P)A(1 \otimes P)) = R_C(A(1 \otimes P)) = R_C(A)P$$

and hence $R_C(A) \in \mathrm{AlgLat}(T)$, for every $C \in C^1(H)$. In a completely analogous fashion it can be shown that $L_D(A) \in \mathrm{AlgLat}(S)$ $(D \in C^1(K))$. Now, a look at the definition of the Fubini product immediately yields the inclusion

$$\mathrm{AlgLat}(S \otimes 1, 1 \otimes T) \subset F(\mathrm{AlgLat}(S), \mathrm{AlgLat}(T)),$$

where, by hypothesis, the right-hand side can be written as $F(\mathcal{A}_S, \mathcal{A}_T)$. Using property S_σ we further obtain that $F(\mathcal{A}_S, \mathcal{A}_T) = \mathcal{A}_S\overline{\otimes}\mathcal{A}_T$. By Lemma 1.1, the latter space coincides with $\mathcal{A}_{(S\otimes 1, 1\otimes T)}$, as desired. □

Due to Kraus [7], Theorem 4.1, we know that the dual operator algebra $\mathcal{A}_S$ generated by a single subnormal operator $S \in B(H)$ has property S_σ. By a classical theorem of Olin and Thomson, $\mathcal{A}_S$ is strongly reflexive in this case. Hence Proposition 1.2 applies to every subnormal operator S. In the special case that S is the unilateral shift, i.e. $S = M_z$ on the Hardy space $H^2(\mathbb{D})$ over the unit disc, short proofs of the above proposition using elementary arguments have been given by M. Ptak ([9], Theorem 2') and J.E. McCarthy ([8], Lemma 6).

Our aim is to extend Kraus' result to the setting of subnormal tuples $S \in B(H)^n$ of class $\mathbb{A}$ on sufficiently nice sets Ω for which $\mathcal{A}_S \cong H^\infty(\Omega)$. The dual algebra generated by a tensor product tuple of the form $(S \otimes 1, 1 \otimes T)$ then corresponds to some space of vector-valued H^∞-functions. The next section is therefore devoted to this kind of function spaces.

2 A tensor product formula for $H^\infty(\Omega, \mathcal{T})$

From now on suppose that $\emptyset \neq \Omega \subset X$ is either a bounded convex open subset of $X = \mathbb{C}^n$ or a relatively compact strictly pseudoconvex open subset of a Stein submanifold $X \subset \mathbb{C}^n$. By the latter we mean that there exist an open subset $U \subset X$ containing the boundary $\partial\Omega$ and a strictly plurisubharmonic C^2-function $\rho : U \to \mathbb{R}$ such that $\Omega \cap U = \{z \in U : \rho(z) < 0\}$. To abbreviate the description of these two cases, let us simply say in the following that Ω is a "bounded convex or strictly pseudoconvex open set".

Let us fix such a set $\Omega \subset X$ now. As a relatively compact submanifold of $\mathbb{C}^n$, the set Ω carries a natural volume measure. After normalization and trivial extension we obtain a Borel probability measure λ on $\overline{\Omega}$ with $\lambda(\Omega) = 1$, $\lambda(\partial\Omega) = 0$ and the property that $\lambda(W) > 0$ for every non-empty open set $W \subset \Omega$. By $H^\infty(\Omega)$ we denote the Banach algebra of all bounded holomorphic functions on Ω equipped with the supremum norm $\|f\|_{\infty,\Omega} = \sup_{z\in\Omega} |f(z)|$. As a consequence of Montel's theorem, the isometric embedding $H^\infty(\Omega) \hookrightarrow L^\infty(\lambda)$ has weak* closed range and thus turns $H^\infty(\Omega)$ into a dual algebra. Via the representation

$$\gamma : L^\infty(\lambda) \to B(L^2(\lambda)), \qquad \varphi \mapsto M_\varphi \qquad \text{with } M_\varphi f = \varphi \cdot f \quad (f \in L^2(\lambda)),$$

which is a weak* continuous isometric $*$-homomorphism, we may identify $H^\infty(\Omega)$ with the dual operator algebra

$$\mathscr{H}^\infty(\Omega) = \gamma(H^\infty(\Omega)) = \{M_\varphi : \varphi \in H^\infty(\Omega))\} \subset B(L^2(\lambda))$$

of all multiplication operators with H^∞-symbol.

