Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 359

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Saarbrücken 2015

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 359
submitted: June 12, 2015

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Existence of generalized minimizers and of dual solutions for a class of variational problems with linear growth related to image recovery

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AMS Subject Classification: 49 Q 20, 49 J 45, 49 N 15
Keywords: image inpainting, denoising of images, variational methods, TV-regularization, dual variational approach

Abstract

We continue the analysis of some modifications of the total variation image inpainting method formulated on the space $BV(\Omega)^M$ in the sense that we generalize some of the main results of [13] to the case of vector-valued images where now we do not impose any structure condition on our density F and the dimension of the domain Ω is arbitrary. Precisely we discuss existence of generalized solutions of the corresponding variational problem and we will also pass to the associated dual variational problem for which we show unique solvability. Among other things, our results are the uniqueness of the absolutely continuous part $\nabla^a u$ of the gradient of BV-solutions u on the entire domain Ω where outside of the damaged region D we even get uniqueness of BV-solutions. As remarkable byproducts we further prove new density results for functions of bounded variation and for Sobolev functions.

1 Introduction

In this note we continue the analysis of some perturbations of the total variation image inpainting model started in [13] from a more theoretical point of view. To become precise we assume that we are given a bounded Lipschitz domain Ω in $\mathbb{R}^n$ with $n \geq 2$ (e.g. a rectangle in the case $n = 2$ or a cuboid in the case $n = 3$) and a $\mathcal{L}^n$-measurable subset D of Ω ($\mathcal{L}^n$ denoting Lebesgue's measure on $\mathbb{R}^n$) satisfying

$$(1.1) \qquad 0 \leq \mathcal{L}^n(D) < \mathcal{L}^n(\Omega).$$

We suppose further that we are given an observed image described through a measurable function $f : \Omega - D \to \mathbb{R}^M$ where we require

$$(1.2) \qquad f \in L^2(\Omega - D)^M.$$

Roughly speaking, the "inpainting domain" D (compare [18]) represents a certain part of this image for which image data are missing or inaccessible and our aim is to restore this missing part from the part which is known, i.e. to generate an image $u : \Omega \to \mathbb{R}^M$ based on the partial observation $f : \Omega - D \to \mathbb{R}^M$.

There are various types of images depending on the dimension of the domain or of the codomain, respectively. In case $n = 2, M = 1$ we are concerned with a classical digital image, i.e. its co-domain specifies the corresponding grey value where normally low grey levels are dark and high grey levels are bright (see, e.g., [2, 31]). The case $n = 3, M = 1$ covers three-dimensional images that are of fundamental meaning in medical imaging, e.g. computerized tomography or magnetic resonance imaging (see, e.g., [30, 31] and the references quoted therein). Examples of vector-valued images are coloured images where each channel (or dimension) represents a corresponding colour (see, e.g., [9] and the references quoted therein).
The kind of image interpolation described above at least in the case $n = 2$ and $M = 1$ is called "inpainting" or "image inpainting" , respectively (compare [18, 34, 35]). There are various different techniques to handle the inpainting problem being of variational or non-variational and of local or non-local kind (see, e.g., [6, 8, 18, 19, 20, 21, 23, 34, 35] and the references quoted therein) where in this note we will concentrate on a TV-like variational approach being of non-local type as proposed in [12, 13, 14, 15, 17]. In most cases one considers the functional

$$J[w] := \int_{\Omega} \psi(|\nabla w|)dx + \frac{\lambda}{2} \int_{\Omega - D} |w - f|^2 dx, \tag{1.3}$$

where λ is a positive regularization parameter and ψ is supposed to be a convex and increasing function with non-negative values. The second term on the right-hand side of (1.3) measures the quality of data fitting, i.e. the deviation of the original image u from the given data f on $\Omega - D$ while the first term allows to incorporate some kind of apriori information of the generated image via some kind of mollification on the entire domain Ω into the minimization process. Note that in the case $\mathcal{L}^n(D) = 0$ the problem reduces to "pure denoising".
In this setup, a common choice of ψ is $\psi(|\nabla u|) := |\nabla u|$ leading to the total variation inpainting model (compare [7, 34]). To discuss this variational problem, one has to work with L^1-functions $\Omega \to \mathbb{R}^M$ of bounded variation, i.e. in the space $BV(\Omega)^M$. In this situation, ∇u denotes the distributional gradient which is respresented by a tensor-valued Radon measure on Ω with finite total variation $\int_{\Omega} |\nabla u|$ (for details we refer to [4] or [28]).
In this paper we follow the basic idea of [13], i.e. we replace the unpleasant quantity $\int_{\Omega} |\nabla u|$ through a functional $\int_{\Omega} F(\nabla u)$ with density F of linear growth being strictly convex w.r.t. the tensor-valued measure ∇u and investigate solvability of the problem

$$\int_{\Omega} F(\nabla w) + \frac{\lambda}{2} \int_{\Omega - D} |w - f|^2 dx \to \min \quad \text{in } BV(\Omega)^M \cap L^2(\Omega - D)^M. \tag{1.4}$$

Furthermore, we will pass to the dual variational problem for which we prove unique solvability under rather weak assumptions and establish a surprising

compactness property for minimizing sequences of the functional I introduced below (see (1.9)).
At this stage we want to emphasize the main difference between the investigations carried out in [13] and the situation being under consideration in the present setting: if Ω is a domain in $\mathbb{R}^2$ as required in [13], then the standard embedding theorem for BV-functions (see, e.g. [4], Corollary 3.49, p.152) implies that $BV(\Omega)^M$ is continuously embedded in the space $L^2(\Omega)^M$, which means that the problem (1.4) can be studied just on the whole space $BV(\Omega)^M$. In case $n \geq 3$ this is no longer true: the requirement "$u \in L^2(\Omega - D)^M$" acts as an additional constraint, and one major effort outlined in this paper consists in adjusting the arguments from [13] to the higherdimensional case.

