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On the structure of Hankel algebras

Michael Didas

Let H^2 denote the Hardy space on the unit disc $\mathbb{D}$ and let A be a closed subalgebra of $L^\infty(\partial\mathbb{D})$ strictly containing H^∞. The Hankel algebra $\mathscr{H}_A$ is by definition the smallest closed subalgebra of $B(H^2)$ containing all Toeplitz and Hankel operators with symbols from A. We establish a short exact sequence of the form $0 \to \mathscr{C} \to \mathscr{H}_A \to A \to 0$ generalizing the corresponding sequence for the underlying Toeplitz algebra, where $\mathscr{C}$ denotes the commutator ideal of $\mathscr{H}_A$. This extends a result of Power [14] to the non-selfadjoint setting. By a similar method we obtain a decomposition theorem for the set of all operators $X \in B(H^2)$ that are simultaneously asymptotically Toeplitz and Hankel (in the sense of Barria-Halmos [2] and Feintuch [11], respectively). As an application of the above short exact sequence we show that every derivation on $\mathscr{H}_A$ is a commutator with an operator $S \in B(H^2)$ and maps into the commutator ideal.

MSC: 47L80, 47B35, 47B47 *Keywords:* Hankel algebras, asymptotic Toeplitz and Hankel operators, Toeplitz projection, derivations

§1 Introduction

Let $T_f = PM_f|H^2$ denote the Toeplitz operator with symbol $f \in L^\infty$ on the Hardy space H^2 of the unit circle. Given a subset $\mathcal{S} \subset L^\infty$, we write $T_\mathcal{S} = \{T_f : f \in \mathcal{S}\}$ for the collection of all Toeplitz operators associated with symbol functions from $\mathcal{S}$. It is well known that the assignment

$$\xi : L^\infty \to T_{L^\infty} \subset B(H^2), \quad f \mapsto T_f$$

is an isometric and involutive linear map onto the space of all Toeplitz operators. The projection involved in the definition of T_f prevents ξ to be multiplicative, but the difference of T_{fg} and T_fT_g can be explicitly expressed in terms of Hankel operators. Using the unitary operator $J : L^2 \to L^2$ mapping z^n to $\overline{z}^n$ for $n \in \mathbb{Z}$, the Hankel operator with symbol $f \in L^\infty$ can be considered as an operator on H^2 via the definition

$$H_f = PJM_{zf}|H^2.$$

Note that under the action of the map $Q = PJM_z : L^2 \to H^2$ involved here, the Fourier sequence $(a_n)_{n\in\mathbb{Z}}$ of a L^2-function is mapped to $(a_{-1}, a_{-2}, a_{-3}, \ldots)$. From this we see that $Q = QP_{(H^2)^\perp}$ and that $Q|(H^2)^\perp$ provides an isometric isomorphism between $(H^2)^\perp$ and H^2.

Abbreviating the function Jf by $\widetilde{f}$ (i.e. $\widetilde{f}(z) = f(\overline{z})$ for every $z \in \mathbb{T}$), the above-mentioned lack of multiplicativity of the map $\xi(f) = T_f$ is controlled by the identity

$$T_{fg} = T_fT_g + H_{\widetilde{f}}H_g \quad (f, g \in L^\infty)$$

which can be verified by direct computation (cf. Lemma 1.1 in [14]). Similarly, one deduces the relation

$$H_{fg} = T_{\tilde{f}} H_g + H_f T_g \quad (f, g \in L^\infty),$$

which plays a central role in the sequel. Given a symbol class $\mathcal{S} \subset L^\infty$, we write

$$H_{\mathcal{S}} = \{H_f : f \in \mathcal{S}\}.$$

Now, let $A \subset L^\infty$ be a subalgebra. The smallest closed subalgebra $\mathscr{T}_A \subset B(H^2)$ containing all operators T_f with $f \in A$ is called the Toeplitz algebra induced by A. Let $\mathscr{C}_A$ denote the commutator ideal of $\mathscr{T}_A$. One of the basic results in the theory of Toeplitz algebras is the existence of a homomorphism $s : \mathscr{T}_A \to A$ with $s(T_f) = f$ $(f \in A)$ such that the sequence

$$0 \longrightarrow \mathscr{C}_A \longrightarrow \mathscr{T}_A \stackrel{s}{\longrightarrow} A \longrightarrow 0$$

is exact for various algebras A, among them L^∞, the continuous functions $C(\mathbb{T})$ and the algebra $H^\infty + C(\mathbb{T})$ (see Theorem 7.11, 7.23 and 7.29 in [8]). Here, as usual, $H^\infty \subset L^\infty$ denotes the subspace of functions with vanishing negative Fourier coefficients (which are precisely the radial boundary values of bounded analytic functions on the unit disc). In the two cases $A = C(\mathbb{T})$ and $A = H^\infty + C(\mathbb{T})$, the commutator ideal $\mathscr{C}_A$ is known to be the ideal of compact operators $\mathcal{K}(H^2)$, while Barria and Halmos (see Theorem 7 in [2]) identified

$$\mathscr{C}_{L^\infty} = \{X \in \mathscr{T}_{L^\infty} : (T_z^*)^n X T_z^n \xrightarrow{n} 0 \text{ (SOT)}\}.$$

Using the terminology of [2], we call the operators $X \in B(H^2)$ belonging to the class

$$\mathscr{T}^\infty = \{X \in B(H^2) : ((T_z^*)^n X T_z^n)_{n \geq 1} \text{ is an SOT-convergent sequence}\}$$

asymptotic Toeplitz operators. To justify this terminology, we should recall that the classical Toeplitz operators $X \in B(H^2)$ are characterized by the so-called Brown-Halmos condition

$$T_z^* X T_z = X$$

which also shows that the SOT-limit of a convergent sequence $(T_z^*)^n X T_z^n$ is of the form T_φ for some unique element $\varphi \in L^\infty$. This φ is then called the (asymptotic) symbol of X and abbreviated by $\sigma(X)$. In the sequel, we write $\mathscr{T}^\infty$ for the set of all asymptotic Toeplitz operators and $\mathscr{T}_0^\infty$ for those with symbol zero. Using this notation, we have $\mathscr{C}_{L^\infty} = \mathscr{T}_{L^\infty} \cap \mathscr{T}_0^\infty$.

