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A density result for Sobolev functions and functions of higher order bounded variation with additional integrability constraints

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Abstract

This note deals with the density of smooth functions in classes of Sobolev functions and functions of higher order bounded variation of type $W^{m,p}(\Omega)\cap L^q(\Omega - D)$ and $BV^m(\Omega) \cap L^q(\Omega - D)$, respectively, where m is a positive integer, $1 \leq p < \infty$, D is an open precompact subset of the domain of definition $\Omega \subset \mathbb{R}^n$ with sufficiently regular boundary and we say a function is of m-th order bounded variation if its m-th order partial derivatives in the sense of distributions are finite radon measures. It takes up earlier results concerning functions with merely one order of differentiability which emerged in the context of a variational problem related to image analysis.

1 Introduction

At the investigation of a variational integral related to a problem in image analysis, C. Tietz and the author encountered the problem of approximating Sobolev and BV-functions with higher summability on a measurable subset of their domain. Namely we considered a functional of type

$$\mathcal{F}_{p,q}[u] = \int_{\Omega} |\nabla u|^p dx + \int_{\Omega - D} |u - f|^q dx$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded with Lipschitz boundary, $D \subset \Omega$ is a measurable subset with $0 < \mathcal{L}^n(D) < \mathcal{L}^n(\Omega)$, $f \in L^q(\Omega - D)$ is a given function and u varies in $W^{1,p}(\Omega) \cap L^q(\Omega - D)$, $1 \leq p < q < \infty$. In case of $p = 1$ one would rather replace the first integral by the total variation $|\nabla u|(\Omega)$ and study the problem $\mathcal{F} \to \min$ in the space $BV(\Omega) \cap L^q(\Omega - D)$ which naturally comes with a useful notion of compactness in contrast to the non-reflexive space $W^{1,1}(\Omega)$ (see, e.g., [AFP], Theorem 3.23). For an outline of how this functional and its minimizers

can be traced back to problems in image analysis, we would like to refer the reader to the introduction of [FT].
The following result revealed to be a key tool towards proving fine properties of solutions of $\mathcal{F} \to \min$:

Theorem 1.1 (cf. [FT], Lemma 2.1 and Lemma 2.2)
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary and $D \subset \Omega$ a measurable subset with $0 < \mathcal{L}^n(D) < \mathcal{L}^n(\Omega)$.

(i) If u is in $W^{1,p}(\Omega) \cap L^q(\Omega - D)$, then there is a sequence of smooth functions $(\varphi_k)_{k=1}^\infty \subset C^\infty(\overline{\Omega})$ such that

$$\|u - \varphi_k\|_{1,p;\Omega} + \|u - \varphi_k\|_{q;\Omega-D} \to 0 \quad \textit{for} \quad k \to \infty.$$

(ii) If u is in $BV(\Omega) \cap L^q(\Omega - D)$, then there is a sequence of smooth functions $(\varphi_k)_{k=1}^\infty \subset C^\infty(\overline{\Omega})$ such that

$$\|u - \varphi_k\|_{1;\Omega} + \|u - \varphi_k\|_{q;\Omega-D} + \left| |\nabla u|(\Omega) - \int\limits_\Omega |\nabla \varphi_k| dx \right|$$
$$+ \left| \sqrt{1 + |\nabla u|^2}(\Omega) - \int\limits_\Omega \sqrt{1 + |\nabla \varphi_k|^2} dx \right| \to 0 \quad \textit{for} \quad k \to \infty.$$

Here, $\sqrt{1 + |\nabla u|^2}(\Omega)$ has to be understood in the sense of convex functions of a measure, see 'Notations & Conventions' in the next section or [DT] for an detailed overview. The aim of this note is to generalize this theorem towards spaces of functions with higher order derivatives. The main results are:

Theorem 1.2
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $D \subset\subset \Omega$ an open precompact subset with minimally smooth boundary[1] and $u \in W^{m,p}(\Omega) \cap L^q(\Omega - D)$. Then there is a sequence of smooth functions $(\varphi_k)_{k=1}^\infty \subset C^\infty(\overline{\Omega})$ such that

$$\|u - \varphi_k\|_{m,p;\Omega} + \|u - \varphi_k\|_{q;\Omega-D} \to 0 \quad \textit{for} \quad k \to \infty.$$

Theorem 1.3
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary, $D \subset\subset \Omega$ an open precompact subset with C^1-boundary which is star-shaped with respect to a point $x_0 \in D$ and $u \in BV^m(\Omega) \cap L^q(\Omega - D)$. Then there is a sequence of smooth functions

[1] The term 'minimally smooth boundary' was introduced by E.M. Stein in his book [St]. For an explanation see the subsequent section 'Notations and Conventions'.

