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Michael Didas and Jörg Eschmeier

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Michael Didas

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
didas@math.uni-sb.de

Jörg Eschmeier

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
eschmei@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

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Let $H^2(\Omega)$ be the Hardy space on a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, and let $A \subset L^\infty(\partial\Omega)$ denote the subalgebra of all L^∞-functions f with compact Hankel operator H_f. Given any closed subalgebra $B \subset A$ containing $C(\partial\Omega)$, we describe the first Hochschild cohomology group of the corresponding Toeplitz algebra $\mathcal{T}(B) \subset B(H^2(\Omega))$. In particular we show that every derivation on $\mathcal{T}(A)$ is inner. These results are new even for $n = 1$, where it follows that every derivation on $\mathcal{T}(H^\infty + C)$ is inner, while there are non-inner derivations on $\mathcal{T}(H^\infty + C(\partial\mathbb{B}_n))$ over the unit ball $\mathbb{B}_n$ in dimension $n > 1$.

Introduction

A recent result of Cao (Theorem 3 in [2]) describes the first Hochschild cohomology group of the Toeplitz C^*-algebra generated by all Toeplitz operators with continuous symbol on the Hardy space over a strictly pseudoconvex domain in $\mathbb{C}^n$. Using a modification of Cao's arguments and a result of Davidson from 1977 (Corollary 4 in [5]) the first author showed in [7] that every continuous derivation of the Toeplitz algebra $\mathcal{T}(H^\infty + C)$ on the Hardy space of the unit disc $\mathbb{D}$ is inner. It seems natural to ask if the first Hochschild cohomology group vanishes in this case, that is, if the latter result remains true without the continuity assumption and, secondly, if a generalization to higher dimensions is possible. In the present note we answer both questions in the affirmative by establishing a description of the first Hochschild cohomology group for a variety of Hardy-space Toeplitz algebras on strictly pseudoconvex domains, including the cases mentioned above.

Throughout this paper, we fix a bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with C^∞-boundary. The Hardy space $H^2(\sigma)$ with respect to the normalized surface measure σ on $\partial\Omega$ can be defined as the norm closure of the set $A(\partial\Omega) = \{f|\partial\Omega : f \in C(\overline{\Omega}), f|\Omega \text{ holomorphic}\}$ in $L^2(\sigma)$. As usual, the Toeplitz operator $T_f \in B(H^2(\sigma))$ with symbol $f \in L^\infty(\sigma)$ is given by the formula

$$T_f = PM_f|H^2(\sigma),$$

where $P : L^2(\sigma) \to H^2(\sigma)$ denotes the orhogonal projection and $M_f : L^2(\sigma) \to L^2(\sigma), g \mapsto fg$, is the operator of multiplication with f. A natural question to ask is to what extent the membership of a function f to some special symbol class $S \subset L^\infty(\sigma)$ determines the behaviour of the corresponding Toeplitz operator T_f. Besides this single-operator point of view, one may ask for the properties of the so-called Toeplitz algebra

$$\mathcal{T}(S) = \overline{\text{alg}}\{T_f : f \in S\} \subset B(H^2(\sigma))$$

associated with the symbol class S. Among the most important choices for S are the bounded holomorphic functions on Ω (more precisely, their non-tangential boundary values) which will be denoted by $H^\infty(\sigma)$ in the sequel and the continuous functions $C(\partial\Omega)$, which give rise to the algebra of all analytic Toeplitz operators $\mathcal{T}(H^\infty(\sigma))$ and the Toeplitz C^*-algebra $\mathcal{T}(C(\partial\Omega))$, respectively. Another natural symbol class, arising intrinsically in the theory of Toeplitz operators, can best be expressed in terms of the corresponding Hankel operators

$$H_f : H^2(\sigma) \to L^2(\sigma), \quad h \mapsto (1-P)(fh).$$

From the work of Davidson [5] for the unit disc, Ding and Sun [9] for the unit ball, and Didas et al [8] for the strictly pseudoconvex case, it is known that an operator $S \in B(H^2(\sigma))$ commutes modulo the compact operators with all analytic Toeplitz operators if and only if $S = T_f + K$ where K is compact and f belongs to the class