In Section 2 of this paper we establish an analogous short exact sequence

$$0 \longrightarrow \mathscr{C}(\mathscr{H}_A) \longrightarrow \mathscr{H}_A \stackrel{s}{\longrightarrow} A \longrightarrow 0$$

in the context of the Hankel algebra

$$\mathscr{H}_A = \overline{\text{alg}}^{\|\cdot\|}(T_A \cup H_A) \subset B(H^2)$$

and give a description of the commutator ideal $\mathscr{C}(\mathscr{H}_A)$ as $\mathscr{H}_A \cap \mathscr{T}_0^\infty$ (see Theorem 4). Our methods work for so-called inner subalgebras $A \subset L^\infty$, a class which has been introduced by Power to study the structure of Hankel C^*-algebras in [14]. Following Power we call a subalgebra $A \subset L^\infty$ inner, if

$$A = \overline{LH}\{\overline{\eta}\varphi : \overline{\eta}, \varphi \in A \text{ with } \eta \text{ inner and } \varphi \in H^\infty\}$$

Recall that a function $\eta \in H^\infty$ is said to be inner, if $|\eta| = 1$ a.e. on $\mathbb{T}$. What at a first look seems to be a rather technical condition on A covers most of the natural examples: By Stone-Weierstraß, $C(\mathbb{T})$ is inner and so is L^∞ itself by Theorem 6.32 in [8]. Moreover, if $\Sigma \subset H^\infty$ denotes a semigroup of inner functions, then the corresponding Douglas algebra $A = \overline{\{\overline{\eta}\varphi : \eta \in \Sigma, \varphi \in H^\infty\}}$ (see Def. 6.35 in [8]) is inner. A result of Marshall and Chang [13] shows that every closed subalgebra $A \subset L^\infty$ strictly containing H^∞ is a Douglas algebra and hence inner. The special role of inner functions in the context of Toeplitz operators relies on the following fact: Given $f, g \in H^2$, an inner function η, and $\varphi \in L^\infty$, we have $\langle T_\eta^* T_\varphi T_\eta f, g\rangle = \langle \varphi\eta f, \eta g\rangle = \langle T_\varphi f, g\rangle$. Hence, in view of the Brown-Halmos condition, an operator $X \in H^2$ is a Toeplitz operator if and only if

$$T_\eta^* X T_\eta = X \quad \text{for every inner function } \eta.$$

Our main tool in the proof of Theorem 4 is the Toeplitz projection introduced by Arveson in [1] (see Proposition 5.2). This is a completely positive unital map

$$\Phi : H^2 \to H^2 \quad \text{satisfying} \quad \Phi^2 = \Phi \quad \text{and} \quad \mathrm{ran}(\Phi) = T_{L^\infty}.$$

It turns out that Φ induces a direct sum decomposition $\mathscr{H}_A = T_A \oplus \mathscr{C}(\mathscr{H}_A)$.

Section 3 is devoted to the study of asymptotic Hankel operators $X \in B(H^2)$ which can be defined via the condition

$$(H_{\overline{z}^n} X T_z^n)_n \quad \text{is an SOT-convergent sequence in } B(H^2).$$

This class of operators has been introduced and studied by Feintuch in [10] and [11]. Let us write $\mathscr{H}^\infty$ for the set of all asymptotic Hankel operators and $\mathscr{H}_0^\infty$ for those where the corresponding limit is the zero operator. It is well known that each element of the full Hankel algebra is simultaneously asymptotically Toeplitz and Hankel, in other words $\mathscr{H}_{L^\infty} \subset \mathscr{T}^\infty \cap \mathscr{H}^\infty$. This motivates a study of the space $\mathscr{T}^\infty \cap \mathscr{H}^\infty$ and its subspace $\mathscr{T}^\infty \cap \mathscr{H}_0^\infty$. The latter one can be interpreted as the set of all "asymptotically analytic Toeplitz operators" (see Section 3 for details) and turns out to be an algebra. Its structure is analyzed in detail in Theorem 7. As a consequence, we deduce that $\mathscr{T}^\infty \cap \mathscr{H}^\infty = T_{L^\infty} \oplus (\mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty)$, see Corollary 8. Along the way, we give an alternative description of the commutator ideal of the Hankel algebra $\mathscr{H}_A$ (Corollary 9).

In Section 4 we consider derivations of $\mathscr{H}_A$ where A is an inner subalgebra of L^∞. We show (see Proposition 11) that each such derivation maps into the commutator ideal and is given by the commutator with an operator $S \in B(H^2)$.

§2 The structure of Hankel algebras

We start with an elementary observation concerning the rearrangement of products consisting solely of Hankel- and Toeplitz-factors which is implicitly contained in the paper [2]. For the convenience of the reader, we have included a proof. Empty products in $B(H^2)$ are assumed to have the value 1_H.

1 Lemma. (Barria-Halmos) *For every finite product $X = \prod_{i=1}^n X_i \neq 0$ with $X_i \in T_{L^\infty} \cup H_{L^\infty}$, there are a number $m \in \mathbb{N}$ and operators $T_j \in B(H^2)$ that are finite products of elements in T_{L^∞}, and $H_j \in B(H^2)$ being finite products of elements from H_{L^∞} such that $T_j \circ H_j \neq 0$ $(j = 1, \ldots, m)$ and*

$$X = \sum_{j=1}^{m} T_j \circ H_j \quad .$$

Moreover, one can achieve that the number of Hankel factors in each summand is equal to the number of Hankel factors contained in X.

Proof. We may of course assume that there is at least one Hankel factor present. In a first step, we suppose that there is exactly one Hankel factor X_k contained in X. If $k = n$, there is nothing to show. Otherwise $X_k X_{k+1} = H_f T_g = H_{fg} - T_{\tilde{f}} H_g$ for some $f, g \in L^\infty$ and hence

$$X = \prod X_i = [X_1 \cdots X_{k-1} H_{fg} X_{k+2} \cdots X_n] - [X_1 \cdots X_{k-1} T_{\tilde{f}} H_g X_{k+2} \cdots X_n].$$

Both of these products contain exactly one Hankel factor which is one step nearer to the right end as in the original term. Note further that one of these summands may vanish if the symbol of the Hankel operator happens to be analytic. Repeating this procedure a finite number of times and cancelling out all zero summands, we obtain a sum

$$X = \sum_{j=1}^{m} (\prod_{T \in M_j} T) \circ B_j \text{ with } M_j \subset T_{L^\infty} \text{ finite and } B_j \in H_{L^\infty},$$

as desired. In the second step, we prove the assertion of the Lemma by induction on the number s of Hankel factors contained in X. The first step settles the case $s = 1$. Now assume that the assertion holds for some $s \leq n - 1$ and that X_k is the right-most Hankel factor in X. From the first step we obtain a decomposition

$$\prod_{i=k}^{n} X_i = \sum_{j=1}^{m} T_j \circ H_j \text{ with single Hankel operators } H_j \in H_{L^\infty}.$$

Hence $X = \sum_{j=1}^m (X_1 \cdots X_{k-1} T_j) \circ H_j$, where the terms in brackets contain $s - 1$ Hankel factors. After rearranging them using the induction hypothesis we obtain a sum of the desired form. In particular, each summand is a product containing exactly s Hankel operators as its right-most factors. □

In the next lemma we collect some basic facts about asymptotic Toeplitz operators from [2]. We give a short reformulation of the proofs here.

