On the marginals of probability contents on lattices

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Abstract. The paper extends the fundamental existence assertion for probability contents and measures with given marginals: The extension is from algebras to lattices, and thus is in accord with an actual trend in measure and integration. The proof of the basic theorem is a rapid application of a former Hahn-Banach type separation theorem.

We start from the well-known theorem of Strassen [6] Theorem 6 on the existence of probability measures with given marginals. We also refer to Jacobs [1] Appendix B. In the version of Fremlin [2] Proposition 457D, that is under the usual product formation and in terms of probability contents on a nonvoid set \( X \), the theorem reads as follows.

**Theorem 1.** Let \( \mathcal{P} \) and \( \mathcal{Q} \) be algebras in \( X \), and \( \varphi : \mathcal{P} \to [0, \infty] \) and \( \psi : \mathcal{Q} \to [0, \infty] \) be probability contents. For an algebra \( \mathcal{A} \) in \( X \) with \( \mathcal{P}, \mathcal{Q} \subseteq \mathcal{A} \) and a content \( \vartheta : \mathcal{A} \to [0, \infty] \), then there exists a probability content \( \gamma : \mathcal{A} \to [0, \infty] \) with \( \gamma \leq \vartheta \) which extends \( \varphi \) and \( \psi \) such that
\[
\varphi(A) + \psi(B) \leq 1 + \vartheta(A \cap B)
\]
for all \( A \in \mathcal{P} \) and \( B \in \mathcal{Q} \).

As to the transition from contents to measures, we restrict ourselves to the obvious remark that \( \gamma \) is upward and downward \( \sigma \)-continuous when \( \vartheta \) is downward \( \sigma \)-continuous at \( \emptyset \). Besides [1] Appendix B we also refer to the results listed in [5] Section 3.

For the subsequent extension we define as usual \( \varphi : \mathcal{S} \to [0, \infty] \) to be a content on a lattice \( \mathcal{S} \) in \( X \) if \( \emptyset \in \mathcal{S} \) and \( \varphi \) is isotone with \( \varphi(\emptyset) = 0 \) and modular: \( \varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B) \) for all \( A, B \in \mathcal{S} \), with submodular and supermodular defined to mean \( \leq \) and \( \geq \) instead of \( = \); and we define \( \varphi : \mathcal{S} \to [0, \infty] \) to be a probability content if in addition \( X \in \mathcal{S} \) and \( \varphi(X) = 1 \). Then our extension reads as follows.

**Theorem 2.** Let \( \mathcal{P} \) and \( \mathcal{Q} \) be lattices in \( X \) which contain \( \emptyset \) and \( X \), and \( \varphi : \mathcal{P} \to [0, \infty] \) and \( \psi : \mathcal{Q} \to [0, \infty] \) be isotone and supermodular with \( \varphi(\emptyset) = \psi(\emptyset) = 0 \) and \( \varphi(X) = \psi(X) = 1 \). For a lattice \( \mathcal{A} \) in \( X \) with \( \mathcal{P}, \mathcal{Q} \subseteq \mathcal{A} \) and a content \( \vartheta : \mathcal{A} \to [0, \infty] \), then there exists a probability content \( \gamma : \mathcal{A} \to [0, \infty] \) with \( \gamma \leq \vartheta \) such that
\[
\varphi(A) + \psi(B) \leq 1 + \vartheta(A \cap B)
\]
for all \( A \in \mathcal{P} \) and \( B \in \mathcal{Q} \).

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Of course \( \varphi \leq \gamma |\mathfrak{P} \leftrightarrow \varphi = \gamma |\mathfrak{P} \) when \( \mathfrak{P} \) is an algebra and \( \varphi \) is a probability content, and the same for \( \psi \). Thus the new assertion contains the former one. Also \( \gamma \) is downward \( \sigma \) continuous at \( \varnothing \) when \( \vartheta \) is so.

The proof of Theorem 2 is quite short. The basic point is the Hahn-Banach type separation result \([4]\) Theorem 1.2 which follows. Its proof in \([4]\) combines the Hahn-Banach version \([4]\) Theorem 1.1 with \([3]\) Theorem 11.11 for the Choquet integral.