Now, we like to fix our setup and state our precise assumptions: suppose that we are given a function $F : \mathbb{R}^{nM} \to [0, \infty)$ being of class $C^1(\mathbb{R}^{nM})$ satisfying the following hypotheses

$$\text{F is strictly convex}, F(0) = 0, \tag{1.5}$$

$$|DF(P)| \leq \nu_1, \tag{1.6}$$

$$F(P) \geq \nu_2|P| - \nu_3 \tag{1.7}$$

with constants $\nu_1, \nu_2 > 0, \nu_3 \in \mathbb{R}$, for all $P \in \mathbb{R}^{nM}$. From (1.6) and using $F(0) = 0$ we immediately obtain

$$F(P) \leq \nu_1|P|$$

for all $P \in \mathbb{R}^{nM}$ which shows that F is of linear growth in the following sense

$$\nu_2|P| - \nu_3 \leq F(P) \leq \nu_1|P|. \tag{1.8}$$

As a starting point we then look at the variational problem

$$\begin{aligned} I[w] := \int_\Omega F(\nabla w)dx + \frac{\lambda}{2} \int_{\Omega - D} |w - f|^2 dx \to \min \\ \text{in } W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M. \end{aligned} \tag{1.9}$$

As a matter of fact, in general one can not expect the solvability of (1.9) in the non-reflexive Sobolev space $W^{1,1}(\Omega)^M$ (compare, e.g., [1] for a definition of these classes), some exceptional cases are discussed in [12] depending on the structure of F. So the question arises how to give a reasonable extension and an interpretation of problem (1.9) in the setting of the more adequate function space $BV(\Omega)^M$. A natural and established approach is to use the concept of convex functions of a measure (see, e.g., [5, 22] or [26]), i.e. we let for $w \in BV(\Omega)^M \cap L^2(\Omega - D)^M$

$$K[w] := \int_\Omega F(\nabla^a w)dx + \int_\Omega F^\infty\Big(\frac{\nabla^s w}{|\nabla^s w|}\Big)d|\nabla^s w| + \frac{\lambda}{2} \int_{\Omega - D} |w - f|^2 dx. \tag{1.10}$$

Here, we denote for vector-valued Radon measures ρ by $\rho^a(\rho^s)$ the regular (singular) part of ρ w.r.t. to Lebesgue's measure $\mathcal{L}^n$. Moreover, F^∞ is the recession function of F, i.e.

$$F^\infty(P) := \lim_{t\to\infty} \frac{F(tP)}{t}, \quad P \in \mathbb{R}^{nM}. \tag{1.11}$$

Since F is (strictly) convex and of linear growth in the sense of (1.8), it follows that F^∞ is well-defined.
Now, the idea is to seek minimizers of the relaxed variational problem

$$K \to \min \quad \text{in } BV(\Omega)^M \cap L^2(\Omega - D)^M \tag{1.12}$$

and to introduce them as generalized solutions of (1.9).

At this point, we will state our first theorem which proves solvability of problem (1.12) in $BV(\Omega)^M \cap L^2(\Omega - D)^M$. Moreover, we will show uniqueness of the absolutely continuous part $\nabla^a u$ of the gradient of BV-solutions on the whole domain Ω and will additionally verify the uniqueness of BV-solutions outside of the damaged region D. In part (c) we justify that each K-minimizer can be seen as a generalized minimizer of the original functional I while in part (d) we prove that each K-minimizer belongs to the set $\mathcal{M}$ of generalized minimizers of the functional I from (1.9) and vice versa.

Theorem 1.1
Suppose that (1.1) *holds and let F satisfy* (1.5)–(1.7). *Further we assume the validity of* (1.2). *Then it holds:*

(a) Problem (1.12) *admits at least one solution.*

(b) Suppose that u and $\widetilde{u}$ are K-minimizing. Then

$$u = \widetilde{u} \text{ a.e. on } \Omega - D \quad \text{and} \quad \nabla^a u = \nabla^a \widetilde{u} \text{ a.e. on } \Omega.$$

(c) $\displaystyle\inf_{W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M} I = \inf_{BV(\Omega)^M \cap L^2(\Omega-D)^M} K.$

(d) Let $\mathcal{M}$ denote the set of all $L^1(\Omega)^M$-cluster points of I-minimizing sequences from the space $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$. Then $\mathcal{M}$ coincides with the set of all K-minimizers from the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$.

Taking into account assertion (b) of Theorem 1.1 we may derive the uniqueness in case of $W^{1,1}$-solvability. Moreover, in the general case, the $L^{\frac{n}{n-1}}$-deviation $\|u-v\|_{L^{\frac{n}{n-1}}}$ of different solutions u, v on the inpainting region can be estimated in terms of $\nabla^s(u-v)$, i.e. it is governed by the total variation of the singular part $\nabla^s(u-v)$ of the tensor-valued Radon measure $\nabla(u-v)$. In case $n = 2, M = 1$ these results can be found in [13], precisely we have:

Corollary 1.1*(a) If there exists $u \in \mathcal{M}$ such that $u \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$, then it follows $\mathcal{M} = \{u\}$.*

(b) Suppose that $\overline{D} \subset \Omega$. Then there is a constant $c = c(n, M)$ such that for $u, v \in \mathcal{M}$ it holds

$$\|u - v\|_{L^{\frac{n}{n-1}}(\Omega)} = \|u - v\|_{L^{\frac{n}{n-1}}(D)} \leq c|\nabla^s(u - v)|(\overline{D})$$

In particular, the constant c on the right-hand side is not depending on the free parameter λ.

Remark 1.1
For the proof of Corollary 1.1 we just note that that Corollary 1.1 in [13] extends to any dimension $n \geq 2$, furthermore, the statements remain valid for vector-valued functions, i.e. for the case $M \geq 2$. The corresponding references are given during the proof of [13], Corollary 1.1.

Remark 1.2

- *Part (b) of Theorem 1.1 shows uniqueness of solutions on $\Omega - D$ and the measures ∇u and $\nabla \widetilde{u}$ of minima $u, \widetilde{u}$ may only differ in their singular parts.*
- *The statements (c) and (d) in Theorem 1.1 reveal that the minimization of K in $BV(\Omega)^M \cap L^2(\Omega - D)^M$ represents a natural extension of the original variational problem (1.9) which in general fails to have solution in $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$. Furthermore, it holds $I = K$ on $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$ and this fact also stresses that we have a reasonable extension of the functional I.*
- *We like to mention that in the case $n = 2$ and $M = 1$, problem (1.9) has been studied extensively in [12, 13, 14, 15, 17]. For the analysis of pure denoising in the case $n = 2$ we like to refer to [11] where also vector-valued images have been discussed and additional boundary data could be included.*

Remark 1.3
The assumptions on our density F in Theorem 1.1 can be weakened in such a way that we just require that F is strictly convex and of linear growth in the sense of (1.8).