2 Lemma. (Barria-Halmos) *The set $\mathscr{T}^\infty$ has the following properties:*

(a) *Every product of the form XH with $X \in B(H^2)$ and $H \in H_{L^\infty}$ is contained in $\mathscr{T}_0^\infty$.*

(b) *For $\varphi_1, \dots, \varphi_n \in L^\infty$, we have $T_{\varphi_1} \cdots T_{\varphi_n} \in \mathscr{T}^\infty$ with symbol $\varphi_1 \cdots \varphi_n$.*

(c) *A finite product of the form $X = X_1 \cdots X_n$ with $X_i \in T_{L^\infty} \cup H_{L^\infty}$ for $i = 1, \dots, n$ belongs to $\mathscr{T}^\infty$. If at least one of the factors is a Hankel operator, then $\sigma(X) = 0$.*

(d) *The full Hankel algebra $\mathscr{H}_{L^\infty}$ is contained in $\mathscr{T}^\infty$.*

Proof. For the proof of part (a), remember the fact that a Hankel operator can be characterized by the identity $HT_z = T_z^* H$. Hence we have

$$\|T_{\overline{z}}^n X H T_z^n f\| \leq \|X\| \|T_{\overline{z}}^n H f\| \qquad (f \in H^2)$$

and the right-hand side converges to 0, since $T_{\overline{z}}^n \to 0$ (SOT). Part (b) follows by considering the finite telesoping sum

$$\begin{aligned} T_{\varphi_1} \cdots T_{\varphi_n} - T_{\varphi_1 \cdots \varphi_n} \quad = \quad & T_{\varphi_1} \cdots T_{\varphi_{n-2}} (T_{\varphi_{n-1}} T_{\varphi_n} - T_{\varphi_{n-1}\varphi_n}) \\ & + T_{\varphi_1} \cdots T_{\varphi_{n-3}} (T_{\varphi_{n-2}} T_{\varphi_{n-1}\varphi_n} - T_{\varphi_{n-2}\varphi_{n-1}\varphi_n}) \\ & + \cdots \end{aligned}$$

Note that all the terms in brackets are semi-commutators, and hence in view of the identity $T_{fg} - T_f T_g = H_{\tilde{f}} H_g$ $(f, g \in L^\infty)$ contain a Hankel factor on the right. Now applying part (a), this yields the assertion of part (b) which is also a special case of (c), namely that all factors X_i are Toeplitz operators $(i = 1, \dots, n)$. Suppose now that a product X under consideration in part (c) contains at leat one Hankel factor. Then representation $X = \sum_{j=1}^m T_j H_j$ obtained in the preceding lemma has non-trivial Hankel parts H_j. Applying part (a) we obtain $\sigma(X) = 0$, as desired. Finally, note that $\mathscr{H}_{L^\infty}$ is the closed linear hull of all products X occurring in part (c). ☐

The main tool for our study of the Hankel algebra $\mathscr{H}_A$ will be a completely positive, unital projection $\Phi : B(H^2) \to B(H^2)$ onto the space T_{L^∞} of all Toeplitz operators. Following Arveson [1] (see Proposition 5.2), such a map Φ can be defined via the formula

$$\langle \Phi(X) f, g \rangle = \mathrm{LIM}\big(\langle T_{\overline{z}}^n X T_z^n f, g \rangle \big)_n \qquad (X \in B(H^2), \quad f, g \in H^2),$$

where $\mathrm{LIM} : \ell^\infty \to \mathbb{C}$ denotes a Banach limit. Obviously, Φ has the additional property that

$$A^* \Phi(X) B = \Phi(A^* X B) \quad (A, B \in (T_z)',\ X \in B(H^2)).$$

Combined with the inner-function criterion for Toeplitz operators mentioned above we have

$$\Phi(X) = \Phi(T_{\overline{\eta}} X T_\eta) \quad (\eta \text{ inner},\ X \in B(H^2)).$$

3 Lemma. *The map Φ from above has the following properties:*

(a) The identity $\Phi(X) = T_{\sigma(X)}$ holds for every $X \in \mathscr{T}^\infty$.

(b) $\ker(\Phi) \supset \{X \in \mathscr{T}^\infty : \sigma(X) = 0\} \supset \{XH : X \in B(H^2), H \in H_{L^\infty}\}$

(c) For every $Y \in \ker(\Phi)$ and every $f \in L^\infty$, we have $YT_f \in \ker(\Phi)$.

(d) Given $\varphi_1, \dots, \varphi_n \in L^\infty$, we have $\Phi(T_{\varphi_1} \cdots T_{\varphi_n}) = T_{\varphi_1 \cdots \varphi_n}$.

Proof. Parts (a) and (b) are obvious, (a) from the definition of Φ and $\mathscr{T}^\infty$ and (b) in view of Lemma 2 (a). For the validity of part (c), fix an operator $Y \in \ker(\Phi)$, an inner function η and $\varphi \in H^\infty$. Then we have

$$\Phi(YT_{\overline{\eta}\varphi}) = \Phi(T_{\overline{\eta}} Y T_{\overline{\eta}\varphi} T_\eta) = \Phi(T_{\overline{\eta}} Y T_\varphi) = T_{\overline{\eta}} \Phi(Y) T_\varphi = 0.$$

The assertion then follows from the fact that L^∞ is inner. Part (d) is a consequence of part (a) and Lemma 2 (b). □

Now we can state our decomposition theorem for Hankel algebras. It extends both Theorem 1.3 (i) of Power [14] and Theorem 7 of Barria-Halmos [2]. Note that every element of $\mathscr{H}_{L^\infty}$ is an asymptotic Toeplitz operator (see Lemma 2), so that the asymptotic symbol $\sigma(X)$ makes sense for every $X \in \mathscr{H}_{L^\infty}$. As before, $\mathscr{C}(\mathscr{H}_A)$ denotes the commutator ideal of $\mathscr{H}_A$.

4 Theorem. *Let $A \subset L^\infty$ be an inner subalgebra.*

(a) There is a direct sum decomposition $\mathscr{H}_A = T_A \oplus \mathscr{C}(\mathscr{H}_A)$ and

$$\mathscr{C}(\mathscr{H}_A) = \ker(\Phi) \cap \mathscr{H}_A = \{X \in \mathscr{H}_A : T_{\overline{z}}^n X T_z^n \xrightarrow{n} 0 \ (SOT)\}.$$

(b) The map $s : \mathscr{H}_A \to A$ given by $s(X) = f$, if $X = T_f + C \in \mathscr{H}_A$ corresponding to the above decomposition, is multiplicative and induces an isometric algebra isomorphism

$$\mathscr{H}_A / \mathscr{C}(\mathscr{H}_A) \xrightarrow{\hat{s}} A.$$

(c) The closed two-sided ideal $(H_A) \subset \mathscr{H}_A$ generated by H_A is contained in $\mathscr{C}(\mathscr{H}_A)$. If $A = \widetilde{A}$, then $\mathscr{C}(\mathscr{H}_A) = (H_A)$.