**Separation Theorem.** On a lattice \( \mathfrak{S} \) in \( X \) with \( \varnothing \in \mathfrak{S} \) let
\[
\alpha : \mathfrak{S} \to [0, \infty] \text{ be isotone with } \alpha(\varnothing) = 0 \text{ and supermodular,}
\beta : \mathfrak{S} \to [0, \infty] \text{ be isotone with } \beta(\varnothing) = 0 \text{ and submodular,}
\]
and \( \alpha \leq \beta \). Then there exists a content \( \gamma : \mathfrak{S} \to [0, \infty] \) such that \( \alpha \leq \gamma \leq \beta \).

An important consequence is \([4]\) Theorem 1.3: Each content \( \vartheta : \mathfrak{S} \to [0, \infty] \) on a lattice \( \mathfrak{S} \) with \( \varnothing \in \mathfrak{S} \) can be extended to a content \( \Theta : \mathfrak{P}(X) \to [0, \infty] \). In fact, this follows from the Separation Theorem applied to the familiar envelopes
\[
\vartheta_* : \mathfrak{P}(X) \to [0, \infty] \text{ defined } \vartheta_*(A) = \sup \{ \vartheta(S) : S \in \mathfrak{S} \text{ with } S \subseteq A \},
\vartheta^* : \mathfrak{P}(X) \to [0, \infty] \text{ defined } \vartheta^*(A) = \inf \{ \vartheta(S) : S \in \mathfrak{S} \text{ with } S \supseteq A \}.
\]
In this context we note that the isotone set functions \( \vartheta : \mathfrak{S} \to [0, \infty] \) with \( \vartheta(\varnothing) = 0 \) fulfill
\[
\vartheta \text{ supermodular } \Rightarrow \vartheta_* \text{ supermodular,}
\vartheta \text{ submodular } \Rightarrow \vartheta^* \text{ submodular.}
\]
It follows that the particular Separation Theorem for the domain \( \mathfrak{P}(X) \) implies the full theorem for all lattices \( \mathfrak{S} \) with \( \varnothing \in \mathfrak{S} \).

The proof of Theorem 2 requires one more lemma.

**Lemma.** Let \( \mathfrak{A} \) be a lattice in \( X \) and \( \vartheta : \mathfrak{A} \to [0, \infty] \) be isotone and modular. For \( A, B, U, V \in \mathfrak{A} \) then
\[
\vartheta(\left( A \cup B \right) \cap (U \cap V)) + \vartheta(\left( A \cap B \right) \cap (U \cup V)) \leq \vartheta(A \cap U) + \vartheta(B \cap V).
\]

Proof of the Lemma. We can assume that \( \vartheta(A \cap U), \vartheta(B \cap V) < \infty \). The left side of the assertion is
\[
= \vartheta(\left( A \cup U \right) \cap (B \cup V)) + \vartheta(\left( A \cap B \right) \cap (U \cap V))
\]
\[
= \vartheta(A \cap U) + \vartheta(B \cap V) - \vartheta(A \cap B \cup U \cup V)
\]
\[
= \vartheta((A \cap U) \cap B) + \vartheta((A \cap U) \cap V) - \vartheta((A \cap U) \cap (B \cap V))
\]
\[
+ \vartheta((A \cap B) \cap V) + \vartheta((A \cap B) \cap U) - \vartheta((A \cap U) \cap (B \cap V))
\]
\[
= \vartheta((A \cap U) \cap (B \cap V)) + \vartheta((A \cap U) \cap (B \cup V)) + \vartheta((A \cap B) \cap (U \cup V)),
\]
and this is \( \leq \vartheta(A \cap U) + \vartheta(B \cap V) \). \( \square \)

Proof of Theorem 2. The implication \( \implies \) is clear. For the proof of \( \Leftarrow \) let \( \Theta : \mathfrak{P}(X) \to [0, \infty] \) be a content which extends \( \vartheta \). 1) For \( A \subseteq X \) and for \( P \in \mathfrak{P} \) with \( P \subseteq A \) and \( Q \in \Omega \) the assumption shows that \( \varphi(P) \leq 1 - \psi(Q) + \vartheta(P \cap Q) \leq 1 - \psi(Q) + \Theta(A \cap Q) \). We define \( \alpha, \beta : \mathfrak{P}(X) \to [0, \infty] \) to be
\[
\alpha(A) = \sup \{ \varphi(P) : P \in \mathfrak{P} \text{ with } P \subseteq A \} = \varphi_*(A),
\beta(A) = \inf \{ 1 - \psi(Q) + \Theta(A \cap Q) : Q \in \Omega \}.
\]
It is clear that $\alpha$ and $\beta$ are isitone with $\alpha(\varnothing) = \beta(\varnothing) = 0$, and the above shows that $\alpha \leq \beta$. From $1 \leq \alpha(X) \leq \beta(X) \leq 1$ then $\alpha(X) = \beta(X) = 1$.