Towards the vector-valud case, fix a separable Banach space E and consider the Banach space $L^\infty(\lambda, E')$ of all equivalence classes of bounded weak*-measurable functions $f : \Omega \to E'$ equipped with the essential supremum norm. Via the bilinear form $\langle g, f\rangle = \int_\Omega \langle g, f\rangle d\lambda$, we can identify $L^\infty(\lambda, E')$ with the dual of the space $L^1(\lambda, E)$ of all equivalence classes of Bochner-integrable functions $g : \Omega \to E$ with $\|g\|_{1,\lambda} = \int_\Omega \|g\| d\lambda < \infty$. In analogy with the $\mathbb{C}$-valued case, the Banach space of all E'-valued bounded holomorphic functions $H^\infty(\Omega, E')$ with the supremum norm can – via the canonical embedding – be thought of as a weak*-closed subspace of $L^\infty(\lambda, E')$ (for details, see [3], Lemma 5.3). It should be mentioned that a sequence (f_k) in $H^\infty(\Omega, E')$ is a weak* zero sequence if and only if (f_k) is norm-bounded and $(f_k(z))$ is a weak* zero sequence in E' for every $z \in \Omega$.

If K is a separable Hilbert space and $\mathcal{T} \subset B(K)$ is a weak* closed subspace, then $H^\infty(\Omega, \mathcal{T})$ and $L^\infty(\lambda, \mathcal{T})$ fit into the above context, since $\mathcal{T}$ can then be identified with the dual space of the separable Banach space $E = C^1(K)/{}^\perp\mathcal{T}$. If, in addition, $\mathcal{T}$ is a subalgebra of $B(K)$, then $H^\infty(\Omega, \mathcal{T})$ and $L^\infty(\lambda, \mathcal{T})$ are dual algebras in a canonical way. If $\mathcal{T} \subset B(K)$ is even a W^*-algebra, then so is $L^\infty(\lambda, \mathcal{T})$.

Again in analogy with the scalar-valued case we obtain a representation of the W^*-algebra $L^\infty(\lambda, B(K))$ via the weak* continuous and isometric $*$-homomorphism

$$\Gamma : L^\infty(\lambda, B(K)) \to B(L^2(\lambda) \otimes K)), \quad \varphi \mapsto M_\varphi,$$

where the operator M_φ acts on the space $L^2(\lambda, K) \cong L^2(\lambda) \otimes K$ as multiplication with symbol φ.

2.1 Proposition. *Let Ω be a bounded convex or strictly pseudoconvex open set, and let $\mathcal{T} \subset B(K)$ be a weak* closed subspace. Then there is a unique dual algebra isomorphism*

$$\Gamma_\mathcal{T} : H^\infty(\Omega, \mathcal{T}) \longrightarrow \mathscr{H}^\infty(\Omega)\overline{\otimes}\mathcal{T} \subset B(L^2(\lambda) \otimes K)$$

mapping $\varphi \cdot T$ to $M_\varphi \otimes T$ whenever $\varphi \in H^\infty(\Omega)$ and $T \in \mathcal{T}$. In fact, $\Gamma_\mathcal{T}$ can be obtained by restricting the map Γ from above to $H^\infty(\Omega, \mathcal{T})$.

Towards a proof of this result, consider the set

$$M = LH\{\varphi \cdot T : \varphi \in H^\infty(\Omega),\, T \in \mathcal{T}\}$$

of elementary functions. From the properties of the map Γ described above and the trivial fact that $\Gamma(M) \subset \mathscr{H}^\infty(\Omega) \otimes \mathcal{T}$ we deduce that the assertion follows as soon as we know that M is dense in $H^\infty(\Omega, \mathcal{T})$.

To realize this claim we first derive an intermediate result which is interesting in its own right. In the following proposition $\mathscr{O}(\overline{\Omega}, E')$ stands for the space of all E'-valued functions that are holomorphic in some open neighbourhood of $\overline{\Omega}$ in $\mathbb{C}^n$.