Motivated by the dual variational formulation of problems in the theory of plasticity (see [25] for a survey), another approach to problem (1.9) seems to be more natural. An essential motivation for studying dual variational problems is the uniqueness of solutions (for more detailed information we refer to section 2.2 in [10]), moreover, the dual solution σ usually admits a clear geometric or physical interpretation. For instance, we can note that in the theory of minimal surfaces, σ corresponds to the normal of the surface and in the theory of plasticity, σ represents the stress tensor. Nevertheless it should be emphasized

that we do not know an adequate interpretation of the dual solution σ in the context of image processing.
Let F satisfy (1.5)–(1.7) and suppose that (1.1) as well as (1.2) hold. Following [24] we define the Lagrangian for functions $w \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$ and $\kappa \in L^\infty(\Omega)^{nM}$ by the equation

$$l(w,\kappa) := \int_\Omega [\kappa : \nabla w - F^*(\kappa)]dx + \frac{\lambda}{2} \int_{\Omega - D} |w - f|^2 dx.$$

Here

$$F^*(Q) := \sup_{P \in \mathbb{R}^{nM}} [P : Q - F(P)], \quad Q \in \mathbb{R}^{nM},$$

represents the conjugate function to F. Quoting [24], Proposition 2.1, p.271, we obtain the representation

$$\int_\Omega F(P)dx = \sup_{\kappa \in L^\infty(\Omega)^{nM}} \int_\Omega [\kappa : P - F^*(\kappa)]dx \tag{1.13}$$

for functions $P \in L^1(\Omega)^{nM}$ and this leads to another formula for the functional I. Precisely, we get

$$I[w] = \sup_{\kappa \in L^\infty(\Omega)^{nM}} l(w,\kappa), \quad w \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M, \tag{1.14}$$

and by virtue of (1.14) we can introduce the dual functional

$$R : L^\infty(\Omega)^{nM} \to [-\infty, \infty],$$
$$R[\kappa] := \inf_{w \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M} l(w,\kappa).$$

Consequently, the dual problem is: to maximize R among all functions $\kappa \in L^\infty(\Omega)^{nM}$.

In the following theorem we summarize our results on the dual variational problem.

Theorem 1.2
Suppose that (1.1) *and* (1.2) *hold. Further let* F *satisfy* (1.5)–(1.7). *Then it holds:*

(a) The dual problem

$$R \to max \quad in \quad L^\infty(\Omega)^{nM}$$

admits at least one solution. Moreover, the inf-sup *relation*

$$\inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M} I[v] = \sup_{\sigma \in L^\infty(\Omega)^{nM}} R[\sigma]$$

is valid.

(b) We have uniqueness of the dual solution if the conjugate function F^* is strictly convex on the set $\{p \in \mathbb{R}^{nM} : F^*(p) < \infty\}$.

(c) Consider any I-minimizing sequence (u_m) from the space $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$. Then it holds

$$u_m \to u \quad \textit{in } L^2(\Omega - D)^M,$$

where u is the unique restriction of any generalized minimizer $\overline{u}$ from Theorem 1.1 to the set $\Omega - D$.

Actually the additional hypothesis imposed on F^* in assertion (b) of Theorem 1.2 can be removed. More precisely it holds:

Theorem 1.3
Let (1.1), (1.2) *hold and assume that we have* (1.5)–(1.7) *for the density F. Then the dual problem*

$$R \to \textit{max} \quad \textit{in } L^\infty(\Omega)^{nM}$$

admits a unique solution σ. We further have the duality formula

$$\sigma = DF(\nabla^a u) \quad \textit{a.e. on } \Omega,$$

where u denotes any K-minimizer from the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$.

We finish the introduction by adding some comments: assume that the density $F : \mathbb{R}^{nM} \to [0, \infty)$ is of the form

$$F(P) = \Phi(|P|), \quad P \in \mathbb{R}^{nM},$$

with $\Phi : [0, \infty) \to [0, \infty)$ of class C^2 satisfying the conditions $(1.3^*) - (1.4^*)$ and (1.4^*_μ) from [13] with exponent $\mu > 1$, e.g.

$$\Phi(t) := \Phi_\mu(t) := \int_0^t \int_0^s (1 + r)^{-\mu} \, dr \, ds, \quad t \geq 0.$$

Suppose further that the data f are from the space $L^\infty(\Omega - D)^M$. Then Theorem 1.1 can be proved completely along the lines of the proof of Theorem 1.2 in [13] without referring to the density results stated in Section 2 by observing that K-minimizing sequences (u_m) can be chosen in such a way that

$$\sup_\Omega |u_m| \leq \sup_{\Omega - D} |f|$$

holds yielding compactness of (u_m) in $BV(\Omega)^M$. In the same spirit, the arguments used during the proof of Theorem 1.4 in [13] now directly imply Theorem 1.2. Moreover, we have uniqueness of the dual solution.

Another important question concerns the regularity of K-minimizers under the particular assumptions mentioned before: in [12] and [17] a satisfying answer is given for the scalar case ($M = 1$) in two dimensions ($n = 2$). In the forthcoming paper [36], it is shown that these regularity results extend to any dimension n and arbitrary codimension M by using much more elaborate arguments.

Our paper is organized as follows: in Section 2 we prove the density of smooth functions in spaces like $BV(\Omega)^M \cap L^2(\Omega - D)^M$. Using these results, generalized solutions are studied in Section 3. In Section 4 we investigate the dual variational problem, whose uniqueness is established in Section 5.

2 Some density results

In this section we provide some approximation results valid for Sobolev functions and for functions of BV-type.

Lemma 2.1
Let $\Omega \subset \mathbb{R}^n$ denote a bounded Lipschitz domain and consider a measurable subset D of Ω such that $\mathcal{L}^n(D) < \mathcal{L}^n(\Omega)$. Consider $p \in [1, n)$ and let $q \in (\frac{np}{n-p}, \infty)$. Suppose further that $u \in W^{1,p}(\Omega)^M \cap L^q(\Omega - D)^M$ is given. Then there exists a sequence $(u_k) \subset C^\infty(\overline{\Omega})^M$ such that (as $k \to \infty$)

$$\|u_k - u\|_{W^{1,p}(\Omega)} + \|u_k - u\|_{L^q(\Omega - D)} \to 0. \tag{2.1}$$

Remark 2.1
By the continuity of Sobolev's embedding $W^{1,p}(\Omega)^M \hookrightarrow L^{\frac{np}{n-p}}(\Omega)^M$ (see, e.g., [1], Theorem 5.4, p.97/98), our choice of q is reasonable, since otherwise we may directly apply [1], Theorem 3.18, p.54.

Lemma 2.2
With Ω and D as in Lemma 2.1 consider $u \in BV(\Omega)^M \cap L^q(\Omega - D)^M$ for some $q \in (\frac{n}{n-1}, \infty)$. Then there exists a sequence $(u_m) \subset C^\infty(\overline{\Omega})^M$ such that (as $m \to \infty$)

(i) $u_m \to u$ *in* $L^1(\Omega)^M$,

(ii) $u_m \to u$ *in* $L^q(\Omega - D)^M$,

(iii) $\int_\Omega \sqrt{1 + |\nabla u_m|^2} dx \to \int_\Omega \sqrt{1 + |\nabla u|^2}$.

Remark 2.2
According to the embedding $BV(\Omega)^M \hookrightarrow L^{\frac{n}{n-1}}(\Omega)^M$ valid for "bounded extension domains" Ω (see, e.g., [4], Corollary 3.49, p.152) it makes sense to consider exponents $q > \frac{n}{n-1}$ in Lemma 2.2.