$(\varphi_k)_{k=1}^{\infty} \subset C^{\infty}(\overline{\Omega})$ *such that*

$$\|u-\varphi_k\|_{m-1,1;\Omega} + \|u-\varphi_k\|_{q;\Omega-D} + \left| |\nabla^m u|(\Omega) - \int_{\Omega} |\nabla^m \varphi_k| dx \right|$$
$$+ \left| \sqrt{1+|\nabla^m u|^2}(\Omega) - \int_{\Omega} \sqrt{1+|\nabla^m \varphi_k|^2} dx \right| \to 0 \quad \textit{for} \quad k \to \infty.$$

The search for a generalization of Theorem 1.1 can be motivated through the consideration of the functional which we get after replacing the gradient operator by its higher order analogue in the definition of $\mathcal{F}_{p,q}$:

$$\mathcal{F}_{m,p,q}[u] = \int_{\Omega} |\nabla^m u|^p dx + \int_{\Omega-D} |u-f|^q dx$$

for $1 < p < \infty$ and

$$\mathcal{F}_{m,1,q}[u] = |\nabla^m u|(\Omega) + \int_{\Omega-D} |u-f|^q dx.$$

for $p = 1$ on $W^{m,p}(\Omega) \cap L^q(\Omega - D)$ and $BV^m(\Omega) \cap L^q(\Omega - D)$, respectively. Solutions of $\mathcal{F} \to \min$ can be interpreted in the context of higher order denoising/inpainting of images, which is a current field of investigation in image analysis, see, e.g., [BKP]. As for $m = 1$, an adequate approximation result in the spirit of Theorem 1.1 is a vital means for the investigation of (generalized) minimizers of $\mathcal{F}_{m,p,q}$. In this note, however, we restrict ourselves to the proofs of Theorems 1.2 and 1.3 and postpone their applications to variational problems of higher order to a separate paper.

One should note at this point, that due to the Sobolev inequality we have that for $mp < n$ any function $u \in W^{m,p}(\Omega)$ is at least $np/(n-mp)$-summable and as a direct consequence of this and the embedding $BV(\Omega) \hookrightarrow L^{n/n-1}(\Omega)$, any $u \in BV^m(\Omega)$ is $n/(n-m)$-summable; so an actual problem does not arise unless q is 'large enough', which we want to propose tacitly from now on.

The methods for proving Theorem 1.1 were customized to grasp the case of merely one order of differentiability and fail for higher orders since they crucially rely on a 'cut-off' procedure which turns out to be unsuitable for higher orders owing to the appearance of higher order terms from the iterated chain rule. So we had to pursue an entirely different approach which, unfortunately, goes along with much more rigorous restrictions on the geometry of Ω and D. The most obvious approach towards proving density of smooth functions in classes of Sobolev and BV-functions would be to simply replicate the classical methods by Meyers and Serrin ('H=W', [MS]), which, however does not work unless we impose q-integrability on all of Ω. At this point I want to express particular thanks to Prof. Dr. M. Bildhauer of Saarland University for many fruitful discussions as well as to Prof. Dr. M. Fuchs, my PhD advisor, for directing my interest upon this topic. Further thanks go to Christian Tietz for valuable feedback and assessment. Finally I would like to thank Prof. Dr. J. Weickert for supporting my research both financially and with his advice whenever it comes to questions from the field of image analysis.

The subsequent paragraph introduces most of our non-standard notation and some basic concepts from the theory of Sobolev and BV-functions. It is followed by a section which gathers some results on the extensibility and boundary values of Sobolev functions, which might be already well known to the reader. The third and fourth section deal with the proofs of Theorems 1.2 and 1.3, respectively.

Notation and conventions

Throughout the following, unless otherwise mentioned, Ω denotes an at least open and bounded subset of $\mathbb{R}^n$ for $n \geq 2$ with Lipschitz-regular boundary and $D \subset\subset \Omega$ is an open, compactly contained subset with Lipschitz-boundary as well. We adopt the notion of 'minimally smooth' boundaries from [St] which is the case when there is an $\varepsilon > 0$, a covering $(U_i)_{i=1}^{\infty}$ of $\partial\Omega$ through open sets, an integer N and a positive real L such that the following three conditions hold:

(i) If $x \in \partial\Omega$, then $B_\varepsilon(x) \subset U_i$ for some i.

(ii) No point of $\mathbb{R}^n$ is contained in more than N of the U_i's.

(iii) For each i, there are coordinates $(x_1, ..., x_n)$ s.t. $\Omega \cap U_i$ can be written as $\{(x_1, ..., x_n) \in \mathbb{R}^n \ : x_n < \varphi_i(x_1, ..., x_{n-1}) \}$ with a Lipschitz-continuous function $\varphi_i : \mathbb{R}^{n-1} \to \mathbb{R}$ and $\mathrm{Lip}(\varphi_i) \leq L$.