2.2 Proposition. *Suppose that Ω is a convex or strictly pseudoconvex open set and that E is a separable complex Banach space. Then*

$$\mathscr{O}(\overline{\Omega}, E')|\Omega \subset H^\infty(\Omega, E')$$

is sequentially weak dense. More precisely, there is a constant $c \geq 1$, such that every function f in the unit ball of $H^\infty(\Omega, E')$ can be approximated (with respect to the weak* topology) by a sequence (f_k) with $f_k \in \mathscr{O}(\overline{\Omega}, E')|\Omega$ and $\|f_k\|_{\infty,\Omega} \leq c$ $(k \geq 1)$.*

Proof. Elementary arguments show that the assertion holds in the convex case. (Translate Ω in such a way that it contains the origin and use radial limits.) To treat the strictly pseudoconvex case we use the embedding theorem of Fornaess saying that, up to a biholomorphic identification, the set Ω can be represented as the intersection $\Omega \cong Y \cap C$ of some closed complex submanifold $Y \subset \mathbb{C}^m$ and a C^2-strictly convex open subset $C \subset \mathbb{C}^m$ for some suitably chosen $m \geq 1$ (see Theorem 10 in [4]).

Theorem 5.11 in [3] says that our assertion holds in the special case where $E' = H^\infty(\Omega_2)$. Following the proof of the cited theorem (setting there $D_1 = \Omega$ and replacing E by E'), we deduce that it suffices to show that the restriction map

$$H^\infty(C, E') \longrightarrow H^\infty(Y \cap C, E')$$

is onto. Towards this end, note that the mapping

$$B : E \times H^\infty(Y \cap C, E') \to H^\infty(Y \cap C), \qquad (x, f) \mapsto \langle x, f(\cdot)\rangle$$

is (norm-) continuous and bilinear.

By the remark following Theorem 4.11.1 in Henkin-Leiterer [5], there is a bounded linear extension operator

$$\theta : H^\infty(Y \cap C) \to H^\infty(C).$$

In order to lift this operator to the E'-valued setting, we start with an arbitrary function $f \in H^\infty(Y \cap C, E')$. By defining

$$\hat{f}(z) : E \to \mathbb{C}, \qquad x \mapsto \mathcal{E}_z \theta B(x, f) \qquad (\text{for every } z \in C),$$

where $\mathcal{E}_z : H^\infty(\Omega, E') \to E'$ denotes the (weak* continuous) point evaluation at z, we obtain a family of vectors $\hat{f}(z) \in E'$ ($z \in C$) satisfying

$$\langle x, \hat{f}(z)\rangle = \theta(\langle x, f(\cdot)\rangle)(z) \qquad (x \in E,\ z \in C).$$

The function $\hat{f} : C \to E'$ constructed this way clearly extends f and is weak* holomorphic and hence holomorphic. Since the estimate

$$\|\hat{f}(z)\| \le \|\mathcal{E}_z\|\|\theta\| \sup_{\|x\|\le 1} \|B(x, f)\|_{\infty,\Omega} \le \|\theta\|\|f\|_{\infty,\Omega} \qquad (z \in C)$$

holds, the assignment

$$\hat{\theta} : H^\infty(Y \cap C, E') \to H^\infty(C, E'), \qquad f \mapsto \hat{f}$$

yields a bounded linear extension operator in the vector-valued case. In particular, the corresponding restriction $H^\infty(C, E') \to H^\infty(Y \cap C, E')$ is onto, as desired. Hence the assertion of the proposition holds with approximation constant $c = \|\theta\|$. □

Now we are able to finish the proof of Proposition 2.1. We use the notation $\mathscr{O}(W)$ ($\mathscr{O}(W, \mathcal{T})$, resp.) to denote the set of all $\mathbb{C}$-valued ($\mathcal{T}$-valued, resp.) holomorphic functions on an open set $W \subset X$.