Remark 2.3
In place of (iii) *we will just prove the validity of*

$$(iii)^* \qquad \int_\Omega |\nabla u_m| dx \to \int_\Omega |\nabla u| = |\nabla u|(\Omega) \quad \text{as } m \to \infty,$$

where -as usual- the symbol $|\nabla u|(\Omega)$ *denotes the total variation of the tensor-valued signed Radon measure* ∇u *on* Ω. *Clearly* $(iii)^*$ *is not enough to quote the continuity theorem of Reshetnyak as stated for example in [27], Theorem 2, p.92, or [5], Proposition 2.2. However, recalling the definition of the quantity* $\int_\Omega \sqrt{1+|\nabla u|^2}$ *given in e.g. [5], Definition 2.1, or [22], p.675, it is not hard to check that the sequence* (u_m) *constructed during the proof of Lemma 2.2 actually satisfies (iii). We leave the details to the reader.*

Let us now come to the

Proof of Lemma 2.1. Let us choose a smooth bounded domain $\widetilde{\Omega}$ such that $\Omega \Subset \widetilde{\Omega}$. According to [33], Remark 1.60, p.34, we may extend $u \in W^{1,p}(\Omega)^M$ to a function $\widetilde{u} \in W^{1,p}(\widetilde{\Omega})^M$ (compare also [3], Fortsetzungssatz A 5.12, p.174). For $m \in \mathbb{N}$ we let $\Phi_m : \mathbb{R}^M \to \mathbb{R}^M$,

$$\Phi_m(y) := \begin{cases} y, & |y| \le m \\ m\frac{y}{|y|}, & |y| \ge m \end{cases}$$

and claim for the sequence $\widetilde{u}_m := \Phi_m \circ \widetilde{u}$ the validity of (as $m \to \infty$)

$$\|\widetilde{u}_m - u\|_{L^q(\Omega - D)} \to 0, \tag{2.2}$$

$$\|\widetilde{u}_m - \widetilde{u}\|_{W^{1,p}(\widetilde{\Omega})} \to 0. \tag{2.3}$$

In fact, from $|\widetilde{u}_m - u| \le 2|u|$ a.e. on $\Omega - D$ together with $\widetilde{u}_m \to \widetilde{u}$ a.e. on $\widetilde{\Omega}$ it follows by dominated convergence that (2.2) is true (recall our assumption $u \in L^q(\Omega - D)^M$). In the same way we obtain $\widetilde{u}_m \to \widetilde{u}$ in $L^p(\widetilde{\Omega})^M$. The chain rule in its general form (see, e.g., [4], Theorem 3.96, p. 189) shows $\widetilde{u}_m \in W^{1,p}(\widetilde{\Omega})^M$ together with $|\nabla \widetilde{u}_m| \le \mathrm{Lip}(\Phi_m)|\nabla \widetilde{u}| = |\nabla \widetilde{u}|$.
We wish to remark that the crucial estimate $|\nabla \widetilde{u}_m| \le \mathrm{Lip}(\Phi_m)|\nabla \widetilde{u}|$ in a slightly weaker form occurs in the paper [32], a complete proof of the inequality can be found in [16], Lemma B.1.
From $\widetilde{u}_m = \widetilde{u}$ a.e. on $\{x \in \widetilde{\Omega} : |\widetilde{u}(x)| \le m\} =: \widetilde{\Omega}_m$ it follows that $\nabla \widetilde{u}_m = \nabla \widetilde{u}$ on $\widetilde{\Omega}_m$ (see [29], Lemma 7.7, p.145), in particular we get $\nabla \widetilde{u}_m \to \nabla \widetilde{u}$ a.e. on $\widetilde{\Omega}$, and $\|\nabla \widetilde{u}_m - \nabla \widetilde{u}\|_{L^p(\widetilde{\Omega})} \to 0$ again is a consequence of dominated convergence.
According to (2.2) and (2.3) we find a subsequence $(\widetilde{u}_{m_k})$, $k \in \mathbb{N}$, such that

$$\|\widetilde{u}_{m_k} - \widetilde{u}\|_{W^{1,p}(\widetilde{\Omega})} + \|\widetilde{u}_{m_k} - u\|_{L^q(\Omega - D)} \le \frac{1}{k} \tag{2.4}$$

for any $k \in \mathbb{N}$. In a next step we consider a suitable sequence of radii $\rho_k \downarrow 0$ such that ($(\cdot)_{\rho_k}$ denoting the mollification operator)

$$\|\widetilde{u}_{m_k} - (\widetilde{u}_{m_k})_{\rho_k}\|_{W^{1,p}(\Omega)} + \|\widetilde{u}_{m_k} - (\widetilde{u}_{m_k})_{\rho_k}\|_{L^q(\Omega)} \leq \frac{1}{k} \tag{2.5}$$

for each integer k. In order to get (2.5) we just observe that from $\widetilde{u}_{m_k} \in W^{1,p}(\widetilde{\Omega})^M \cap L^q(\widetilde{\Omega})^M$ (it actually holds $\widetilde{u}_{m_k} \in L^\infty(\widetilde{\Omega})^M$) and by recalling $\Omega \Subset \widetilde{\Omega}$ it follows $(\widetilde{u}_{m_k})_\rho \to \widetilde{u}_{m_k}$ as $\rho \downarrow 0$ in $W^{1,p}(\Omega)^M \cap L^q(\Omega)^M$. Obviously the functions $u_k := (\widetilde{u}_{m_k})_{\rho_k}$ belong to the class $C^\infty(\overline{\Omega})^M$, and (2.1) is a consequence of (2.4) and (2.5). $\square$

Remark 2.4
For future applications to higher order problems it would be desirable to prove a variant of Lemma 2.1 for the case $u \in W^{k,p}(\Omega)^M \cap L^q(\Omega - D)^M$ *with* $k \geq 2$, $q > \frac{np}{n-kp}$, $p \in [1, \frac{n}{k})$.

Proof of Lemma 2.2. Let $u_0 \in L^1(\partial\Omega)^M$ denote the trace of the given function $u \in BV(\Omega)^M \cap L^q(\Omega - D)^M$ whose properties are summarized in e.g. [4], Theorem 3.87, p.180/181. With $\widetilde{\Omega}$ as in the proof of Lemma 2.1 we let $u_0 := 0$ on $\partial\widetilde{\Omega}$, thus $u_0 \in L^1(\partial G)^M$ where $G := \widetilde{\Omega} - \overline{\Omega}$. Since $\partial\Omega$ is Lipschitz we may refer to [28], Theorem 2.16, p.39 and can find $v \in W^{1,1}(G)^M$ having trace u_0 on ∂G satisfying in addition

$$\|v\|_{W^{1,1}(G)} \leq c\|u_0\|_{L^1(\partial G)} \tag{2.6}$$

with c depending on ∂G but independent of u_0 and v. We then let

$$\widetilde{u} := \begin{cases} u, & \text{on } \Omega \\ v, & \text{on } \widetilde{\Omega} - \overline{\Omega} \end{cases}$$