$$A = \{f \in L^\infty(\sigma) : H_f \text{ is compact}\}.$$

The identity $H_{fg} = H_f T_g + (1-P)M_f H_g$ valid for $f, g \in L^\infty(\sigma)$ shows that A is a closed subalgebra of $L^\infty(\sigma)$. Moreover, since Hankel operators with continuous symbol are compact in our setting (Theorem 4.2.17 in [16]), A always contains the space $H^\infty(\sigma) + C(\partial\Omega)$. According to [1] the latter space is also a closed subalgebra of $L^\infty(\sigma)$. By a classical result of Hartman [11], the equality $A = H^\infty + C$ holds on the open unit disc in $\mathbb{C}$, while the inclusion $H^\infty(\sigma) + C(\partial\mathbb{B}_n) \subset A$ is known [6] to be strict in the case of the open unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ for $n > 1$.

Given any closed subalgebra $B \subset L^\infty(\sigma)$ with $C(\partial\Omega) \subset B \subset A$, our main result characterizes the first Hochschild cohomology group of the Toeplitz algebra $\mathcal{T}(B)$. We briefly recall the definition of the first Hochschild cohomology. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{E}$ be a Banach-$\mathcal{A}$-bimodule. A derivation from $\mathcal{A}$ into $\mathcal{E}$ is a (not necessarily continuous) linear map $D : \mathcal{A} \to \mathcal{E}$ satisfying the identity

$$D(AB) = D(A)B + AD(B) \qquad (A, B \in \mathcal{A}).$$

For a given element $S \in \mathcal{E}$, the commutator with S

$$D : \mathcal{A} \to \mathcal{E}, \quad D(X) = [X, S] = XS - SX$$

defines a derivation from $\mathcal{A}$ into $\mathcal{E}$. Derivations arising in this way are called inner. Writing $Z^1(\mathcal{A}, \mathcal{E})$ for the space of all derivations from $\mathcal{A}$ into $\mathcal{E}$ and $N^1(\mathcal{A}, \mathcal{E}) \subset Z^1(\mathcal{A}, \mathcal{E})$ for the subspace consisting of all inner derivations, the first Hochschild cohomology group can defined as the quotient

$$H^1(\mathcal{A}, \mathcal{E}) = Z^1(\mathcal{A}, \mathcal{E})/N^1(\mathcal{A}, \mathcal{E}).$$

In particular, $H^1(\mathcal{A}, \mathcal{E})$ vanishes if and only if every derivation from $\mathcal{A}$ into $\mathcal{E}$ is inner.

Let us finally mention that, for a given Hilbert space H, we write $\mathcal{K}(H)$ for the ideal of all compact operators on H and that, for a subset $\mathcal{S} \subset B(H)$, we denote its essential commutant by

$$\mathcal{S}^{ec} = \{X \in B(H) : [X, S] \in \mathcal{K}(H) \text{ for all } S \in \mathcal{S}\}.$$

Now we have gathered all the notations required for an adequate formulation of our main result.

A description of H^1 for Toeplitz algebras

The following theorem can be thought of as a Banach-algebra version of Cao's result (Theorem 3 in [2]) on the Toeplitz C^*-algebra.