Proof of Proposition 2.1. As pointed out above, it remains to check that the set $M = LH\{\varphi \cdot T : \varphi \in H^\infty(\Omega),\ T \in \mathcal{T}\}$ is weak* dense in $H^\infty(\Omega, \mathcal{T})$. Towards this end, fix an arbitrary function $f \in H^\infty(\Omega, \mathcal{T})$. Then, by the preceding proposition, there is a sequence (f_k) in $\mathscr{O}(\overline{\Omega}, \mathcal{T})|\Omega$ such that $f_k \xrightarrow{k} f$

pointwise weak* on Ω and $\sup_k \|f_k\|_{\infty,\Omega} \leq c\|f\|_{\infty,\Omega}$. For each $k \geq 1$ we may choose an open neighborhood U_k of $\overline{\Omega}$ in such a way that f_k can be extended to a function in $\mathscr{O}(U_k, \mathcal{T})$ again denoted by f_k. In view of the well-known identification $\mathscr{O}(U_k, \mathcal{T}) \cong \mathscr{O}(U_k)\widehat{\otimes}\mathcal{T}$, there are elementary functions

$$g_k = \sum_{i=1}^{r_k} h_i^{(k)} \otimes A_i^{(k)} \in M \qquad \text{with } h_i^{(k)} \in H^\infty(\Omega),\ A_i^{(k)} \in \mathcal{T} \quad (k \geq 1)$$

satisfying $\|g_k - f_k\|_{\infty,\Omega} < 1/k$. The sequence $(g_k)_k$ is norm-bounded and converges to f pointwise weak*. Therefore (g_k) is the desired sequence in M approximating f in the weak* topology of $H^\infty(\Omega, \mathcal{T})$. □

3 Property S_σ for $\mathscr{H}^\infty(\Omega)$ and applications

In the special case that Ω is the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, Kraus remarked in [7] (Example 3.3, p.399) that $H^\infty(\mathbb{D})$ has property S_σ. Since Kraus' original proof involves W^*-dynamical systems and makes use of the group structure of $\partial\mathbb{D} = \mathbb{T}$, it cannot be extended to our situation. Via the tensor product formula established in Proposition 2.1, we show directly and with elementary arguments that $\mathscr{H}^\infty(\Omega)$ satisfies property S_σ.

3.1 Theorem. *For every bounded convex or strictly pseudoconvex open set $\Omega \subset X$, the dual operator algebra $\mathscr{H}^\infty(\Omega)$ has property S_σ.*

Proof. Fix an arbitrary separable complex Hilbert space K and a weak* closed subspace $\mathcal{T} \subset B(K)$. We have to show that

$$F(\mathscr{H}^\infty(\Omega), \mathcal{T}) \subset \mathscr{H}^\infty(\Omega)\overline{\otimes}\mathcal{T}.$$

Towards this end, we start with an arbitrary element $A \in F(\mathscr{H}^\infty(\Omega), \mathcal{T})$, which means by definition that

$$A \in \mathscr{H}^\infty(\Omega)\overline{\otimes}B(K) \quad \text{and} \quad R_C(A) \in \mathcal{T} \quad \text{for all} \quad C \in C^1(L^2(\lambda)).$$

Proposition 2.1 says that, using the canonical weak* continuous isometry

$$\Gamma : L^\infty(\lambda, B(K)) \to B(L^2(\lambda) \otimes K),$$

the operator A can be written as $A = \Gamma(f_A)$ with some bounded holomorphic function $f_A \in H^\infty(\Omega, B(K))$. Suppose for a moment that f_A takes its values in $\mathcal{T}$ only. Then we could again use Proposition 2.1 to finish the proof with the observation that

$$\Gamma(f_A) \in \Gamma(H^\infty(\Omega, \mathcal{T})) \subset \mathscr{H}^\infty(\Omega)\overline{\otimes}\mathcal{T}.$$

Therefore, our aim is to show that $f_A(z) \in \mathcal{T}$ for all $z \in \Omega$.