The class of all sets with minimally smooth boundary contains, e.g., open and bounded convex sets or open and bounded sets with C^1-boundary. With Ω^ε (Ω_ε) we denote the outer (inner) parallel set of Ω in distance ε:

$$\Omega^\varepsilon := \{x \in \mathbb{R}^n \ : \ \mathrm{dist}(x, \Omega) < \varepsilon\}, \quad \Omega_\varepsilon := \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) > \varepsilon\}.$$

By $\rho_\varepsilon * u$ we abbreviate the convolution of a function $u \in L^1_{\mathrm{loc}}(\Omega)$ with a symmetric mollifier $\rho_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, which is supported in the closure of the ball $B_\varepsilon(0)$. $\mathcal{H}^s$ designates the s-dimensional Hausdorff measure and by $L^p(\Omega)$, $1 \leq p < \infty$ we mean the space of (real-valued) functions which are p-integrable w.r.t. the n-dimensional Lebesgue measure $\mathcal{L}^n$, normed in the usual way by $\|\cdot\|_{p;\Omega}$. Further, $W^{m,p}(\Omega)$, $m \in \mathbb{N}$, designates the Sobolev space of (real-valued) functions whose distributional derivatives up to order m are represented by p-integrable functions, together with the norm

$$\|u\|_{m,p;\Omega} := \sum_{\substack{\nu \in \mathbb{N}_0^n \\ |\nu| \le m}} \|\partial^\nu u\|_{p;\Omega}.$$

The notion $\nabla^k u$ means the k-th iterated (distributional) gradient of a function u, i.e. the k-th order symmetric tensor-valued function with components $(\nabla^k u)_{i_1,...,i_k} = \partial_{i_1} \cdots \partial_{i_k} u$, $i_1, ..., i_k \in \{1, ..., n\}$. $S^k(\mathbb{R}^n)$ denotes the set of all symmetric tensors of order k with real components, which is naturally isomorphic to the set of all k-linear symmetric maps $\mathbb{R}^k \to \mathbb{R}$.
We declare by

$$BV^m(\Omega) := \left\{ u \in W^{m-1,1}(\Omega) \ : \ \nabla^{m-1} u \in BV(\Omega, S^{m-1}(\mathbb{R})) \right\}$$

the space of (real valued) functions of m-th order bounded variation, i.e. the set of all functions, whose distributional gradients up to order $m-1$ are represented through 1-integrable tensor-valued functions and whose m-th distributional gradient is a tensor-valued Radon measure of finite total variation

$$|\nabla^m u|(\Omega) = \sup \left\{ \int_\Omega u \Big(\sum_{|\nu| = m} \partial^\nu g_\nu \Big) dx \ : \ g \in C_0^m(\Omega, \mathbb{R}^M), \ \|g\|_\infty \le 1 \right\}$$

with $M := \#\{\nu \in \mathbb{N}_0^n \ : \ |\nu| := \nu_1 + ... + \nu_n = m\}$. Together with the norm

$$\|u\|_{BV^m(\Omega)} := \|u\|_{m-1,1;\Omega} + |\nabla^m u|(\Omega),$$

$BV^m(\Omega)$ becomes a Banach space. Further, we adopt the concept of convex functions of measures from [DT] but restrict our considerations to the special case of the function $f : \mathbb{R}^n \to [1, \infty)$, $x \mapsto \sqrt{1 + |x|^2}$. Then, given a finite Radon measure μ in its Lebesgue-decomposition $\mu = \mu^a \mathcal{L}^n + \mu^s$, with $\mu^a \in L^1(\Omega)$ and $\mu^s \perp \mathcal{L}^n$, we can define a measure $f(\mu)$ by declaring

$$f(\mu)(B) := \big(\sqrt{1 + |\mu^a|} \mathcal{L}^n \big)(B) + |\mu^s|(B) = \int_B \sqrt{1 + |\mu^a|} dx + |\mu^s|(B)$$

for any Borel-set $B \subset \mathbb{R}^n$.
Following [DT], apart from the norm topology, we will provide $BV^m(\Omega)$ with the topology induced by the following distance:

$$d_f(u, v) := \\ \|u - v\|_{m-1,1;\Omega} + \Big| |\nabla^m u|(\Omega) - |\nabla^m v|(\Omega) \Big| + \Big| f(\nabla^m u)(\Omega) - f(\nabla^m v)(\Omega) \Big|$$

for $u, v \in BV^m(\Omega)$. Then convergence with respect to this distance refines strict BV-convergence (see [AFP], Definition 3.14) and $C^\infty(\Omega)$ is a dense subspace of $\big(BV^m(\Omega), d_f(.,.)\big)$ (see [DT], Theorem 2.2).
To simplify matters, all results are formulated in terms of real valued functions and extend component-wise to the vector-valued case.