1 Theorem. *Let $B \subset L^\infty(\sigma)$ be a closed subalgebra with $C(\partial\Omega) \subset B \subset A$. Then every derivation $D : \mathcal{T}(B) \to B(H^2(\sigma))$ is inner and the map*

$$\delta : H^1(\mathcal{T}(B), \mathcal{T}(B)) \longrightarrow \mathcal{T}(B)^{ec}/\mathcal{T}(B), \quad \delta([D]) = [S] \quad \textit{if} \quad D = [\cdot, S],$$

is a well-defined isomorphism of linear spaces. □

We postpone the proof of this theorem for a moment in order to demonstrate some of its consequences. Let us first remark that, as A contains $C(\partial\Omega)$, the algebra $\mathcal{T}(A)$ contains the Toeplitz C^*-algebra $\mathcal{T}(C(\partial\Omega))$ and hence all compact operators on $H^2(\sigma)$ (Theorem 4.2.24 in [16]). Together with the description of $\mathcal{T}(H^\infty(\sigma))^{ec}$ established in Corollary 4.8 of [8], we obtain the chain of inclusions

$$\mathcal{T}(A)^{ec} \subset \mathcal{T}(H^\infty(\sigma))^{ec} = \{T_f + K : f \in A, K \in \mathcal{K}(H^2(\sigma))\} \subset \mathcal{T}(A).$$

The identity $T_{fg} - T_f T_g = -PM_f H_g$ for $f, g \in L^\infty(\sigma)$ shows that $\mathcal{T}(A)$ is essentially commutative. Hence we can complete the above chain with the inclusion $\mathcal{T}(A) \subset \mathcal{T}(A)^{ec}$, which shows that in fact equality holds at each stage. In particular, we have

$$\mathcal{T}(A)^{ec} = \mathcal{T}(H^\infty(\sigma))^{ec} = \mathcal{T}(A).$$

As a consequence we obtain the following special case of Theorem 1 which applies for example to the algebra $B = H^\infty(\sigma) + C(\partial\Omega)$.

2 Corollary. *If the algebra $B \subset A$ from Theorem 1 contains $H^\infty(\sigma)$, then we have $H^1(\mathcal{T}(B), \mathcal{T}(B)) \cong \mathcal{T}(A)/\mathcal{T}(B) \cong A/B$ as linear spaces.*

Proof. For the first identification, it suffices to observe that $\mathcal{T}(A) = \mathcal{T}(A)^{ec} \subset \mathcal{T}(B)^{ec} \subset \mathcal{T}(H^\infty(\sigma))^{ec} = \mathcal{T}(A)$ holds and to apply Theorem 1. The second identification is given by the map $A/B \to \mathcal{T}(A)/\mathcal{T}(B)$, $[f] \mapsto [T_f]$, which is easily seen to be a vector-space isomorphism. For the details, see the remarks following Lemma 8. □

For $B = A = \{f \in L^\infty(\sigma) : H_f \text{ compact}\}$ the assertion of Corollary 2 deserves to be stated separately.

3 Corollary. *The first Hochschild cohomology group $H^1(\mathcal{T}(A), \mathcal{T}(A))$ vanishes on every bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with C^∞-boundary. In particular, every derivation on $\mathcal{T}(H^\infty(\sigma) + C(\partial\mathbb{D}))$ is inner on the unit disc $\mathbb{D}$.* □

As mentioned before, it was observed by Davie and Jewell in [6] that the inclusion $H^\infty(\sigma) + C(\partial\mathbb{B}_n) \subset A$ is strict for every $n > 1$. Thus in contrast to the case $n = 1$, by Corollary 2, there exist non-inner derivations on the Toeplitz algebra $\mathcal{T}(H^\infty(\sigma) + C(\partial\mathbb{B}_n))$ for every $n > 1$.

Finally, we obtain the result of Cao mentioned at the beginning which was the starting point of our considerations. Note that $\mathcal{T}(C(\partial\Omega))^{ec} = \{T_{z_1}, \dots T_{z_n}\}^{ec}$ (see, e.g., Lemma 4.1 in [8]).