2) It is obvious that $\alpha$ is supermodular. We show that $\beta$ is submodular: For $A, B \subset X$ and $U, V \in \Omega$ we obtain from the Lemma

$$
(1 - \psi(U) + \Theta(A \cap U)) + (1 - \psi(V) + \Theta(B \cap V)) \\
\geq 1 - \psi(U \cup V) + \Theta((A \cap B) \cap (U \cup V)) + 1 - \psi(U \cap V) + \Theta((A \cup B) \cap (U \cap V)) \\
\geq \beta(A \cap B) + \beta(A \cup B),
$$

and hence the assertion.

3) Now the Separation Theorem furnishes a content $\Gamma : \mathcal{P}(X) \to [0, \infty]$ with $\alpha \leq \Gamma \leq \beta$. Thus $\Gamma(X) = 1$, so that $\Gamma$ is a probability content. From $\alpha \leq \Gamma$ we obtain $\varphi \leq \Gamma|\mathcal{P}$. And $\Gamma \leq \beta$ means that $\Gamma(A) \leq 1 - \psi(Q) + \Theta(A \cap Q)$ for $A \subset X$ and $Q \in \Omega$. Thus on the one hand $Q := X$ furnishes $\Gamma(A) \leq \Theta(A)$ for $A \subset X$. On the other hand we obtain for $Q \in \Omega$ and $A := Q'$ that $1 - \Gamma(Q) = \Gamma(Q') \leq 1 - \psi(Q)$ or $\psi(Q) \leq \Gamma(Q)$. It follows that $\gamma := \Gamma|\mathcal{A}$ is as required. $\square$

In conclusion we want to transform our theorem into the traditional version in terms of marginals. The notations will be as follows. Let $H : X \to Y$ be a map between nonvoid sets $X$ and $Y$. For a set system $\mathcal{A}$ in $X$ one defines the image set system $\bar{\mathcal{H}}\mathcal{A} := \{B \subset Y : H^{-1}(B) \in \mathcal{A}\}$ in $Y$. Then

- $\mathcal{A}$ lattice in $X \Rightarrow \bar{\mathcal{H}}\mathcal{A}$ lattice in $Y$,
- $\varnothing \in \mathcal{A} \Rightarrow \varnothing \in \bar{\mathcal{H}}\mathcal{A}$ and $X \in \mathcal{A} \Rightarrow Y \in \bar{\mathcal{H}}\mathcal{A}$,
- $\mathcal{A}$ algebra in $X \Rightarrow \bar{\mathcal{H}}\mathcal{A}$ algebra in $Y$.

For a set function $\alpha : \mathcal{A} \to [0, \infty]$ one defines the image set function $\bar{H}\alpha : \bar{\mathcal{H}}\mathcal{A} \to [0, \infty]$ to be $\bar{H}\alpha(B) = \alpha(H^{-1}(B))$. Then $\alpha$ content $\Rightarrow \bar{H}\alpha$ content, etc.

After this we fix nonvoid sets $X$ and $Y$, with the product set $Z = X \times Y$ and the canonical projections $I : Z \to X$ and $J : Z \to Y$. We assume

- $\mathcal{P}$ lattice in $X$ which contains $\varnothing$ and $X$,
- $\mathcal{Q}$ lattice in $Y$ which contains $\varnothing$ and $Y$,
- $\mathcal{A}$ lattice in $Z$ such that $\mathcal{P} \subset \bar{\mathcal{H}}\mathcal{A}$ and $\mathcal{Q} \subset \bar{\mathcal{H}}\mathcal{A}$; the last two relations mean $A \times Y = I^{-1}(A) \in \mathcal{A}$ $\forall A \in \mathcal{P}$ and $X \times B = J^{-1}(B) \in \mathcal{A}$ $\forall B \in \mathcal{Q}$, and hence combine to $\mathcal{P} \times \mathcal{Q} \subset \mathcal{A}$. Then a probability content $\gamma : \mathcal{A} \to [0, \infty]$ produces the probability contents $\bar{I}\gamma|\mathcal{P}$ on $\mathcal{P}$ and $\bar{J}\gamma|\mathcal{Q}$ on $\mathcal{Q}$, the so-called marginals of $\gamma$. In these terms the transformed theorem reads as follows.