and observe $\widetilde{u} \in BV(\widetilde{\Omega})^M$, which follows from [4], Corollary 3.89, p.183, and the fact that (2.6) implies $v \in BV(G)^M$. Viewing ∇u (resp. ∇v) as measures on $\widetilde{\Omega}$ concentrated on Ω (resp. $\widetilde{\Omega} - \overline{\Omega}$) and recalling the definition of v we further deduce from the above reference the identity

$$\nabla\widetilde{u} = \nabla u + \nabla v \tag{2.7}$$

as measures on $\widetilde{\Omega}$. As in the proof of Lemma 2.1 we finally let ($m \in \mathbb{N}$)

$$\widetilde{u}_m := \Phi_m \circ \widetilde{u}$$

and observe (compare the first part of the proof of Theorem 3.96 on p.189 in [4])

$$\widetilde{u}_m \in BV(\widetilde{\Omega})^M, \quad |\nabla\widetilde{u}_m| \leq \mathrm{Lip}(\Phi_m)|\nabla\widetilde{u}| = |\nabla\widetilde{u}|, \tag{2.8}$$

which means $|\nabla\widetilde{u}_m|(E) \leq |\nabla\widetilde{u}|(E)$ for any Borel set $E \subset \widetilde{\Omega}$. In particular, from $|\nabla\widetilde{u}|(\partial\Omega) = 0$ (recall (2.7)) it follows that

$$|\nabla\widetilde{u}_m|(\partial\Omega) = 0, \quad m \in \mathbb{N}. \tag{2.9}$$

As before it holds (as $m \to \infty$)

$$\widetilde{u}_m \to \widetilde{u} \quad \text{in } L^1(\widetilde{\Omega})^M, \tag{2.10}$$

$$\widetilde{u}_m \to u \quad \text{in } L^q(\Omega - D)^M, \tag{2.11}$$

and (2.10) combined with a standard lower semicontinuity result (see [28], Theorem 1.9, p.7 or [4], Proposition 3.6, p.120) implies

$$|\nabla \widetilde{u}|(\widetilde{\Omega}) \le \liminf_{m\to\infty} |\nabla \widetilde{u}_m|(\widetilde{\Omega}).$$

From (2.8) we get

$$|\nabla \widetilde{u}_m|(\widetilde{\Omega}) \le |\nabla \widetilde{u}|(\widetilde{\Omega}),$$

thus

$$|\nabla \widetilde{u}_m|(\widetilde{\Omega}) \to |\nabla \widetilde{u}|(\widetilde{\Omega}), \quad m \to \infty. \tag{2.12}$$

Clearly we can replace $\widetilde{\Omega}$ in (2.12) by the domain Ω, so that in combination with (2.10) and (2.11) it holds for a subsequence (recall that by (2.7) $|\nabla \widetilde{u}|(\Omega) = |\nabla u|(\Omega)$)

$$\begin{aligned} &\|\widetilde{u}_{m_k} - u\|_{L^1(\Omega)} + \|\widetilde{u}_{m_k} - u\|_{L^q(\Omega - D)} \\ &+ \Big| |\nabla \widetilde{u}_{m_k}|(\Omega) - |\nabla u|(\Omega) \Big| \le \frac{1}{k}, \quad k \in \mathbb{N}. \end{aligned} \tag{2.13}$$

From [28], Proposition 1.15, p.12, we see on account of (2.9) that there exists a sequence $\rho_k \downarrow 0$ such that the functions $u_k := (\widetilde{u}_{m_k})_{\rho_k}$ satisfy

$$\Big| |\nabla u_k|(\Omega) - |\nabla \widetilde{u}_{m_k}|(\Omega) \Big| \le \frac{1}{k}, \tag{2.14}$$

moreover due to the convergence $(\widetilde{u}_{m_k})_\rho \to \widetilde{u}_{m_k}$ in $L^p_{\mathrm{loc}}(\widetilde{\Omega})^M$ as $\rho \downarrow 0$ for any $p \in [1, \infty)$ we can arrange

$$\|u_k - \widetilde{u}_{m_k}\|_{L^1(\Omega)} + \|u_k - \widetilde{u}_{m_k}\|_{L^q(\Omega - D)} \le \frac{1}{k} \tag{2.15}$$

for any $k \in \mathbb{N}$. Putting together (2.13)–(2.15) and recalling Remark 2.3, we see that the sequence $(u_k) \subset C^\infty(\overline{\Omega})^M$ has the desired properties. □

3 Weak minimizers. Proof of Theorem 1.1

From now on we assume the validity of the hypotheses from Theorem 1.1. Before we start proving Theorem 1.1 we recall the following auxiliary result which can be found in [13] (compare Lemma 2.2 in this reference).

Lemma 3.1
For $w \in BV(\Omega)^M$ let

$$\widetilde{K}[w] := \int_\Omega F(\nabla^a w)dx + \int_\Omega F^\infty\Big(\frac{\nabla^s w}{|\nabla^s w|}\Big)d|\nabla^s w|.$$

(a) *Suppose that $w_m, w \in BV(\Omega)^M$ are such that $w_m \to w$ in $L^1(\Omega)^M$. Then it holds:*

$$\widetilde{K}[w] \le \lim_{m\to\infty} \widetilde{K}[w_m]. \tag{3.1}$$

(b) *If we know in addition*

$$\int_\Omega \sqrt{1+|\nabla w_m|^2} \to \int_\Omega \sqrt{1+|\nabla w|^2},$$

then it follows

$$\lim_{m\to\infty} \widetilde{K}[w_m] = \widetilde{K}[w]. \tag{3.2}$$

Remark 3.1
The reader should note that Lemma 2.2 of [13] clearly extends to any any dimension $n \ge 2$, moreover, the statement remains valid for vector-valued functions, i.e. for the case $M \ge 2$. The corresponding references are given during the proof of [13], Lemma 2.2.