2 Some Auxiliary Results

The following results may be considered trivial, but since references in literature are hardly found, we decided to give their proofs here:

Proposition 2.1
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary and $u \in W^{m,p}(\Omega)$. With $T : W^{1,p}(\Omega) \to L^p(\partial\Omega, \mathcal{H}^{n-1})$ denoting the boundary operator for real-valued Sobolev functions, we have that for any $\varepsilon > 0$ given, there is smooth function $\varphi \in C^\infty(\Omega)$ with

$$\|u - \varphi\|_{m,p;\Omega} < \varepsilon$$

and such that $T\nabla^k u = T\nabla^k \varphi$ for $0 \le k \le m-1$, where the action of T on a tensor-valued function is component-wise.

Proof. Exhaust Ω with open sets as given by

$$\Omega_j := \{x \in \Omega \; : \; \mathrm{dist}(x, \partial\Omega) > 1/j\}$$

and consider the open covering of Ω through

$$A_0 := \emptyset, \quad A_j := \Omega_{j+1} - \overline{\Omega_{j-1}}.$$

Let $(\eta_j)_{j=1}^\infty$ be a partition of unity with respect to the covering $(A_j)_{j=1}^\infty$ and take a sequence $(\varepsilon_j)_{j=1}^\infty$ of positive reals s.t. $(\mathrm{spt}\,\eta_j)^{\varepsilon_j} \subset\subset A_j$ and

$$\|\eta_j u - \rho_{\varepsilon_j} * (\eta_j u)\|_{m,p;\Omega} < \varepsilon/2^j.$$

It is obvious that $\varphi := \sum_{j=1}^\infty \rho_{\varepsilon_j} * (\eta_j u)$ is a smooth function which approximates u in the right manner.
Now let $T_j : W^{1,p}(\Omega - \overline{\Omega_j}) \to L^p(\partial\Omega)$ denote the trace operator on $W^{1,p}(\Omega - \overline{\Omega_j})$. Note that $T_j u_{|\Omega - \overline{\Omega_j}} = T(u)$ whenever $u \in W^{1,p}(\Omega)$. Furthermore, since the trace operators are continuous, there are positive constants c_j s.t.

$$\|T_j u\|_{p;\partial\Omega} \le c_j \|u\|_{1,p;\Omega - \overline{\Omega_j}}. \tag{1}$$

With
$$a_j := \frac{1}{\max\{c_i \,:\, i \leq j\}},$$
we can choose ε_j small enough such that
$$\|\eta_j u - \rho_{\varepsilon_j} * (\eta_j u)\|_{m,p;\Omega} < a_j/2^j.$$
Now let $0 \leq k \leq m-1$. Thus $\nabla^k u \in W^{1,p}(\Omega, S^{k-1}(\mathbb{R}))$ and by (1) we have
$$\|T\nabla^k u - T\nabla^k \varphi\|_{p;\partial\Omega} = \|T_j \nabla^k u_{|\Omega - \overline{\Omega_j}} - T_j \nabla^k \varphi_{|\Omega-\overline{\Omega_j}}\|_{p;\partial\Omega}$$
$$\leq c_j \|\nabla^k u - \nabla^k \varphi\|_{1,p;\Omega-\overline{\Omega_j}}$$
$$\leq c_j \sum_{l \geq j} \|\eta_l u - \rho_{\varepsilon_l} * (\eta_l u)\|_{m,p;\Omega} < c_j \sum_{l\geq j} \frac{a_l}{2^l} \leq \frac{1}{2^{j-1}}.$$

Since this holds for any $j \in \mathbb{N}$, the result follows. □

Proposition 2.2
Let $\Omega \subset \mathbb{R}^n$ be open and $u \in W^{m,p}(\Omega) \cap L^q(\Omega)$. Then, for any $\varepsilon > 0$ given, there is a smooth function $\varphi \in C^\infty(\Omega)$ satisfying
$$\|u - \varphi\|_{m,p;\Omega} + \|u - \varphi\|_{q;\Omega} < \varepsilon.$$

Proof. If we construct φ in the same manner as in the prove of Proposition 2.1, it follows trivially from the properties of mollification (see e.g. [Ad], Theorem 2.29) that φ approximates u in $L^q(\Omega)$. □

Proposition 2.3
Let $u \in L^p(\mathbb{R}^n)$. For $\alpha > 1$ define $u_\alpha(x) := u(\alpha x)$. Then $u_\alpha \to u$ in $L^p(\Omega)$ for any sequence $\alpha \downarrow 1$ and any measurable set $\Omega \subset \mathbb{R}^n$.