Proof. If $X = \prod_{i=1}^n X_i$ is one of the products occurring in the representation

$$\mathscr{H}_A = \overline{\mathrm{LH}} \left\{ \Pi_{i=1}^n X_i : n \in \mathbb{N}, X_i \in T_A \cup H_A (1 \leq i \leq n) \right\},$$

then, by Lemma 2 and Lemma 3 we have

$$\Phi(X) = \begin{cases} 0 & \text{if at least one factor } X_i \in H_A \\ T_{f_1 \cdots f_n} & \text{if } X_i = T_{f_i} \text{ for all } i = 1, \dots, n. \end{cases}$$

Consequently, Φ maps $\mathscr{H}_A$ into itself, and the restriction $\Phi_0 = \Phi|\mathscr{H}_A \in B(\mathscr{H}_A)$ is a projection ($\Phi_0^2 = \Phi_0$) with $\mathrm{ran}(\Phi_0) = T_A$. As a projection, Φ_0 gives rise to a direct sum decomposition

$$\mathscr{H}_A = \mathrm{ran}(\Phi_0) + \mathrm{ran}(1 - \Phi_0) = T_A + \ker(\Phi_0).$$

Now consider the composition $s = \xi_0^{-1} \circ \Phi_0 : \mathscr{H}_A \to A$, where ξ_0 is the linear isometric isomorphism $\xi_0 : A \to T_A$, $f \mapsto T_f$. Then s is a contractive linear map with $s(T_f + C) = f$ for every $T_f + C \in T_A + \ker(\Phi_0) = \mathscr{H}_A$ and $\ker(s) = \ker(\Phi_0)$. To see that s is multiplicative, it suffices, by continuity and linearity, to check that s is multiplicative on products of the form $\Pi_{i=1}^n X_i$ with $X_i \in T_A \cup H_A$ $(1 \le i \le n)$. But if $X_m \in H_A$ for some $m \in \{1, \dots, n\}$, then we have $\Phi(\Pi_{i=1}^n X_i) = 0 = \Pi_{i=1}^n \Phi(X_i)$ by Lemma 3. Otherwise, we have $X_i = T_{f_i} \in T_A$ for all i, and hence

$$s(X) = \xi_0^{-1}(\Phi_0(T_{f_1} \cdots T_{f_n})) = \xi_0^{-1}(T_{f_1 \cdots f_n}) = f_1 \cdots f_n = s(X_1) \cdots s(X_n)$$

in view of Lemma 3 (d). By the above, s is a contractive algebra homomorphism, and so is the induced map $\hat{s} : \mathscr{H}_A / \ker(\Phi_0) \to A$. Since $\xi_0 : A \to T_A$ is an isometry and the Toeplitz projection is contractive, we have the estimate

$$\|[T_f]\| \le \|f\| = \|T_f\| \le \|\Phi(T_f + C)\| \le \|T_f + C\| \qquad (C \in \ker(\Phi_0))$$

which proves that $\|f\| = \|[T_f]\|$ for every $f \in A$, and hence $\widehat{s}$ is an isometry. To finish the proof of parts (a) and (b), we verify that the following inclusions hold:

$$\ker(\Phi_0) \subset \{X \in \mathscr{H}_A : \sigma(X) = 0\} \subset \mathscr{C}(\mathscr{H}_A) \subset \ker(\Phi_0).$$

Concerning the first one, note that $\ker(\Phi_0) = \mathrm{ran}(1 - \Phi_0)$ and hence it suffices to show that

$$\sigma(X - \Phi_0(X)) = 0$$

for every product $X = \Pi_{i=1}^n X_i$ with $X_i \in T_A \cup H_A$ $(i = 1, \dots, n)$. If all factors are Toeplitz operators $X_i = T_{f_i}$, then $\Phi_0(X) = T_{f_1 \cdots f_n}$ by Lemma 3 (d), hence the assertion follows from Lemma 2 (b). Otherwise there is at least one Hankel factor X_m, forcing both $\sigma(X)$ and $\Phi_0(X)$ to be zero (see Lemma 2 (c) and Lemma 3 (b)).

For the second inclusion, we have to show that

$$X \in \mathscr{H}_A \text{ with } \sigma(X) = 0 \quad \Rightarrow \quad X \in \mathscr{C}(\mathscr{H}_A).$$

To begin with, fix an arbitrary operator $X \in \mathscr{H}_A$ with $\sigma(X) = 0$. Then there is an approximating sequence $X_n \stackrel{n}{\to} X$ of the form

$$X_n = \sum_{i=1}^{k_n} X_i^{(n)}$$

where each $X_i^{(n)}$ is a finite product of elements of $T_A \cup H_A$. From the fact that the symbol map $\sigma : \mathscr{H}_{L^\infty} \to L^\infty$ is a contractive homomorphism (see Corollary 5 in [2]), we deduce that

$$\sum_{i=1}^{k_n} \big(X_i^{(n)} - T_{\sigma(X_i^{(n)})}\big) = X_n - T_{\sigma(X_n)} \stackrel{n}{\longrightarrow} X - T_{\sigma(X)} = X.$$

Hence it remains to check that every summand occurring on the left-hand side belongs to $\mathscr{C}(\mathscr{H}_A)$. Towards this, fix indices n and i and write $Y = X_i^{(n)} = X_1 \cdots X_m$ with $X_j \in T_A \cup H_A$. If all factors X_j are Toeplitz operators (with corresponding symbol f_j), then we have

$$Y - T_{\sigma(Y)} = T_{f_1} \cdots T_{f_m} - T_{f_1 \cdots f_m}$$

by Lemma 2 (b). A look at the proof of the cited lemma even shows that

$$Y - T_{\sigma(Y)} \in \mathscr{S},$$

where $\mathscr{S}$ denotes the closed ideal in $\mathscr{T}_A = \overline{\mathrm{alg}}(T_A)$ generated by all semi-commutators $T_f T_g - T_{fg}$ with $f, g \in A$. But if η_1, η_2 are inner and $\varphi_1, \varphi_2 \in H^\infty$, we have