Proceeding with the proof of Theorem 1.1 we first state that assertion (b) is immediate. Next, we let $(u_m) \subset BV(\Omega)^M \cap L^2(\Omega - D)^M$ denote a K-minimizing sequence. It holds

$$\sup_m \int_\Omega |\nabla u_m| < \infty, \tag{3.3}$$

$$\sup_m \int_{\Omega-D} |u_m|^2 dx < \infty \tag{3.4}$$

where (3.3) is valid on account of (1.7).
As stated on p.380 of [4] we have the inequality (compare also [33], Lemma 1.65, p. 39)

$$\int_\Omega |v - (v)_{\Omega-D}|dx \le c \int_\Omega |\nabla v| dx$$

valid for $v \in W^{1,1}(\Omega)^M$, c denoting a positive constant independent of v. By standard approximation (see, e.g. [4], Theorem 3.9, p.122) this estimate extends to $v \in BV(\Omega)^M$, thus (3.3) and (3.4) imply

$$\sup_m \int_\Omega |u_m| dx < \infty. \tag{3.5}$$

Combining (3.3) and (3.5), the BV-compactness theorem (see, e.g., [4], Theorem 3.23, p.132) gives the existence of a function $\overline{u} \in BV(\Omega)^M$ such that $u_m \to: \overline{u}$ in $L^1(\Omega)^M$ and a.e. up to a subsequence. Further, (3.4) combined with Fatou's lemma implies $\overline{u} \in L^2(\Omega - D)^M$, i.e. we have $\overline{u} \in BV(\Omega)^M \cap L^2(\Omega - D)^M$ and $K[\overline{u}]$ is well-defined.
By (3.1) we obtain

$$\widetilde{K}[\overline{u}] \leq \liminf_{m\to\infty} \widetilde{K}[u_m]$$

whereas Fatou's lemma gives

$$\int\limits_{\Omega-D} |\overline{u} - f|^2 dx \leq \liminf_{m\to\infty} \int\limits_{\Omega-D} |u_m - f|^2 dx.$$

This yields

$$\begin{aligned} K[\overline{u}] &\leq \liminf_{m\to\infty} \widetilde{K}[u_m] + \liminf_{m\to\infty} \frac{\lambda}{2} \int\limits_{\Omega-D} |u_m - f|^2 dx \\ &\leq \liminf_{m\to\infty} K[u_m] = \inf_{BV(\Omega)^M \cap L^2(\Omega-D)^M} K, \end{aligned}$$

i.e. $\overline{u}$ is K-minimizing. This shows assertion (a) of Theorem 1.1.

For proving assertion (c) we set

$$\alpha := \inf_{BV(\Omega)^M \cap L^2(\Omega-D)^M} K, \quad \beta := \inf_{W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M} I$$

and observe that $\alpha \leq \beta$ is obvious since $I = K$ on $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$. To establish the reverse inequality we fix an arbitrary K-minimizer $u \in BV(\Omega)^M \cap L^2(\Omega - D)^M$ and choose a sequence (u_m) according to Lemma 2.2. Quoting Lemma 3.1 we have $\widetilde{K}[u_m] \to \widetilde{K}[u]$, and, by Lemma 2.2 (ii) we finally get $K[u_m] \to K[u]$. This yields

$$\beta = \inf_{W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M} I \leq I[u_m] = K[u_m] \longrightarrow K[u] = \alpha$$

which shows (c).

To establish part (d) we first consider $u \in \mathcal{M}$, i.e. $u_m \to u$ in $L^1(\Omega)^M$ for an I-minimizing sequence (u_m) from $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$, where may assume in addition $u_m \to u$ a.e. on Ω. Fatou's lemma then implies

$$\int\limits_{\Omega-D} |u - f|^2 dx \leq \liminf_{m\to\infty} \int\limits_{\Omega-D} |u_m - f|^2 dx,$$

whereas by Lemma 3.1 (a)

$$\widetilde{K}[u] \leq \liminf_{m\to\infty} \widetilde{K}[u_m].$$

Thus we arrive at

$$K[u] \leq \liminf_{m\to\infty} K[u_m] = \liminf_{m\to\infty} I[u_m]$$

and the K-minimality of u follows from assertion (c).
Conversely consider a K-minimizer $u \in BV(\Omega)^M \cap L^2(\Omega - D)^M$. If we choose u_m according to Lemma 2.2 and apply Lemma 3.1 (b), we obtain (as $m \to \infty$)

$$I[u_m] = K[u_m] \to K[u].$$

This shows that (u_m) is an I-minimizing sequence for which (see Lemma 2.2 (i)) $u_m \to u$ in $L^1(\Omega)^M$. This proves $u \in \mathcal{M}$. □

4 Dual solutions. Proof of Theorem 1.2

Let the assumptions of Theorem 1.2 hold. We first like to note that a proof of assertion (a) probably can be deduced from [25], Theorem 1.2.1, p.15/16 or [24], Proposition 2.3, Chapter III, p.52. As in [13], proof of Theorem 1.4, we decide to give a more constructive proof relying on an approximation of our original variational problem (1.9) by a sequence of more regular problems admitting smooth solutions with suitable convergence properties. Consequently, this sequence might be of interest for numerical computations. To become more precise we consider for fixed $\delta \in (0,1]$ the problem

$$(4.1) \qquad I_\delta[w] := \int_\Omega F_\delta(\nabla w)dx + \frac{\lambda}{2}\int_{\Omega-D} |w-f|^2 dx \to \min \quad \text{in } W^{1,2}(\Omega)^M$$

where

$$(4.2) \qquad F_\delta(P) := \frac{\delta}{2}|P|^2 + F(P), \quad P \in \mathbb{R}^{nM}.$$

Clearly (4.1) admits at most one solution $u_\delta \in W^{1,2}(\Omega)^M$. In fact, if u_1, u_2 are solutions of (4.1), then we have $\nabla u_1 = \nabla u_2$ on Ω together with $u_1 = u_2$ on $\Omega - D$. But then $u_1 = u_2$ on Ω on account of (1.1). Next, with δ being fixed, we consider a minimizing sequence (u_m) for (4.1). It holds

$$\sup_m \|\nabla u_m\|_{L^2(\Omega)} \leq c(\delta) < \infty,$$
$$\sup_m \|\nabla u_m\|_{L^1(\Omega)} < \infty,$$
$$\sup_m \|u_m - f\||_{L^2(\Omega-D)} < \infty.$$

The quadratic variant of the Poincaré inequality from Section 3 then yields

$$\sup_m \|u_m\||_{W^{1,2}(\Omega)} < \infty,$$

so that $u_m \rightharpoondown: u_\delta$ in $W^{1,2}(\Omega)^M$ at least for a subsequence of (u_m). Standard theorems on lower semicontinuity then show that u_δ solves (4.1). From $I_\delta[u_\delta] \le I_\delta[0]$ we immediately deduce

$$
\begin{aligned}
&\sup_\delta \|\nabla u_\delta\|_{L^1(\Omega)} < \infty,\\
&\sup_\delta \|u_\delta - f\|_{L^2(\Omega - D)} < \infty, \qquad (4.3)\\
&\sup_\delta \ \delta \int_\Omega |\nabla u_\delta|^2 dx < \infty,
\end{aligned}
$$

where of course the linear growth of F has been used. The Poincaré-inequality from Section 3 combined with (4.3) additionally yields

$$
\sup_\delta \|u_\delta\|_{L^1(\Omega)} < \infty. \qquad (4.4)
$$

Now, from (4.3) and (4.4) it follows (at least for a suitable sequence $\delta \downarrow 0$)

$$
\begin{aligned}
&u_\delta \to: \overline{u} \quad \text{in } L^1(\Omega)^M \text{ and a.e.,}\\
&u_\delta \rightharpoondown \overline{u} \quad \text{in } L^2(\Omega - D)^M
\end{aligned}
$$

for a function $\overline{u} \in BV(\Omega)^M$, and the weak $L^2(\Omega - D)^M$-convergence additionally implies

$$
\int_{\Omega - D} |\overline{u} - f|^2 dx \le \liminf_{\delta \downarrow 0} \int_{\Omega - D} |u_\delta - f|^2 dx.
$$

Altogether it is shown that we obtain a limit function $\overline{u}$ from the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$.