Proof. W.l.o.g. we assume $\Omega = \mathbb{R}^n$. The set of smooth functions with compact support $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Thus, we can choose a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ converging to u. Then $\varphi_k(\alpha x)$ approximates u_α in $L^p(\mathbb{R}^n)$ and the result follows since $\phi_k(\alpha x) \to \varphi_k(x)$ converges uniformly for $\alpha \downarrow 1$ and k fixed. □

The following extension result will be a key tool towards proving approximation theorems in both $W^{m,p}(\Omega) \cap L^q(\Omega - D)$ and $BV^m(\Omega) \cap L^q(\Omega - D)$:

Proposition 2.4 (*Extension of $W^{m,p}(\Omega) \cap L^q(\Omega)$-functions*)
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with minimally smooth boundary and $u \in W^{m,p}(\Omega) \cap L^q(\Omega)$. Then there is a continuous linear operator $\mathfrak{E}$, mapping u to a function $\tilde{u} \in W^{m,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ and such that $\tilde{u} = u$ (a.e.) on Ω.

Proof. We claim, that the operator $\mathfrak{E} : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n)$, as defined in part 3.3 of [St] performs an extension in the right manner. Indeed, this is a mere consequence of the universality of this operator in the sense that it simultaneously extends all orders of differentiability by the same construction. $\square$

3 Proof of Theorem 1.2

We start by proving another version of Theorem 1.2 under stronger assumptions on the geometry of Ω and D in order to clarify the main idea and then apply similar arguments to a more general setting.

Lemma 3.1 ($C^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega) \cap L^q(\Omega - D)$ for star-shaped D)
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with minimally smooth boundary, $D \subset\subset \Omega$ an open and precompact subset with Lipschitz boundary which is star-shaped with respect to a point $x_0 \in D$ and $u \in W^{m,p}(\Omega) \cap L^q(\Omega - D)$. Given an arbitrary $\varepsilon > 0$, there is a function $\varphi \in C^\infty(\overline{\Omega})$ s.t.

$$\| u - \varphi \|_{m,p;\Omega} + \| u - \varphi \|_{q;\Omega - D} < \varepsilon.$$

Proof. W.l.o.g. we may assume $x_0 = 0$.
Applying Proposition 2.4, we can extend u outside of Ω to a function $u' \in W^{m,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n - D)$. Then, by Proposition 2.3, $u'_\alpha(x) := u'(\alpha x)$ converges to u' in $W^{m,p}(\Omega) \cap L^q(\Omega - D)$ for $\alpha \downarrow 1$. Fix $\alpha > 1$ with

$$\| u' - u'_\alpha \|_{m,p;\Omega} + \| u' - u'_\alpha \|_{q;\Omega - D} < \varepsilon/3 \tag{1}$$

Due to the star shape, $D_\alpha := 1/\alpha\, D$ is a precompact subset of D and u'_α is q-integrable on $\mathbb{R}^n - D_\alpha$. By Proposition 2.2, we can construct a smooth function $\varphi' \in C^\infty(\mathbb{R}^n - \overline{D_\alpha})$ with

$$\|u'_\alpha - \varphi'\|_{m,p;\mathbb{R}^n - \overline{D_\alpha}} + \|u'_\alpha - \varphi'\|_{q;\mathbb{R}^n - D_\alpha} < \varepsilon/3 \tag{2}$$

and such that $T\nabla^k \varphi' = T\nabla^k u'_\alpha$ in $L^p(\partial D_\alpha, \mathcal{H}^{n-1})$ for every $0 \le k \le m-1$. Consequently, φ' can be extended to D_α by $u'_{\alpha|D_\alpha}$ to a function $v \in W^{m,p}(\Omega) \cap L^q(\Omega - D_\alpha)$. On D, we can construct a smooth function $\varphi'' \in C^\infty(D)$ with

$$\|v - \varphi''\|_{m,p;D} < \varepsilon/3 \tag{3}$$

and such that $\partial^\nu \varphi''_{|\partial D} = \partial^\nu \varphi'_{|\partial D}$ for every multi-index $\nu \in \mathbb{N}_0^n$. Therefore, and by (1)-(3)

$$\varphi(x) := \begin{cases} \varphi''(x), & x \in D, \\ \varphi'(x), & x \in \overline{\Omega} - D \end{cases}$$

is a smooth function that approximates u in the right manner. $\square$

We now come to the *proof of Theorem 1.2:*

Let $\{x_1, x_2, x_3, ...\} \subset \partial D$ be a dense subset of ∂D. For every $i \in \mathbb{N}$ choose an open ball $B_{r_i}(x_i)$ such that $B_{r_i}(x_i) \cap D$ is Lipschitz-equivalent to $B_1(0) \cap \mathbb{R}^{n-1} \times [0, -\infty)$ via a bi-Lipschitz-map $\phi_i : B_{r_i}(x_i) \to B_1(0)$ and such that $\inf_i r_i > 0$. Let p_i denote the preimage of $(0, ..., 0, -1)$ with respect to ϕ_i. W.l.o.g. we can assume $p_i = 0$ for i fixed. Note that $B_{r_i}(x_i)$ is star shaped with respect to p_i.
Now let $\eta_i \in C_0^\infty(B_{r_i}(x_i))$ be a smooth function with $0 \leq \eta_i \leq 1$, $\eta_i \equiv 1$ on $B_{r_i/2}(x_i)$. We successively construct a sequence $(u_i)_{i=1}^\infty$ of $W^{m,p}(\Omega) \cap L^q(\Omega - D)$-functions in the following way:
For $i = 1$, take $\alpha_1 > 1$ small enough such that $u_1(x) := (\eta_1 u)(\alpha_1 x) + (1 - \eta_1(x))u(x)$ fulfills