4 Corollary. (Cao) *For the Toeplitz C^*-algebra on a bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with C^∞-boundary we have the isomorphism*

$$H^1(\mathcal{T}(C(\partial\Omega)), \mathcal{T}(C(\partial\Omega))) \cong \{T_{z_1}, \dots T_{z_n}\}^{ec}/\mathcal{T}(C(\partial\Omega)).$$

□

Proof of Theorem 1

Theorem 1 can be proved using a general structure theorem due to Chernoff for derivations of operator algebras $\mathcal{A} \subset B(E)$ that contain the finite-rank operators on a normed linear space E (see Theorem 3.3 in [3]). Combining ideas of Cao [2] and Davidson [5] we give a short C^*-theoretic proof of the Hilbert-space version of Chernoff's theorem.

5 Proposition. *If $\mathcal{B}$ is a closed subalgebra of $B(H)$ containing the compact operators $\mathcal{K}(H)$ and the identity 1_H, then every derivation from $\mathcal{B}$ into $B(H)$ is inner, that is, $D = [\cdot, S]$ for some operator $S \in B(H)$.*

Proof. First observe that the set of operators $\mathcal{K}_1 = \mathcal{K}(H) + \mathbb{C} \cdot 1_H \subset \mathcal{B}$ is a self-adjoint closed subalgebra of $B(H)$ and hence is a unital C^*-algebra. So if $D : \mathcal{B} \to B(H)$ is any derivation, then its restriction $D_1 : \mathcal{K}_1 \to B(H)$, $D_1 = D|\mathcal{K}_1$, is a derivation from the C^*-algebra $\mathcal{K}_1$ into $B(H)$ (viewed as a Banach $\mathcal{K}_1$-bimodule) and therefore is continuous by a result of Ringrose (see Theorem 2 in [13] or Corollary 5.3.7 in [4]).
Our first aim is to show that D maps the space $\mathcal{K}(H)$ into itself. Since every element of the C^*-algebra $\mathcal{K}_1$ is a finite linear combination of unitary elements in $\mathcal{K}_1$, it suffices to show that $D(U) \in \mathcal{K}(H)$ for every unitary operator $U \in \mathcal{K}_1$. Since $\mathcal{K}_1/\mathcal{K}(H)$ is contained in the centre of the Calkin algebra $\mathcal{C}(H) = B(H)/\mathcal{K}(H)$, for every integer $m > 0$, we obtain the identity

$$[D(U^m)] = [\sum_{i=1}^{m} U^{i-1} D(U) U^{m-i}] = m[U^{m-1} D(U)]$$

in $\mathcal{C}(H)$. Using the fact that multiplication with a unitary element in a C^*-algebra is isometric, we find that

$$m\|[D(U)]\| = \|[D(U^m)]\| \le \|D_1\|$$

for all integers $m > 0$. Hence $D(U) \in \mathcal{K}(H)$ as was to be shown.

As a derivation of $\mathcal{K}(H)$ into itself, the restriction $D|\mathcal{K}(H)$ can be written as the commutator with some fixed operator $S \in B(H)$ (Corollary 4.1.7 in [14]). More explicitly, there exists an operator $S \in B(H)$ such that

$$D(K) = KS - SK \qquad (K \in \mathcal{K}(H)).$$

Since the identity

$$\begin{aligned} D(A)K + AD(K) &= D(AK) \\ &= AKS - SAK \\ &= (AKS - ASK) + (ASK - SAK) \\ &= AD(K) + (AS - SA)K \end{aligned}$$

holds for every $A \in \mathcal{B}$ and $K \in \mathcal{K}(H)$, it follows that $D = [\cdot, S]$. This observation completes the proof. □

6 Corollary. *If in the setting of the last proposition the quotient algebra $\mathcal{B}/\mathcal{K}(H)$ is commutative and semi-simple, then every derivation $D : \mathcal{B} \to \mathcal{B}$ has the form $D(X) = XS - SX$ $(X \in \mathcal{B})$ for some fixed operator $S \in \mathcal{B}^{ec}$ in the essential commutant of $\mathcal{B}$.*

Proof. By the preceding proposition, there is an operator $S \in B(H)$ with $D = [\cdot, S]$. In particular D induces a continuous derivation

$$\widehat{D} : \mathcal{B}/\mathcal{K}(H) \to \mathcal{B}/\mathcal{K}(H), \quad [X] \mapsto [D(X)].$$