Let

$$
\tau_\delta := DF(\nabla u_\delta) \quad \text{and} \quad \sigma_\delta := DF_\delta(\nabla u_\delta) = \delta \nabla u_\delta + \tau_\delta \qquad (4.5)
$$

and observe that (4.3) implies

$$
||\delta \nabla u_\delta||^2_{L^2(\Omega)} = \delta \Big(\delta \int_\Omega |\nabla u_\delta|^2 dx \Big) \to 0 \quad \text{as } \delta \to 0, \qquad (4.6)
$$

whereas (1.6) shows that τ_δ is uniformly bounded w.r.t. δ, i.e.

$$
\sup_\delta ||\tau_\delta||_{L^\infty(\Omega)} < \infty. \qquad (4.7)
$$

After passing to suitable sequences $\delta \to 0$ we get from (4.5)–(4.7)

$$
\sigma_\delta \rightharpoondown: \sigma \text{ in } L^2(\Omega)^{nM} \quad \text{and} \quad \tau_\delta \overset{*}{\rightharpoondown}: \tau \text{ in } L^\infty(\Omega)^{nM} \qquad (4.8)
$$

and by combining (4.8) with (4.6), it follows $\sigma = \tau$.

We claim that $\sigma \in L^\infty(\Omega)^{nM}$ is a solution of the dual variational problem. To justify this we first observe that u_δ solves the Euler equation

$$\int_\Omega \tau_\delta : \nabla\varphi dx + \delta \int_\Omega \nabla u_\delta : \nabla\varphi dx + \lambda \int_{\Omega-D} (u_\delta - f)\cdot\varphi dx = 0 \tag{4.9}$$

for all $\varphi \in W^{1,2}(\Omega)^M$.

Note that (4.9) exactly corresponds to (3.8) in [13], and as done there (compare (3.10) in [13]) we can use (4.9) to deduce

$$\begin{aligned} \sup_{\rho\in L^\infty(\Omega)^{nM}} R[\rho] \le & \inf_{v\in W^{1,1}(\Omega)^M\cap L^2(\Omega-D)^M} I[v] \le I[u_\delta] \le I_\delta[u_\delta] \\ &= -\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 dx + \int_\Omega (-F^*(\tau_\delta))dx \\ &- \frac{\lambda}{2}\int_{\Omega-D} |u_\delta|^2 dx + \frac{\lambda}{2}\int_{\Omega-D} |f|^2 dx. \end{aligned} \tag{4.10}$$

We wish to remark that the quadratic structure of the data fitting term is essential for the derivation of (4.10).

Neglecting the quantity $-\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 dx$ in (4.10) for the moment, we pass to the limit $\delta \to 0$. This gives by using upper semicontinuity of $\int_\Omega (-F^*(\cdot))dx$ w.r.t. weak-$*$ convergence and by recalling $\int_{\Omega-D} |\overline{u}|^2 dx \le \liminf_{\delta\to 0} \int_{\Omega-D} |u_\delta|^2 dx$

$$\begin{aligned} \sup_{L^\infty(\Omega)^{nM}} R \le & \inf_{W^{1,1}(\Omega)^M\cap L^2(\Omega-D)^M} I \\ & \le \int_\Omega (-F^*(\tau))dx - \frac{\lambda}{2}\int_{\Omega-D} |\overline{u}|^2 dx + \frac{\lambda}{2}\int_{\Omega-D} |f|^2 dx. \end{aligned} \tag{4.11}$$

Passing to the limit $\delta \to 0$ in Euler's equation (4.9) we obtain (recall (4.6), (4.8) and $u_\delta \rightharpoonup \overline{u}$ in $L^2(\Omega - D)^M$)

$$\int_\Omega \tau : \nabla\varphi dx + \lambda \int_{\Omega-D} (\overline{u} - f)\cdot\varphi dx = 0 \tag{4.12}$$

for any $\varphi \in W^{1,2}(\Omega)^M$ and by approximation, equation (4.12) extends to $\varphi \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$ (we refer to Lemma 2.1).

At the same time, it holds

$$
\begin{aligned}
R[\tau] &:= \inf_{v\in W^{1,1}(\Omega)^M\cap L^2(\Omega-D)^M} l(v,\tau)\\
&= \int_\Omega (-F^*(\tau))dx\\
&+ \inf_{v\in W^{1,1}(\Omega)^M\cap L^2(\Omega-D)^M}\Big[\int_\Omega \tau : \nabla v dx + \frac{\lambda}{2}\int_{\Omega-D} |v-f|^2 dx\Big]\\
&= \int_\Omega (-F^*(\tau))dx\\
&+ \inf_{v\in W^{1,1}(\Omega)^M\cap L^2(\Omega-D)^M}\Big[-\lambda\int_{\Omega-D} (\overline{u}-f)\cdot v dx + \frac{\lambda}{2}\int_{\Omega-D} |v-f|^2 dx\Big]\\
&= \int_\Omega (-F^*(\tau))dx\\
&+ \inf_{v\in W^{1,1}(\Omega)^M\cap L^2(\Omega-D)^M}\Big[\frac{\lambda}{2}\int_{\Omega-D} |\overline{u}-v|^2 dx + \frac{\lambda}{2}\int_{\Omega-D} |f|^2 dx - \frac{\lambda}{2}\int_{\Omega-D} |\overline{u}|^2 dx\Big]
\end{aligned}
$$

where we have used (4.12) with the admissible choice $\varphi = v$ as well as the quadratic structure of the data fitting term. As a consequence we obviously get

$$
R[\tau] \geq \int_\Omega (-F^*(\tau))dx + \frac{\lambda}{2}\int_{\Omega-D} |f|^2 dx - \frac{\lambda}{2}\int_{\Omega-D} |\overline{u}|^2 dx
$$

which implies (recall (4.11))

$$
\sup_{L^\infty(\Omega)^{nM}} R \leq \inf_{W^{1,1}(\Omega)^M\cap L^2(\Omega-D)^M} I \leq R[\tau].
$$

Hence τ is R-maximizing and the inf-sup relation is valid which proves assertion (a) of Theorem 1.2. Additionally by means of the above chain of inequalities we have shown that

$$
\delta \int_\Omega |\nabla u_\delta|^2 dx \to 0 \tag{4.13}
$$

$$
(u_\delta) \text{ is an } I-\text{minimizing sequence} \tag{4.14}
$$

at least for a subsequence $\delta_m \to 0$. Thanks to Theorem 1.1, (d) and (4.14) it further follows that $\overline{u}$ is K-minimizing in $BV(\Omega)^M \cap L^2(\Omega - D)^M$.