In view of the dual algebra isomorphism $\gamma : H^\infty(\Omega) \to \mathscr{H}^\infty(\Omega)$, $\varphi \mapsto M_\varphi$, each $g \in L^1(\lambda)$ induces a weak* continuous linear form

$$\mathscr{H}^\infty(\Omega) \to \mathbb{C}, \quad M_\varphi \mapsto \langle g, \varphi \rangle = \int_\Omega g\varphi d\lambda,$$

which, by the Hahn-Banach theorem, can be extended from $\mathscr{H}^\infty(\Omega)$ to a weak* continuous linear form on all of $B(L^2(\lambda))$. Hence via trace-duality we find an operator $C_g \in C^1(L^2(\lambda))$ satisfying

$$\langle C_g, M_\varphi \rangle = \int_\Omega g\varphi d\lambda \qquad (\varphi \in H^\infty(\Omega)).$$

From the very definition of the right slice map associated with C_g we deduce that, for every $D \in C^1(K)$, every $\varphi \in H^\infty(\Omega)$ and every $T \in B(K)$, the identity

$$\langle D, R_{C_g}(\Gamma(\varphi T)) \rangle = \langle D, \langle C_g, M_\varphi \rangle T \rangle = \langle D, \left(\int_\Omega g\varphi d\lambda \right) T \rangle = \int_\Omega g \langle D, \varphi T \rangle d\lambda$$

holds. Since, according to Proposition 2.1, the linear span of $H^\infty(\Omega) \cdot B(K)$ is weak* dense in $H^\infty(\Omega, B(K))$, this implies that

$$\langle D, R_{C_g}(\Gamma(f_A)) \rangle = \int_\Omega g \langle D, f_A(\cdot) \rangle d\lambda \qquad (D \in C^1(L^2(\lambda)),\ g \in L^1(\lambda)).$$

By hypothesis, we have $R_C(A) \in \mathcal{T}$ for every $C \in C^1(L^2(\lambda))$, and consequently

$$0 = \langle D, R_{C_g}\Gamma(f_A)) \rangle = \int_\Omega g \langle D, f_A(\cdot) \rangle d\lambda \qquad (D \in {}^\perp\mathcal{T},\ g \in L^1(\lambda)).$$

From this we conclude that the scalar-valued H^∞-function $\langle D, f_A(\cdot) \rangle$ vanishes identically on Ω for every $D \in {}^\perp\mathcal{T}$. But this means precisely that

$$f_A(z) \in ({}^\perp\mathcal{T})^\perp = \overline{\mathcal{T}}^{w^*} = \mathcal{T} \qquad (z \in \Omega),$$

as was to be shown. □

For the rest of this article, we specialize to the case where $\Omega \subset \mathbb{C}^n$ is a bounded symmetric and circled domain or a relatively compact strictly pseudoconvex open subset $\Omega \subset X$ of a Stein submanifold $X \subset \mathbb{C}^n$, and assume that the closure $\overline{\Omega} \subset \mathbb{C}^n$ is polynomially convex.

Fix a subnormal tuple $S \in B(H)^n$ of class $\mathbb{A}$ over Ω. This means by definition that S possesses an extension to a commuting tuple $\hat{S} \in B(\hat{H})^n$ of normal operators on some Hilbert space $\hat{H} \supset H$ and that there exists an isometric and weak* continuous functional calculus $\Phi : H^\infty(\Omega) \to B(H)$ for S. Furthermore, the normal extension $\hat{S}$ can be chosen to be minimal in the sense that if $M \subset \hat{H}$ is any reducing subspace for $\hat{S}$ containing H, then $M = \hat{H}$.

Spectral theory for the minimal normal extension $\hat{S}$ of S then yields a regular Borel probability measure μ on $\overline{\Omega}$ having the following properties (see e.g. [3]):

(a) There is an isometric and weak* continuous algebra homomorphism

$$r_\mu : H^\infty(\Omega) \to L^\infty(\mu)$$

extending the canonical map $\mathbb{C}[z] \to L^\infty(\mu)$, $p \mapsto [p|\Omega]$. In other words, μ is a faithful Henkin measure.

(b) The normal tuple $\hat{S}$ possesses an isometric, weak* continuous and involutive functional calculus

$$\Psi : L^\infty(\mu) \to B(\hat{H}).$$

The mappings Φ, r_μ and Ψ will be used now to show that $\mathcal{A}_S$ has property S_σ.