$$T_{\overline{\eta}_1 \varphi_1} T_{\overline{\eta}_2 \varphi_2} - T_{\overline{\eta}_1 \overline{\eta}_2 \varphi_1 \varphi_2} = T_{\overline{\eta}_1} (T_{\varphi_1} T_{\overline{\eta}_2} - T_{\overline{\eta}_2} T_{\varphi_1}) T_{\varphi_2} \in \mathscr{C}(\mathscr{H}_A),$$

so $\mathscr{S} \subset \mathscr{C}(\mathscr{H}_A)$ since A is inner. Now we turn to the second case, namely that one factor X_l of $Y = \Pi_{i=1}^m X_i$ belongs to H_A. Suppose for a moment that the symbol of $X_l = H_f$ has the special form $f = \overline{\eta}\varphi$ with $\overline{\eta}, \varphi \in A$, η inner and $\varphi \in H^\infty$. Then the decomposition

$$H_{\overline{\eta}\varphi} = H_{\overline{\eta}\varphi} - T_{\overline{\eta}} H_{\overline{\eta}\varphi} T_\eta = T_{\overline{\eta}} (T_\eta H_{\overline{\eta}\varphi} - H_{\overline{\eta}\varphi} T_\eta) \in \mathscr{C}(\mathscr{H}_A).$$

shows that $X_l \in \mathscr{C}(\mathscr{H}_A)$. For a general symbol $f \in A$ this follows from the hypothesis on A to be inner. So we finally obtain that $Y \in \mathscr{C}(\mathscr{H}_A)$, while $\sigma(Y) = 0$ by Lemma 2 (c). Hence, in any case, $Y - T_{\sigma(Y)} \in \mathscr{C}(\mathscr{H}_A)$, as desired.

To finish the proof of (a) and (b) note that the last inclusion in the above chain, namely $\mathscr{C}(\mathscr{H}_A) \subset \ker(\Phi_0)$, holds trivially since by the above $\mathscr{H}_A / \ker(\Phi_0)$ is commutative.

Towards part (c) observe that the inclusion $\mathscr{C}(\mathscr{H}_A) \supset (H_A)$ has just been shown. For the reverse one, we have to verify that all commutators of the form

$$T_f H_g - H_g T_f, \quad H_f H_g - H_g H_f \quad \text{and} \quad T_f T_g - T_g T_f \qquad (f, g \in A)$$

belong to (H_A). Only the last one requires an argument. Lemma 1.1 in Power guarantees that, for all $f, g \in A$, we have

$$T_f T_g - T_g T_f = (T_f T_g - T_{fg}) + (T_{gf} - T_g T_f) = -H_{\tilde{f}} H_g + H_{\tilde{g}} H_f \in (H_A),$$

since, by hypothesis, $\widetilde{A} = A$. □

It should be remarked that if the subalgebra $A \subset L^\infty$ is self-adjoint and satisfies $A = \widetilde{A}$, then $\mathscr{H}_A$ is a C^*-algebra (note that $H_f^* = H_{\tilde{\overline{f}}}$ for $f \in L^\infty$). Hence $\mathscr{H}_A$ coincides with $C^*(T_A \cup H_A)$ in this case. If, in addition, A is inner, then we have $\mathscr{H}_A = T_A + C^*(H_A)$ by Theorem 1.3 (i) from [14]. Combined with the above we obtain the identity $\mathscr{C}(\mathscr{H}_A) = (H_A) = C^*(H_A)$ in this special situation.

§3 Hankel algebras and asymptotic Hankel operators

In this section we study operator algebras associated with so called asymptotic Hankel operators. A first systematic study of this class of operators can be found in Feintuch [10] and [11]. We briefly recall the definition given there. For $n \in \mathbb{N}$, let $H^2_n = \mathrm{LH}\{z^i : 0 \le i \le n\}$ and let $P_n \in B(H^2)$ be the orthogonal projection onto H^2_n. Define $J_n \in B(H^2)$ by $J_n z^i = z^{n-i}$ for $i = 0, \ldots, n$ and $J_n|(H^2_n)^\perp = 0$. In other words, J_n is the partial isometry with initial and final space H^2_n which reverses just the order of the standard basis elements of H^2_n. An operator $X \in B(H^2)$ is called an asymptotic Hankel operator, if the sequence

$$H_n(X) = J_n X T_z^{n+1} \quad (n \ge 1)$$

is SOT-convergent. In case that the corresponding limit exists, it is denoted by $H(X)$. As shown by Feintuch in [11] the set

$$\mathscr{H}^\infty = \{X \in B(H^2) : X \text{ is asymptotically Hankel}\}$$

is norm closed. By $\mathscr{H}^\infty_0$ we mean the subset of all $X \in \mathscr{H}^\infty$ with $H(X) = 0$.

It should be remarked that the map J_n occuring in the above definition is nothing else than the Hankel operator $H_{\overline{z}^{n+1}}$. To see this, note that for $n, k \ge 0$ we have $H_{\overline{z}^{n+1}} z^k = PJM_{z\overline{z}^{n+1}} z^k = PJz^{k-n} = Pz^{n-k} = J_n z^k$.

The following observation (see Section 3 in [11]) shows that the notion of an asymptotic Hankel operator extends the classical definition in a natural way.

5 Lemma. (Feintuch) *(a) Every Toeplitz operator T_f is asymptotically Hankel with $H(T_f) = H_f$ $(f \in L^\infty)$.*

(b) Every product $T_f T_g$ of Toeplitz operators is asymptotically Hankel with $H(T_f T_g) = H_f T_g + T_{\tilde{f}} H_g = H_{fg}$. □

Let $H^\infty \subset L^\infty$ be the closed subalgebra of all functions having vanishing negative Fourier coefficients. Toeplitz operators with symbols from H^∞ are called analytic. Our first result gives a necessary condition for an operator $X \in B(H^2)$ to belong to $\mathscr{H}^\infty_0$, namely that the image of X under the Toeplitz projection Φ is an analytic Toeplitz operator.