**Theorem 3.** Let $\varphi : \mathcal{P} \to [0, \infty]$ and $\psi : \mathcal{Q} \to [0, \infty]$ be isitone and supermodular with $\varphi(\varnothing) = \psi(\varnothing) = 0$ and $\varphi(X) = \psi(Y) = 1$, and $\vartheta : \mathcal{A} \to [0, \infty]$ be a content. Then

- there exists a probability content $\gamma : \mathcal{A} \to [0, \infty]$ with $\gamma \leq \vartheta$ such that $\varphi \leq \bar{I}\gamma|\mathcal{P}$ and $\psi \leq \bar{J}\gamma|\mathcal{Q}$

$$
\iff \varphi(A) + \psi(B) \leq 1 + \vartheta(A \times B) \text{ for all } A \in \mathcal{P} \text{ and } B \in \mathcal{Q}.
$$

Proof. By assumption $\bar{\mathcal{P}} := \{A \times Y : A \in \mathcal{P}\}$ and $\bar{\mathcal{Q}} := \{X \times B : B \in \mathcal{Q}\}$ are lattices in $Z$ which contain $\varnothing$ and $Z$ and fulfil $\mathcal{P}, \mathcal{Q} \subset \mathcal{A}$. And

- $\bar{\varphi} : \bar{\mathcal{P}} \to [0, \infty]$ defined to be $\bar{\varphi}(A \times Y) = \varphi(A)$ for $A \in \mathcal{P}$ and
- $\bar{\psi} : \bar{\mathcal{Q}} \to [0, \infty]$ defined to be $\bar{\psi}(X \times B) = \psi(B)$ for $B \in \mathcal{Q}$
are isotone and supermodular with \( \tilde{\varphi}(\emptyset) = \tilde{\psi}(\emptyset) = 0 \) and \( \tilde{\varphi}(Z) = \tilde{\psi}(Z) = 1 \).

For a probability content \( \gamma : \mathfrak{A} \to [0, \infty] \) we have

\[
\tilde{\varphi} \leq \gamma|\mathfrak{P} \quad \text{or} \quad \tilde{\varphi}(A \times Y) \leq \gamma(I^{-1}(A)) \quad \forall A \in \mathfrak{P} \iff \varphi \leq \tilde{I}\gamma|\mathfrak{P},
\]

\[
\tilde{\psi} \leq \gamma|\mathfrak{Q} \quad \text{or} \quad \tilde{\psi}(X \times B) \leq \gamma(J^{-1}(B)) \quad \forall B \in \mathfrak{Q} \iff \psi \leq \tilde{J}\gamma|\mathfrak{Q}.
\]

Now in Theorem 2 the condition for \( \gamma \leq \vartheta \) combined with \( \tilde{\varphi} \leq \gamma|\tilde{\mathfrak{P}} \) and \( \tilde{\psi} \leq \gamma|\tilde{\mathfrak{Q}} \) reads

\[
\tilde{\varphi}(A \times Y) + \tilde{\psi}(X \times B) \leq 1 + \vartheta((A \times Y) \cap (X \times B)) \quad \text{for} \ A \in \mathfrak{P} \quad \text{and} \ B \in \mathfrak{Q},
\]

that is \( \varphi(A) + \psi(B) \leq 1 + \vartheta(A \times B) \) for \( A \in \mathfrak{P} \) and \( B \in \mathfrak{Q} \). Thus Theorem 2 turns at once into the present assertion. \( \square \)

As before we also have \( \varphi \leq \tilde{I}\gamma|\mathfrak{P} \iff \varphi = \tilde{I}\gamma|\mathfrak{P} \) when \( \mathfrak{P} \) is an algebra and \( \varphi \) is a probability content, and the same for \( \psi \). And of course \( \gamma \) is downward \( \sigma \)-continuous at \( \emptyset \) when \( \vartheta \) is so.

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