$$\|u - u_1\|_{m,p;\Omega} + \|u - u_1\|_{q;\Omega-D} < \varepsilon/2.$$

Then (provided α_1 is small enough) u_1 is q-integrable across $\partial D \cap B_{r_1/4}(x_1)$. In the second step, we find $\alpha_2 > 1$ for which the function $u_2 := (\eta_2 u_1)(\alpha_2 x) + (1 - \eta_2(x))u_1(x)$ satisfies

$$\|u_1 - u_2\|_{m,p;\Omega} + \|u_1 - u_2\|_{q;\Omega-D} < \varepsilon/4.$$

Then u_2 is q-integrable across $\partial D \cap \big(B_{r_1/4}(x_1) \cup B_{r_2/4}(x_2)\big)$.
By continuing this process, we recursively define a sequence (u_i) s.t.

$$\|u_{i-1} - u_i\|_{m,p;\Omega} + \|u_{i-1} - u_i\|_{q;\Omega-D} < \varepsilon/2^i$$

and u_i is q-integrable across $\partial D \cap \big(\bigcup_{j=1}^i B_{r_j/4}(x_j)\big)$. Since ∂D is compact in Ω, after finitely many steps N, $\bigcup_{i=1}^N B_{r_i/4}$ covers ∂D. Then u_N is a function with

$$\|u - u_N\|_{m,p;\Omega} + \|u - u_N\|_{q;\Omega} < \varepsilon$$

and that is q-integrable outside an inner parallel set of D. From this point on, the result follows by the same arguments as used in the proof of Lemma 3.1. $\square$

4 Proof of Theorem 1.3

In this section we are concerned with generalizing our previous results for Sobolev functions towards the space $BV^m(\Omega) \cap L^q(\Omega - D)$.

Definition 4.1. In the following, we will keep saying "*φ approximates $u \in BV^m(\Omega) \cap L^q(\Omega - D)$ in the sense of $(\mathcal{A}_\varepsilon)$*" for a given $\varepsilon > 0$, if φ approximates u with respect to the metric $d_f(.,.)$ as well as in $L^q(\Omega - D)$, where $D \subset\subset \Omega$ might be empty:

$$(\mathcal{A}_\varepsilon) \begin{cases} \|u-\varphi\|_{m-1,1;\Omega} + \|u-\varphi\|_{q,\Omega-D} \\ \\ +\big||\nabla^m u|(\Omega) - |\nabla^m \varphi|(\Omega)\big| \\ \\ +\Big|\sqrt{1+|\nabla^m u|^2}(\Omega) - \int_\Omega \sqrt{1+|\nabla^m \varphi|^2} dx\Big| < \varepsilon. \end{cases}$$

Notice, that versions of Proposition 2.1 and 2.2 can be proven in the context of $BV^m(\Omega)$:

Proposition 4.2
Let $\Omega \subset \mathbb{R}^n$ have C^1-boundary[2] *and $u \in BV^m(\Omega) \cap L^q(\Omega)$. Then, for any $\varepsilon > 0$ given there is a smooth function $\varphi \in C^\infty(\Omega)$ satisfying*

$$d_f(u, \varphi) + \|u - \varphi\|_{q;\Omega} < \varepsilon.$$

Proof. In [DT], Theorem 2.2 it is shown, that $C^\infty(\Omega)$ lies dense in $BV^m(\Omega)$ with respect to the distance $d_f(.,.)$. The construction of such a smooth approximation follows basically the same steps as in case of a Sobolev function (i.e. the classical Meyers-Serrin argument as also seen in the proof of Proposition 2.1), and thus it is clear that additional integrability constraints are respected by the approximation thanks to the properties of mollification. □

Proposition 4.3
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary and $u \in BV^m(\Omega)$. Define $T : W^{1,1}(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$ as in Proposition 2.1 and let $S : BV(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$ denote the trace operator on $BV(\Omega)$. Then, for any $\varepsilon > 0$ there is a smooth function $\varphi \in C^\infty(\Omega)$ which approximates u in the sense of $(\mathcal{A}_\varepsilon)$ and such that

$$T\nabla^k u = T\nabla^k \varphi, \ \textit{for all}\ 0 \le k < m-2 \quad \textit{and} \quad S\nabla^{m-1} u = S\nabla^{m-1}\varphi$$

[2]The author is not particularly sure to what extend it is necessary to request actual smoothness of the boundary, since Demengel and Temam in [DT] only speak of a 'sufficiently smooth' boundary, for which it seems to be adequate to assume it to be once differentiable.

in $L^1(\partial\Omega, \mathcal{H}^{n-1})$ (hold in mind that T and S act component-wise on tensor-valued functions).