6 Proposition. *Every $X \in \mathscr{H}^\infty_0$ satisfies $\Phi(X) \in T_{H^\infty}$.*

Proof. Fix an operator $X \in \mathscr{H}^\infty_0$ and write $\Phi(X) = T_\varphi$. We want to show that the symbol φ is analytic. Towards this end, let $X \oplus 0$ denote the trivial extension of X to an operator on $L^2 = H^2 \oplus (H^2)^\perp$. Making use of the isometric identification $(H^2)^\perp \to H^2$, $h \mapsto PJM_z h$ (see the introduction for details), we have the following norm identity for $f \in H^2$. (Recall that P denotes the projection onto H^2.)

$$\begin{aligned} \|(M_{\overline{z}}^{n+1}(X \oplus 0)M_z^{n+1} - T_{\overline{z}}^{n+1} X T_z^{n+1})f\|_{L^2} &= \|P_{(H^2)^\perp} M_{\overline{z}}^{n+1} X T_z^{n+1} f\|_{L^2} \\ &= \|PJM_z M_{\overline{z}}^{n+1} X T_z^{n+1} f\| \\ &= \|PM_{z^n} JXT_z^{n+1} f\| = \ldots \end{aligned}$$

Since X maps into H^2 and $PM_z^n Jz^k = PM_z^n z^{-k} = Pz^{n-k} = J_n z^k$ for $k \geq 0$, we can complete the above chain with

$$\ldots = \|J_n X T_z^{n+1} f\| \xrightarrow{n} 0. \qquad (*)$$

Now define an operator $\Psi(X) \in B(L^2)$ in complete analogy to the Toeplitz projection $\Phi(X)$ by means of

$$\langle \Psi(X) f, g\rangle = \mathrm{LIM}\langle M_{\overline{z}}^n (X \oplus 0) M_z^n f, g\rangle \qquad (f, g \in H^2).$$

Note that by the translation invariance of LIM (which should be the same as in the definition of Φ), the operator $\Psi(X)$ belongs to the commutant of M_z on L^2 and hence is of the form $\Psi(X) = M_\psi$ for some $\psi \in L^\infty$. Moreover, the above norm convergence $(*)$ implies that $\langle (\Psi(X) - \Phi(X)) f, g\rangle = 0$ for $f, g \in H^2$ and hence $T_\varphi = \Phi(X) = P_H \Psi(X)|H = T_\psi$. This implies that $\varphi = \psi$. Now we verify that ψ is analytic. For $k \geq 0$, we have

$$\begin{aligned} \langle \psi, z^{-(k+1)}\rangle &= \langle M_\psi 1, z^{-(k+1)}\rangle \\ &= \mathrm{LIM}\langle M_{\overline{z}}^{n+1}(X \oplus 0) M_z^{n+1} 1, z^{-(k+1)}\rangle \\ &= \mathrm{LIM}\langle X T_z^{n+1} 1, z^{n-k}\rangle. \end{aligned}$$

For $n \geq k \geq 0$ the scalar product in the preceding line can be rewritten in the form

$$\langle X T_z^{n+1} 1, z^{n-k}\rangle = \langle X T_z^{n+1} 1, J_n z^k\rangle = \langle J_n X T_z^{n+1} 1, z^k\rangle \qquad (n \geq k \geq 0)$$

since J_n is unitary on H_n^2 and commutes with the projection onto H_n^2. By the hypothesis on X, this is a zero sequence, so $\psi = \varphi$ is indeed analytic. $\square$

In [11], Feintuch asks for an analytic description of $\mathscr{H}_0^\infty$. It would be interesting to know if the above necessary condition could be extended to such a description.

The preceding proposition allows us to describe the structure of the space $\mathscr{T}^\infty \cap \mathscr{H}_0^\infty$. Note that, for a Toeplitz operator T_φ with $\varphi \in L^\infty$, we have $H(T_\varphi) = H_\varphi = 0 \Leftrightarrow \varphi \in H^\infty$. The elements of $\mathscr{T}^\infty \cap \mathscr{H}_0^\infty$ are asymptotically Toeplitz and satisfy $H(X) = 0$, thus they can be thought of as "asymptotically analytic" Toeplitz operators. The following theorem describes their structure. The proof of part (a) relies on the identity $1 - P_n = T_z^{n+1} T_{\overline{z}}^{n+1}$ $(n \in \mathbb{N})$.

7 Theorem. *The set $\mathscr{T}^\infty \cap \mathscr{H}_0^\infty$ of all asymptotically analytic Toeplitz operators is a closed subalgebra of $B(H^2)$, and the following assertions hold:*

(a) The set $\mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty$ is equal to $\{X \in B(H^2) : X T_z^n \xrightarrow{n} 0 \text{ (SOT)}\}$, and this is a closed left ideal of $B(H^2)$ containing $C^*(H_{L^\infty})$.

(b) There is a direct sum decomposition $\mathscr{T}^\infty \cap \mathscr{H}_0^\infty = T_{H^\infty} \oplus (\mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty)$.

(c) The generalized symbol map yields an isometric algebra isomorphism $(\mathscr{T}^\infty \cap \mathscr{H}_0^\infty)/(\mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty) \xrightarrow{\widehat{\sigma}} H^\infty$, $[X] \mapsto \sigma(X)$.

Proof. For every $X \in \mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty$ and $n \in \mathbb{N}$, consider the decomposition

$$XT_z^{n+1} = (1 - P_n)XT_z^{n+1} + P_nXT_z^{n+1} = T_z^{n+1}(T_{\overline{z}}^{n+1}XT_z^{n+1}) + P_nXT_z^{n+1}.$$

Now apply the sum on the right to a vector $x \in H^2$. Then the first summand converges to zero, since $(T_z^{n+1})_n$ is norm-bounded and $X \in \mathscr{T}_0^\infty$. Concerning the second summand, observe that $\|P_nXT_z^{n+1}x\| = \|J_nXT_z^{n+1}x\|$, since J_n is unitary on $\operatorname{ran}(P_n)$ and vanishes on $\operatorname{ran}(P_n)^\perp$. Thus we have $XT_z^{n+1}x \overset{n}{\to} 0$, and the inclusion "$\subset$" from part (a) follows. The other direction and the assertion that this set is a closed left ideal are obvious. To finish the proof of part (a) note that, for every Hankel operator $H \in H_{L^\infty}$, we have $HT_z^n = T_{\overline{z}}^n H \overset{n}{\longrightarrow} 0$ (SOT), and that $H_f^* = H_{\widetilde{\overline{f}}}$.

The inclusion "$\supset$" from part (b) holds trivially. For the other direction, write a given operator $X \in \mathscr{T}^\infty \cap \mathscr{H}_0^\infty$ as

$$X = \Phi(X) + (1 - \Phi)(X) = T_{\sigma(X)} + (X - T_{\sigma(X)}).$$

The first summand has the desired form since, according to the preceding proposition, we have $\sigma(X) \in H^\infty$. Moreover, under the given assumptions on X, this also implies that the second summand belongs to $\mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty$, as desired.