Since $\mathcal{B}/\mathcal{K}(H)$ is supposed to be commutative and semi-simple, the Singer-Wermer theorem (Theorem 1 in [15]) implies that $\widehat{D} = 0$. Hence $D(\mathcal{B}) \subset \mathcal{K}(H)$ and $S \in \mathcal{B}^{ec}$. □

7 Corollary. *Let $\mathcal{B} \subset B(H)$ be a unital closed subalgebra containing the compact operators $\mathcal{K}(H)$ such that the quotient algebra $\mathcal{B}/\mathcal{K}(H)$ is commutative and semi-simple. Then the mapping $\delta : H^1(\mathcal{B}, \mathcal{B}) \to \mathcal{B}^{ec}/\mathcal{B}$,*

$$\delta([D]) = [S] \text{ if } D = [\cdot, S]$$

is a well-defined vector-space isomorphism.

Proof. Let $D : \mathcal{B} \longrightarrow \mathcal{B}$ be a given derivation. By Corollary 6 there is an operator $S \in \mathcal{B}^{ec}$ in the essential commutant of $\mathcal{B}$ such that $D = [\cdot, S]$. If $T \in B(H)$ is another operator with $D = [\cdot, T]$, then $T - S \in \mathcal{B}^c \subset \mathcal{K}(H)^c = \mathbb{C}1_H \subset \mathcal{B}$ and hence the equivalence classes of T and S in $\mathcal{B}^{ec}/\mathcal{B}$ coincide. If D is inner, then it follows that the operator S chosen above belongs to $\mathcal{B}$. Thus the map δ is well defined. Obviously, it is linear and injective. To complete the proof, observe that every operator $S \in \mathcal{B}^{ec}$ in the essential commutant of $\mathcal{B}$ induces a well defined derivation $D : \mathcal{B} \to \mathcal{B}$, $A \mapsto [A, S]$. Hence δ is also surjective. □

In the setting of Corollary 7 the first Hochschild cohomology group $H^1(\mathcal{B},\mathcal{B})$ of $\mathcal{B}$ vanishes if and only if $\mathcal{B}$ is equal to its essential commutant $\mathcal{B}^{ec}$ in $B(H)$. It is elementary to check that this happens if and only if the quotient algebra $\mathcal{B}/\mathcal{K}(H)$ is a maximal abelian subalgebra of the Calkin algebra $\mathcal{C}(H) = B(H)/\mathcal{K}(H)$. This remark shows in particular that the quotient $\mathcal{T}(A)/\mathcal{K}(H)$ is a maximal abelian subalgebra of the Calkin algebra $\mathcal{C}(H)$.

Moreover, the proof of the main theorem can be completed by showing that the Toeplitz algebra $\mathcal{T}(B)$ induced by the symbol class $B \subset L^\infty(\sigma)$ occurring in the statement of Theorem 1 satisfies the requirements of Corollary 7. This will be done in the following lemma.

8 Lemma. *Let $B \subset L^\infty(\sigma)$ be a closed subalgebra with $C(\partial\Omega) \subset B \subset A$. Then the mapping $\tau :\ B \to \mathcal{T}(B)/\mathcal{K}(H^2(\sigma))$ defined by*

$$\tau(f) = T_f + \mathcal{K}(H^2(\sigma))$$

is an isometric isomorphism between commutative semi-simple Banach algebras.