For assertion (b) of Theorem 1.2 we may proceed exactly as in [13], proof of Theorem 1.4. As a consequence of uniqueness the convergences (4.8) and (4.13) hold for any sequence $\delta \to 0$.

For proving Theorem 1.2 (c) we proceed similar to the proof of Theorem 1.7

in [11]: let (u_m) denote an I-minimizing sequence from the space $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$. Using the previous notation, we deduce from (4.11) and (4.12) (with admissible choice $\varphi = u_m$)

$$\inf_{W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M} I \leq \int_\Omega [\tau : \nabla u_m - F^*(\tau)]dx - \frac{\lambda}{2} \int_{\Omega-D} |\overline{u}|^2 dx + \frac{\lambda}{2} \int_{\Omega-D} |f|^2 dx + \lambda \int_{\Omega-D} (\overline{u} - f) \cdot u_m \, dx$$

where $\overline{u}, \tau$ have the same meaning as before. This gives

$$\begin{aligned} \inf_{W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M} I \leq & \int_\Omega F(\nabla u_m)dx + \frac{\lambda}{2} \int_{\Omega-D} |u_m - f|^2 dx \\ & - \frac{\lambda}{2} \int_{\Omega-D} |u_m - \overline{u}|^2 dx \\ =& I[u_m] - \frac{\lambda}{2} \int_{\Omega-D} |u_m - \overline{u}|^2 dx, \end{aligned}$$

and we obtain our claim by recalling that $\overline{u}$ is K-minimizing and that by Theorem 1.1 (b) we have uniqueness of K-minimizers on $\Omega - D$. Altogether the proof of Theorem 1.2 is complete. $\square$

5 Uniqueness of the dual solution and the duality formula: proof of Theorem 1.3

Let the assumptions of Theorem 1.3 hold and consider a K-minimizing function u from the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$, whose existence is guaranteed by Theorem 1.1. Remembering the decomposition $\nabla u = \nabla^a u \llcorner \mathcal{L}^n + \nabla^s u$ with density $\nabla^a u$ being independent of the particular minimizer (recall Theorem 1.1 (b)) we claim

Lemma 5.1
The tensor $\rho := DF(\nabla^a u)$ is a maximizer of the dual problem.

Proof of Lemma 5.1. On account of (1.6) we have that ρ is in the space $L^\infty(\Omega)^{nM}$, hence $R[\rho]$ is defined and given by

$$R[\rho] = \inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M} l(v, \rho). \tag{5.1}$$

For $v \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$ it holds

$$
\begin{aligned}
l(v,\rho) &= \int\limits_\Omega [DF(\nabla^a u) : \nabla v - F^*(DF(\nabla^a u))]dx \\
&+ \frac{\lambda}{2} \int\limits_{\Omega - D} |v-f|^2 dx \\
&= \int\limits_\Omega F(\nabla^a u)dx + \int\limits_\Omega (\nabla v - \nabla^a u) : DF(\nabla^a u)dx \\
&+ \frac{\lambda}{2} \int\limits_{\Omega - D} |v-f|^2 dx,
\end{aligned} \tag{5.2}
$$

where we have made use of the formula

$$F(P) + F^*(DF(P)) = P : DF(P), \quad P \in \mathbb{R}^{nM}.$$

Since u is K-minimizing, we get (compare (1.10))

$$0 = \frac{d}{dt_{|0}} K[u+tv] = \int\limits_\Omega DF(\nabla^a u) : \nabla v dx + \lambda \int\limits_{\Omega - D} v \cdot (u-f)dx. \tag{5.3}$$

Note that obviously $\nabla^s(u+tv) = \nabla^s u$ holds for the singular parts of the measures. Clearly $\nabla(u+tu) = (1+t)\nabla u$, hence again by the K-minimality of u

$$
\begin{aligned}
0 = \frac{d}{dt_{|0}} K[u+tu] &= \int\limits_\Omega DF(\nabla^a u) : \nabla^a u dx + \int\limits_\Omega F^\infty\Big(\frac{\nabla^s u}{|\nabla^s u|}\Big) d|\nabla^s u| \\
&+ \lambda \int\limits_{\Omega - D} u \cdot (u-f)dx.
\end{aligned} \tag{5.4}
$$

Inserting (5.3) and (5.4) into (5.2) we find

$$
\begin{aligned}
l(v,\rho) &= \int\limits_\Omega F(\nabla^a u)dx + \int\limits_\Omega F^\infty\Big(\frac{\nabla^s u}{|\nabla^s u|}\Big) d|\nabla^s u| \\
&- \lambda \int\limits_{\Omega - D} v \cdot (u-f)dx + \lambda \int\limits_{\Omega - D} u \cdot (u-f)dx \\
&+ \frac{\lambda}{2} \int\limits_{\Omega - D} |v-f|^2 dx.
\end{aligned} \tag{5.5}
$$

Obsering that a.e. on $\Omega - D$ it holds

$$-\lambda v \cdot (u-f) + \lambda u \cdot (u-f) + \frac{\lambda}{2}|v-f|^2 = \frac{\lambda}{2}|u-f|^2 + \frac{\lambda}{2}|u-v|^2,$$

we deduce from (5.5)

$$l(v,\rho) \geq K[u],$$

and (5.1) implies $R[\rho] \geq K[u]$. But then the claim of Lemma 5.1 is a consequence of Theorem 1.2 (a). $\square$

The dual solution ρ from Lemma 5.1 by definition takes its values in the open set $\mathrm{Im}(DF)$. If the dual problem would admit a second solution $\widetilde{\rho} \neq \rho$, then exactly the same arguments as used during the proof of Theorem 2.15 in [10] would lead to a contradiction. In fact, as demonstrated in this reference, the assumption $\rho \neq \widetilde{\rho}$ (on a set of positive measure) yields the strict inequality

$$\int_\Omega (-F^*)\left(\frac{\rho+\widetilde{\rho}}{2}\right)dx > \frac{1}{2}\int_\Omega (-F^*)(\rho)dx + \frac{1}{2}\int_\Omega (-F^*)(\widetilde{\rho})dx.$$

At the same time we observe that

$$L^\infty(\Omega)^{nM} \ni \kappa \mapsto \inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M} \int_\Omega [\kappa : \nabla v - \mathbb{1}_{\Omega-D}|v-f|^2]dx$$

is a concave function, hence

$$R\left[\frac{\rho+\widetilde{\rho}}{2}\right] > \frac{1}{2}R[\rho] + \frac{1}{2}R[\widetilde{\rho}],$$

which is not possible.
Thus, $DF(\nabla^a u)$ is the only dual solution and the proof of Theorem 1.3 is complete. $\square$

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