In view of part (b), the map $\widehat{\sigma}$ considered in part (c) is a linear isomorphism. Since the Toeplitz projection is contractive, we have the following estimate for an arbitrary element $X \in \mathscr{T}^\infty \cap \mathscr{H}_0^\infty$:

$$\|\sigma(X)\| = \|T_{\sigma(X)}\| = \|\Phi(X + C)\| \leq \|X + C\| \qquad (C \in \mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty).$$

Consequently, $\|\sigma(X)\| \leq \|[X]\| = \|[T_{\sigma(X)}]\| \leq \|\sigma(X)\|$, proving that $\widehat{\sigma}$ is an isometry. To see that $\mathscr{T}^\infty \cap \mathscr{H}_0^\infty$ is an algebra and that $\widehat{\sigma}$ is multiplicative, fix $X_i \in \mathscr{T}^\infty \cap \mathscr{H}_0^\infty$ $(i = 1, 2)$. Using a representation of the form $X_i = T_{\sigma(X_i)} + C_i$ with $C_i \in \mathscr{T}^\infty \cap \mathscr{T}_0^\infty$, we consider the product

$$X_1X_2 = T_{\sigma(X_1)}T_{\sigma(X_2)} + T_{\sigma(X_1)}C_2 + C_1T_{\sigma(X_2)} + C_1C_2.$$

Since $\sigma(X_2) \in H^\infty$, we have $C_1T_{\sigma(X_2)}T_z^n = C_1T_z^nT_{\sigma(X_2)}$ $(n \in \mathbb{N})$, and hence all but the first summand belong to $\mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty$ in view of part (a). The first summand can be written as

$$T_{\sigma(X_1)}T_{\sigma(X_2)} = T_{\sigma(X_1)\sigma(X_2)} - H_{\widetilde{\sigma(X_1)}}H_{\sigma(X_2)}.$$

This finally shows that X_1X_2 is an element of $T_{H^\infty} \oplus (\mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty)$ and that $\sigma(X_1X_2) = \sigma(T_{\sigma(X_1)}T_{\sigma(X_2)}) = \sigma(X_1)\sigma(X_2)$, as desired. □

As a consequence, we obtain the following description of $\mathscr{T}^\infty \cap \mathscr{H}^\infty$:

8 Corollary. *The identity $\mathscr{T}^\infty \cap \mathscr{H}^\infty = T_{L^\infty} \oplus (\mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty)$ holds.*

Proof. Let $X \in \mathscr{T}^\infty \cap \mathscr{H}^\infty$. Then, by definition, $H_n(X) \overset{n\to\infty}{\longrightarrow} H_\varphi$ for some $\varphi \in L^\infty$. Since $H_n(T_\varphi) \overset{n\to\infty}{\longrightarrow} H_\varphi$, the first summand of the decomposition

$$X = (X - T_\varphi) + T_\varphi$$

belongs to $\mathscr{T}^\infty \cap \mathscr{H}_0^\infty$, and thus has the form $T_f + C \in T_{H^\infty} \oplus (\mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty)$ by the preceding theorem. So we have a decomposition

$$X = T_f + T_\varphi + C \in T_{L^\infty} + (\mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty),$$

proving the inclusion

$$\mathscr{T}^\infty \cap \mathscr{H}^\infty \subset T_{L^\infty} \oplus (\mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty).$$

The sum is direct since the left-hand summand is the range of Φ while the one on the right-hand side is contained in the kernel of Φ. Since the reverse inclusion is obvious, the proof is complete. □

Part (a) of Theorem 7 yields an alternative description of the commutator ideal of Hankel algebras. An analogue in the context of Toeplitz algebras for multi-variable isometries was recently obtained by Everard (see [9]).

9 Corollary. *For an inner subalgebra $A \subset L^\infty$, the commutator ideal $\mathscr{C}(\mathscr{H}_A)$ can be expressed as*

$$\mathscr{C}(\mathscr{H}_A) = \mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty \cap \mathscr{H}_A = \{X \in \mathscr{H}_A : XT_z^n \xrightarrow{n} 0 \text{ (SOT)}\}.$$

Proof. Due to Theorem 7 (a), only the first equality requires an argument. Moreover, their "$\supset$"-part follows from Theorem 4. For the reverse inclusion, we use the fact that a dense subset of $\mathscr{C}(\mathscr{H}_A)$ is given by operators of the form

$$X - \Phi(X) \quad \text{where } X = \prod_{i=1}^n X_i \text{ with } X_i \in T_A \cup H_A.$$

If X contains at least one Hankel factor then $\Phi(X) = 0$ and the representation $X = \sum_{j=1}^m T_j H_j$ of Lemma 1 shows that $X{T_z}^n = \sum_{j=1}^m T_j T_{\bar{z}}^n H_j \xrightarrow{n} 0$, as desired. So it remains to consider the case where X consists solely of Toeplitz factors. But then $X - \Phi(X) = X - T_\sigma(X)$ can be shown to belong to $\mathscr{C}(\mathscr{H}_A)$ as in the proof of Theorem 4. □

In view of Theorem 4 and the above corollary we have the inclusions

$$\mathscr{H}_{L^\infty} = T_{L^\infty} + \mathscr{C}(\mathscr{H}_{L^\infty}) \subset T_{L^\infty} + (\mathscr{T}_0^\infty \cap \mathscr{H}_0^\infty) = \mathscr{T}^\infty \cap \mathscr{H}^\infty.$$

This shows that the full Hankel algebra $\mathscr{H}_{L^\infty}$ is contained in $\mathscr{T}^\infty \cap \mathscr{H}^\infty$.

10 Corollary. *The identity $\mathscr{H}_{L^\infty} \cap \mathscr{H}_0^\infty = T_{H^\infty} \oplus C^*(H_{L^\infty})$ holds.*

Proof. Given $X \in \mathscr{H}_{L^\infty} \cap \mathscr{H}_0^\infty$, Theorem 4 and the subsequent remark yield a decomposition $X = \Phi(X) + (1 - \Phi)(X) \in T_{L^\infty} \oplus C^*(H_{L^\infty})$. In view of Proposition 6, the inclusion "$\subset$" of the asserted identity holds. The reverse inclusion follows form Theorem 7 (a) and (b). □

§4 Derivations on Hankel algebras

Theorem 4 can be applied to study the structure of derivations on Hankel algebras. Unsurprisingly our results are limited by the lack of a concrete description of the commutator ideal. In [5] similar arguments were used to describe the H^1-group for a certain class of Toeplitz algebras. In our context we can at least show that the derivations of $\mathscr{H}_A$ all map into the commutator ideal. As usual, we write $[X, Y]$ to denote the commutator $XY - YX$ of two operators $X, Y \in B(H^2)$.