Proof. Obviously the map τ is linear. By Corollary 3.6 in [8] the equality of norms

$$\|f\|_{L^\infty(\sigma)} = \|T_f + \mathcal{K}(H^2(\sigma))\|$$

holds for every function $f \in L^\infty(\sigma)$. Hence τ is isometric. Since $B \subset A$, the formula

$$T_f T_g - T_{fg} = -PM_f H_g \quad (f, g \in L^\infty(\sigma))$$

shows that τ is an algebra homomorphism. The identity

$$T_{f_1} \cdots T_{f_r} - T_{f_1 \cdots f_r} = T_{f_1}(T_{f_2} \cdots T_{f_r} - T_{f_2 \cdots f_r}) + (T_{f_1} T_{f_2 \cdots f_r} - T_{f_1 \cdots f_r})$$

together with an elementary induction implies that

$$T_{f_1} \cdots T_{f_r} + \mathcal{K}(H^2(\sigma)) = T_{f_1 \cdots f_r} + \mathcal{K}(H^2(\sigma))$$

belongs to the range of τ for all $f_1, \dots f_r \in B$. Since the range of τ is closed, this argument yields the surjectivity of τ. As a unital closed subalgebra of the commutative C^*-algebra $L^\infty(\sigma)$, the Banach algebra B is semi-simple. This observation completes the proof. □

Let $B \subset L^\infty(\sigma)$ be a closed subalgebra as in Lemma 8. If $f \in L^\infty(\sigma)$ is a function with $T_f + \mathcal{K}(H^2(\sigma)) \in \mathcal{T}(B)/\mathcal{K}(H^2(\sigma))$, then there is a function $g \in B$ with $T_{f-g} = T_f - T_g \in \mathcal{K}(H^2(\sigma))$ and hence $f = g \in B$. Therefore in the setting of Corollary 2, the mapping

$$A/B \to \mathcal{T}(A)/\mathcal{T}(B),\ f + B \to T_f + \mathcal{T}(B)$$

is a vector-space isomorphism as we claimed there.

References

[1] A. Aytuna, A.M. Chollet, *Une extension d'une resultat de W. Rudin*, Bull. Soc. Math. France 104 (1976), 383-388.

[2] G. Cao, *Toeplitz algebras on strongly pseudoconvex domains*, Nagoya Math. J. 185 (2007), 171-186.

[3] P.R. Chernoff, *Representations, automorphisms, and derivations of some operator algebras*, J. Funct. Anal. 12 (1973), 275-289.

[4] H.G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs, New Series, vol. 24, Clarendon Press, Oxford, 2000.

[5] K.R. Davidson, *On operators commuting with Toeplitz operators modulo the compact operators*, J. Funct. Anal. 24 (1977), 291-302.

[6] A.M. Davie, N.P. Jewell, *Toeplitz operators in several variables*, J. Funct. Anal. 24 (1977), 291-302.

[7] M. Didas, *Every continuous derivation of* $\mathcal{T}(H^\infty + C(\mathbb{T}))$ *is inner*, Preprint Nr. 317, Department of Mathematics, Saarland University.

[8] M. Didas, J. Eschmeier, K. Everard, *On the essential commutant of analytic Toeplitz operators associated with spherical isometries*, J. of Funct. Anal. 261 (2011), 1361-1383.

[9] X. Ding, S. Sun, *Essential commutant of analytic Toeplitz operators*, Chinese Science Bulletin 42, No.7 (1997), 548-552.

[10] R.G. Douglas, *Banach algebra techniques in operator theory,* Second edition. Graduate Texts in Mathematics 179, Springer, New York, 1998.

[11] P. Hartman, *On completely continuous Hankel operators*, Proc. Amer. Math. Soc. 9 (1958), 862-866.

[12] N.P. Jewell, S.G. Krantz, *Toeplitz operators and related function algebras on certain pseudoconvex domains*, Transactions of the AMS, Volume 252 (1979), 297-312.

[13] J.R. Ringrose, *Automatic continuity of derivations of operator algebras*, J. London Math. Soc. (2) 5 (1972), 432-438.

[14] S. Sakai, *C^*-algebras and W^*-algebras*, Springer, Berlin, 1971.

[15] I.M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Annalen, Bd. 129 (1955), S. 260-264.

[16] H. Upmeier, *Toeplitz operators and index theory in several complex variables*, Oper. Theory Adv. Appl., vol. 81, Birkhäuser, Basel, 1996.