Proof. The result follows by the same arguments we used in the proof of Proposition 2.1 since by Theorem 3, page 483 in [GMS], S is continuous with respect to the metric $d_f(.,.)$ (see also [Giu], Theorem 2.11 and Remark 2.12 as well as [DT], Theorem 2.3). $\square$

Corollary 4.4 (Extension of $BV^m(\Omega) \cap L^q(\Omega)$-functions)
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary and $u \in BV^m(\Omega) \cap L^q(\Omega)$. Then there is a function $\tilde{u} \in BV^m(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ such that $u = \tilde{u}$ a.e. on Ω and

$$|\nabla^m \tilde{u}|(\partial\Omega) = 0.$$

Proof. According to Propositions 4.3 and 4.2 above, we can choose a function $\varphi \in C^\infty(\Omega) \cap BV^m(\Omega) \cap L^q(\Omega)$ with $T\nabla^k\varphi = T\nabla^k u$ for $0 \le k \le m-2$ and $S\nabla^{m-1}\varphi = S\nabla^{m-1}u$ in $L^1(\partial\Omega, \mathcal{H}^{n-1})$. In particular, $\varphi \in W^{m,1}(\Omega) \cap L^q(\Omega)$ and we can therefore apply Proposition 2.4 to extend φ to a function $\tilde{\varphi} \in W^{m,1}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$. But then

$$\tilde{u}(x) := \begin{cases} u(x), & x \in \Omega, \\ \tilde{\varphi}(x), & x \in \mathbb{R}^n - \overline{\Omega} \end{cases}$$

is an extension of u as claimed. $\square$

With these results at hand, there now follows the *proof of Theorem 1.3*:

Without loss of generality, we may assume $x_0 = 0$.
By Proposition 4.2 we can construct a smooth function $\psi \in C^\infty(\Omega - \overline{D})$ having the same traces as u on ∂D at any order and with

$$d_f(u_{|\Omega - D}, \psi) + \|u - \psi\|_{q;\Omega - D} < \varepsilon/3. \tag{1}$$

In particular, ψ is in $W^{m,1}(\Omega - \overline{D}) \cap L^q(\Omega - D)$ and by Theorem 2.4, we can extend ψ outside of Ω to a function $\psi' \in W^{m,1}(\mathbb{R}^n - \overline{D}) \cap L^q(\mathbb{R}^n - D)$. Due to Proposition 4.3, the function ψ' can be extended by $u_{|D}$ to a function u' in $BV^m(\mathbb{R}^n) \cap L^q(\mathbb{R}^n - D)$ s.t.

$$|\nabla^m u|(\partial D) = |\nabla^m u'|(\partial D) \tag{2}$$

and since $|\nabla^m u|(\mathcal{N}) = \sqrt{1 + |\nabla^m u|}(\mathcal{N})$ for any $\mathcal{L}^n$-null set $\mathcal{N}$ we also get

$$\sqrt{1 + |\nabla^m u|}(\partial D) = \sqrt{1 + |\nabla^m u'|}(\partial D). \tag{3}$$

Altogether, (1)-(3) imply that u' approximates u in the sense that

$$d_f(u,u') + \|u-u'\|_{q;\Omega-D} < \varepsilon/3.$$

Now we consider $u'_\alpha(x) := u'(\alpha x)$ for $\alpha > 1$. Then, by the star shape of D, u'_α is q-integrable outside of $D_\alpha := (1/\alpha)D \subset\subset D$.
It obliges to show $u'_\alpha \to u'$ in the sense of $(\mathcal{A}_\varepsilon)$ for $\alpha \downarrow 1$.
With $h : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto (1/\alpha)\,x$, we have $\nabla^m(u'_\alpha) = \alpha^{m-n} h_* \nabla^m u$, where $h_*\mu(B) := \mu(h^{-1}(B))$ denotes the image measure.
Further we get:

$$\begin{aligned}
|\nabla^m u'_\alpha|(\Omega) &= \sup\Bigg\{ \int\limits_\Omega u'(\alpha x)\Bigg(\sum_{|\nu|=m} \partial^\nu g_\nu(x)\Bigg)dx \ : \ g \in C_0^m(\Omega,\mathbb{R}^M), \|g\|_\infty \leq 1 \Bigg\} \\
&= \alpha^{-n} \sup\Bigg\{ \int\limits_{\alpha\Omega} u'(x)\Bigg(\sum_{|\nu|=m} \partial^\nu g_\nu\Bigg)(x/\alpha)dx \ : \ g \in C_0^m(\Omega,\mathbb{R}^M), \|g\|_\infty \leq 1 \Bigg\} \\
&= \alpha^{m-n} \sup\Bigg\{ \int\limits_{\alpha\Omega} u'(x)\Bigg(\sum_{|\nu|=m} \partial^\nu g_\nu(x/\alpha)\Bigg)dx \ : \ g \in C_0^m(\Omega,\mathbb{R}^M), \|g\|_\infty \leq 1 \Bigg\} \\
&\leq \alpha^{m-n} |\nabla^m u'|(\alpha\Omega) \xrightarrow{\alpha\downarrow 1} |\nabla^m u'|(\overline{\Omega}) = |\nabla^m u'|(\Omega),
\end{aligned}$$

since $u' \in W^{m,1}(\mathbb{R}^n - \overline{D})$ and therefore $|\nabla^m u'|(\partial\Omega) = 0$. This proves

$$\limsup_{\alpha\downarrow 1} |\nabla^m u'_\alpha|(\Omega) \leq |\nabla^m u'|(\Omega)$$

and convergence follows from $\nabla^{m-1} u'_\alpha \xrightarrow{\alpha\downarrow 1} \nabla^{m-1} u'$ in $L^1(\mathbb{R}^n)$ and lower semicontinuity of the total variation.
Moreover, if

$$\nabla^m u' = \nabla_a^m u' \mathcal{L}^n + \nabla_s^m u'$$

denotes the Lebesgue-decomposition of the tensor valued Radon measure $\nabla^m u'$, we have that

$$\alpha^{m-n} h_* \nabla^m u' = \alpha^m \nabla_a^m u' \circ h^{-1} \mathcal{L}^n + h_* \nabla_s^m u'$$

is the Lebesgue-decomposition of $\nabla^m u'_\alpha$, and by definition it follows:

$$\sqrt{1+|\nabla^m u'_\alpha|^2}(\Omega) = \int\limits_\Omega \sqrt{1+|\alpha^m \nabla_a^m u'(\alpha x)|^2}dx + |h_* \nabla_s^m u'|(\Omega).$$

As above, for the total variation of the singular part we have

$$|h_* \nabla_s^m u'|(\Omega) = |\nabla_s^m u'|(\alpha\Omega) \xrightarrow{\alpha\downarrow 1} |\nabla_s^m u'|(\overline{\Omega}) = |\nabla_s^m u'|(\Omega).$$

To the first part, we can apply the transformation formula:

$$\int\limits_{\Omega} \sqrt{1+|\alpha^m \nabla_a^m u'(\alpha x)|^2}dx = \alpha^{-n} \int\limits_{\alpha\Omega} \sqrt{1+|\alpha^m \nabla_a^m u'(x)|^2}dx.$$

Due to $\alpha^m \nabla_a^m u' \xrightarrow{\alpha\downarrow 1} \nabla_a^m u'$ point wise a.e. and $|\alpha^m \nabla_a^m u'(x)| \leq 2|\nabla_a^m u'(x)|$ (we may assume $\alpha^m < 2$), by Lebesgue's theorem on dominated convergence we conclude

$$\alpha^{-n} \int\limits_{\alpha\Omega} \sqrt{1+|\alpha^m \nabla_a^m u'(x)|^2}dx \xrightarrow{\alpha\downarrow 1} \int\limits_{\Omega} \sqrt{1+|\nabla_a^m u'(x)|^2}dx.$$

Hence, we can choose $\alpha > 1$ small enough with

$$d_f(u', u'_\alpha) + \|u' - u'_\alpha\|_{q,\Omega-D} < \varepsilon/3 \tag{4}$$

and u'_α is q-integrable outside D_α. From that point on, we may proceed just like in the proof of Lemma 3.1 and construct a smooth function $\varphi \in C^\infty(\overline{\Omega})$ with

$$\|u'_\alpha - \varphi\|_{q;\Omega-D} + d_f(u'_\alpha, \varphi) < \varepsilon/3. \tag{5}$$

by conjoining C^∞-approximations of u'_α on $\mathbb{R}^n - \overline{D_\alpha}$ and D. Altogether, we have that φ approximates u as claimed. □

Remark 4.5
At this point, one might guess that, using similar arguments as in the proof of Theorem 1.2, we might generalize the above result towards weaker assumptions on Ω and D; but this is not the case. This seems to ground on the fact that the metric $d_f(.,.)$ is not translation invariant, and addition is not a continuous action w.r.t. the topology it induces on $BV^m(\Omega)$. Put simply: minor changes of a function $u \in BV^m(\Omega)$ on a small set can have a major effect on its global behavior.

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