11 Proposition. *For a closed subalgebra $A \subset L^\infty$ strictly containing H^∞ and a (not necessarily continuous) map $D : \mathscr{H}_A \to \mathscr{H}_A$, the following assertions are equivalent:*

(a) *D is a derivation on $\mathscr{H}_A$.*

(b) *The map D has the form $D = [S, \cdot]$ for some operator $S \in B(H^2)$ satisfying $SX - XS \in \mathscr{C}(\mathscr{H}_A)$ for every $X \in \mathscr{H}_A$. In particular,*

$$(SX - XS)T_z^n \xrightarrow{n} 0 \text{ (SOT)} \quad \text{for all } X \in \mathscr{H}_A.$$

(c) *There is an operator $S \in B(H^2)$ such that $D = [S, \cdot]$ and the commutators $[S, T_\theta]$, $[S, T_{\overline{\eta}}]$ and $[S, H_{\overline{\eta}}]$ all belong to $\mathscr{H}_A$ whenever $\theta \in H^\infty$ is inner and $\overline{\eta} \in A$, η inner.*

Proof. $(a) \Rightarrow (b)$. The hypotheses on A guarantee that A is an inner subalgebra which contains $H^\infty + C(\mathbb{T})$ (see the remarks concerning inner subalgebras in the introduction and Corollary 6.40 in [8]). Therefore, $\mathscr{H}_A$ contains the ideal of all compact operators $\mathcal{K}(H^2)$. Then a theorem of Chernoff ([4], Corollary 3.4) says that each derivation $D : \mathscr{H}_A \to \mathscr{H}_A$ has the form $D = [S, \cdot]$ for some operator $S \in B(H^2)$. In particular, D is continuous. Our next aim is to show that D maps the commutator ideal into itself: If we choose any $X, Y \in \mathscr{H}_A$, then we have

$$\begin{aligned} D(XY - YX) &= D(X)Y + XD(Y) - D(Y)X - YD(X) \\ &= (D(X)Y - YD(X)) + (XD(Y) - D(Y)X) \in \mathscr{C}(\mathscr{H}_A). \end{aligned}$$

Using the fact that $\mathscr{C}(\mathscr{H}_A)$ is generated by elements of the form $S[X, Y]T$ with $X, Y, S, T \in \mathscr{H}_A$, the derivation identity, the continuity of D and the fact that $\mathscr{C}(\mathscr{H}_A)$ is an ideal imply that $D\mathscr{C}(\mathscr{H}_A) \subset \mathscr{C}(\mathscr{H}_A)$, as desired. Hence the induced map

$$\widehat{D} : \mathscr{H}_A/\mathscr{C}(\mathscr{H}_A) \to \mathscr{H}_A/\mathscr{C}(\mathscr{H}_A), \qquad [X] \mapsto [D(X)]$$

is a well-defined continuous derivation. By Theorem 4, the quotient is isometrically isomorphic to $A \subset L^\infty$ and hence a commutative and semi-simple Banach algebra. Therefore, the Singer-Wermer theroem says that $\widehat{D} = 0$ or, equivalently, $D(\mathscr{H}_A) \subset \mathscr{C}(\mathscr{H}_A)$. This observation completes the the proof of the first implication. The second one, $(b) \Rightarrow (c)$, is trivial. For $(c) \Rightarrow (a)$ we have to show that the operator $D = [S, \cdot]$ maps $\mathscr{H}_A$ into $\mathscr{H}_A$. Since $\mathscr{H}_A$ is the closed linear hull of all finite products $X = X_1 \cdots X_n$ with $X_i \in T_A \cup H_A$ and since, for each such product X, the commutator $SX - XS$ can be written as a telesoping sum of the form

$$(SX_1 - X_1S)X_2 \cdots X_n + X_1(SX_2 - X_2S)X_3 \cdots X_n + \ldots + X_1 \cdots X_{n-1}(SX_n - X_nS),$$

it suffices to prove that $SX - XS \in \mathscr{H}_A$ for each single factor $X \in T_A \cup S_A$. Towards this, fix $\overline{\eta}, \varphi \in A$ with η inner and $\varphi \in H^\infty$. Then we have

$$ST_{\overline{\eta}\varphi} - T_{\overline{\eta}\varphi}S = (ST_{\overline{\eta}} - T_{\overline{\eta}}S)T_\varphi + T_{\overline{\eta}}(ST_\varphi - T_\varphi S).$$

The first summand belongs to $\mathscr{H}_A$ by hypothesis. In view of a theorem of Marshall (see [12]) saying that the closed linear hull of all inner functions is norm-dense in H^∞, we see that the hypothesis also guarantees that the second summand belongs to $\mathscr{H}_A$. So finally $ST_f - T_fS \in \mathscr{H}_A$ for all $f \in A$, since the assumption on A implies that A is inner. Next, consider X to be a Hankel operator of the form $H_{\overline{\eta}\varphi} = PJM_{z\overline{\eta}}M_\varphi | H^2 = H_{\overline{\eta}}T_\varphi$. Then, again by hypothesis,

$$SH_{\overline{\eta}\varphi} - H_{\overline{\eta}\varphi}S = (SH_{\overline{\eta}} - H_{\overline{\eta}}S)T_\varphi + H_{\overline{\eta}}(ST_\varphi - T_\varphi S) \in \mathscr{H}_A.$$

Since A is inner, this observation finishes the proof. □

In view of [5], a quite obvious condition on a derivation of $\mathscr{H}_A$ to be inner is to map into $\mathcal{K}(H^2)$. More precisely, we have the following result.

12 Corollary. *Let A be as above and $D : \mathscr{H}_A \to \mathscr{H}_A$ be a derivation. Then the following assertions are equivalent:*

(a) D leaves the Toeplitz algebra $\mathscr{T}_{H^\infty + C}$ invariant.

(b) $D(T_\eta) \in \mathcal{K}(H^2)$ for every inner function $\eta \in H^\infty$.

(c) $D = [S, \cdot]$ on $\mathscr{H}_A$ with $S = T_f + K$ where $f \in H^\infty + C$ and $K \in \mathcal{K}(H^2)$.

Proof. It is well known that every element of $\mathscr{T}_{H^\infty + C}$ can be written in the form $T_f + K$ with $f \in H^\infty + C$ and $K \in \mathcal{K}(H^2)$ (see Theorem 7.29 in [8]). In view of the preceding corollary and the fact that $\mathscr{C}(\mathscr{H}_A) \cap T_A = (0)$ by Theorem 4, we have the inclusion $D(\mathscr{T}_{H^\infty + C}) \subset \mathscr{T}_{H^\infty + C} \cap \mathscr{C}(\mathscr{H}_A) \subset \mathcal{K}(H^2)$ proving that (a) implies (b). Now assume that (b) holds. By the preceding proposition, there is an operator $S \in B(H^2)$ such that $D = [S, \cdot]$. By hypothesis, we have $ST_\eta - T_\eta S \in \mathcal{K}(H^2)$ for every inner function $\eta \in H^\infty$. Since the closed linear span of all inner functions is dense in H^∞ (see [12]) we deduce that S belongs to the essential commutant of all analytic Toeplitz operators. A result of Davidson (see [7]) then says that S has the form $S = T_f + K$ with $f \in H^\infty + C$ and $K \in \mathcal{K}(H^2)$. This proves (c) and obviously implies (a). □

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