3.2 Corollary. *Suppose that Ω is a bounded symmetric and circled domain in $\mathbb{C}^n$ or a relatively compact strictly pseudoconvex open subset of a Stein submanifold $X \subset \mathbb{C}^n$ possessing polynomially convex closure $\overline{\Omega} \subset \mathbb{C}^n$. Then the dual algebra $\mathcal{A}_S$ generated by a subnormal tuple $S \in B(H)^n$ of class $\mathbb{A}$ over Ω has property S_σ.*

Proof. From the hypothesis on Ω to have polynomially convex closure we deduce that the polynomials $\mathbb{C}[z]|\overline{\Omega}$ are dense in $\mathscr{O}(\overline{\Omega})$ with respect to the supremum norm $\|\cdot\|_{\infty,\overline{\Omega}}$. Combining this with the assertion of Proposition 2.2 (with $E' = \mathbb{C}$) we see that $\mathbb{C}[z]|\Omega \subset H^\infty(\Omega)$ is weak* dense. Consequently, we have $\Phi(H^\infty(\Omega)) = \mathcal{A}_S$ and $\Psi \circ r_\mu(H^\infty(\Omega)) = \mathcal{A}_{\hat{S}}$. In particular, the composition $\Phi \circ r_\mu^{-1} \circ \Psi^{-1}|\mathcal{A}_{\hat{S}}$ yields a dual algebra isomorphism

$$\tau : \mathcal{A}_{\hat{S}} \to \mathcal{A}_S \quad \text{with} \quad \Psi(r_\mu(f)) \mapsto \Phi(f) \qquad (f \in H^\infty(\Omega)).$$

In view of the identity

$$\Phi(f) = \Psi(r_\mu(f))|H,$$

extending from $f \in \mathbb{C}[z]$ to all of $H^\infty(\Omega)$ by a weak* density argument, the mapping τ is nothing else than the restriction map $\tau(A) = A|H$, for $A \in \mathcal{A}_{\hat{S}}$. This shows that τ is completely bounded. Since the range of τ^{-1} is contained in the abelian C^*-algebra $W^*(\hat{S})$, the inverse of τ is also completely bounded.

Next observe that the two isometric and weak* continuous embeddings

$$\psi : H^\infty(\Omega) \xrightarrow{r_\mu \circ \Psi} \mathcal{A}_{\hat{S}} \subset W^*(\hat{S}) \quad \text{and} \quad \gamma_0 : H^\infty(\Omega) \xrightarrow{\gamma} \mathscr{H}^\infty(\Omega) \subset W^*(M_z)$$

both induce the same operator space structure on $H^\infty(\Omega)$ as their ranges both are contained in abelian C^*-algebras. The composition

$$\Delta : \mathscr{H}^\infty(\Omega) \xrightarrow{\gamma_0^{-1}} H^\infty(\Omega) \xrightarrow{\psi} \mathcal{A}_{\hat{S}} \xrightarrow{\tau} \mathcal{A}_S$$

therefore is a completely bounded dual algebra isomorphism having a completely bounded inverse. Proposition 4.2 in Kraus [7] now guarantees that, via Δ, property S_σ carries over from $\mathscr{H}^\infty(\Omega)$ to $\mathcal{A}_S$. $\square$

3.3 Corollary. *Suppose that Ω is a bounded symmetric and circled domain in $\mathbb{C}^n$ or a relatively compact strictly pseudoconvex open subset of a Stein submanifold $X \subset \mathbb{C}^n$ possessing polynomially convex closure $\overline{\Omega} \subset \mathbb{C}^n$. Given a subnormal tuple $S \in B(H)^n$ of class $\mathbb{A}$ over Ω and a commuting tuple $T \in B(K)^m$ which is strongly reflexive (i.e. $AlgLat(T) = \mathcal{A}_T$), the tensor product tuple*

$$(S \otimes 1, 1 \otimes T) \in B(H \otimes K)^{n+m}$$

is strongly reflexive.

Proof. Theorem 1.4 in [3] says that $\mathcal{A}_S$ is strongly reflexive. Hence the assertion follows from the preceding corollary and Proposition 1.2. □

The last corollary in particular applies to the tuple $S = (M_{z_1}, \ldots M_{z_n})$ of multiplication with the coordinate functions on the classical Hardy or Bergman spaces, $H = H^2(\Omega)$ or $H = A^2(\Omega)$, on a strictly pseudoconvex or a bounded symmetric and circled domain Ω.

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