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Abstract

In this paper we study a submodule filtration for symmetric powers of the tautological representation V of $\mathrm{GL}(2, \mathbb{F}_q)$. Adapting results from Bardoe and Sin, we describe the properties of the successive quotients in detail.

In particular, this allows for an algorithm that determines the multiplicities of the composition factors for arbitrary symmetric powers of V.

The results for the present situation are applicable in future work on Drinfeld modular forms of level T.

0 Introduction

Modular representation theory differs fundamentally from representation theory over the complex numbers. In finite characteristic many classical results or techniques do not apply; the most obvious example being the question of semi-simplicity.

In the present paper we study representations of $G := \mathrm{GL}(2, \mathbb{F}_q)$ over the field $\overline{\mathbb{F}}_q$. (We make no difference between representations and G-modules, i.e., $\overline{\mathbb{F}}_q$-vector spaces with a structure as (left-)modules for the group algebra $\overline{\mathbb{F}}_q[G]$.)

The fundamental building blocks for representation theory in this setting have been described explicitly in prior work, for example by Bonnafé [Bon11] or Wack [Wac96]. In particular, we have explicit descriptions of the simple G-modules as well as the projective indecomposable G-modules.

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On the other hand, even the theory of symmetric powers of the tautological G-module V has not yet been fully developed. Some initial results can be found in [BS00] and play an important role in the present paper.

Related, but slightly different settings have been studied for example by Doty [Dot85] and Rust [Rus95]. Doty studies the representation theory of the algebraic group $\mathrm{GL}(n, \overline{\mathbb{F}}_q)$ while Rust studies certain arithmetically defined representations of G in characteristic 0. Neither of these theories can be directly applied to the situation at hand, however.

The aim of the present paper is to determine the composition factors of arbitrary symmetric powers of V. The key to our approach is the construction of a specific G-module filtration for symmetric powers. Together with results by Bardoe and Sin, this filtration allows for a systematic counting of composition factors.

The present paper summarizes parts of the author's dissertation [Var15], in particular chapters 5, 6, 10, and 11. The algorithms provided in the dissertation have been translated into the language of symmetric powers for the present work.

The principal results of this paper are Theorem 3.6, which describes the filtration in question and identifies its successive quotients, Theorem 3.7, which states a sufficient condition for the largest non-trivial submodule in the filtration to be non-split, Theorem 4.6, in which the composition factors of a reoccurring building block are described in detail, and Algorithms 4.13 and 4.14, which together determine explicitly the multiplicities of the composition factors of symmetric powers of V.

It is worth mentioning that our results have many interesting applications for the representation theory of Drinfeld modular forms of level T, since the latter theory heavily involves symmetric powers of V.

In the first section of the present paper, we gather the necessary basic concepts from representation theory without any claim to completeness.

The second section introduces a class of G-modules that has previously been studied in great detail by Bardoe and Sin. We adapt these results to the situation at hand.

In the third section, we study symmetric powers of V. In particular, we describe a special submodule that in turn allows us to construct a submodule filtration with desirable properties.

In the final section, we use this filtration to count multiplicities of composition factors of symmetric powers.

1 Preliminaries

This section provides a brief outline of the necessary basic concepts of modular representation theory. Our aim is not to give an exhaustive overview but to fix the language that we are going to use in the present paper.

As a general reference to modular representation theory see, for example, [Alp86], [Ben91], or [Fei82]. With regard to the situation for the group $\mathrm{GL}(2, \mathbb{F}_q)$, we refer to Bonnafé [Bon11] and Wack [Wac96].

Consider first the following, closely related concepts:

1.1 Definition. Let K be a field and M a K-vector space. Further let G be a group.

1. A group homomorphism $\rho : G \to \mathrm{Aut}_K(M)$ is called a *representation* of G on M. The space M is called the *representation space* of ρ. The dimension of M is called the dimension of the representation ρ.

   One-dimensional representations are also called *characters* of G.

2. If M is equipped with a structure as a module for the group algebra $K[G]$, we call M a *G-module*. We also say G acts on M.

Remark. Here and in the following, all modules are *left*-modules and all group actions are actions from the left.

For a fixed group G we will use the terms representation and G-module interchangeably. In particular, we call a vector space itself a representation if the action by G is uniquely determined from the context.

Direct sums and tensor products of representations are defined in the natural way and follow the usual conventions.

1.2 Definition. Let M and N be G-modules. A linear map $\varphi : M \to N$ is called *G-equivariant* or a *G-homomorphism* if it satisfies

$$\varphi(gx) = g\varphi(x)$$

for all $g \in G$, $x \in M$.

For the rest of this paper we fix the following situation:

1.3 Notation. For a prime p let $\mathbb{F}_q$ be the field with $q = p^r$ elements. Further let

$$G = \mathrm{GL}(2, \mathbb{F}_q).$$

We are going to study exclusively representations of G over the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$.

Remark. Concerning applications of this paper's results, it should be noted that one may replace $\overline{\mathbb{F}}_q$ by the field $\mathcal{C}_\infty$ encountered in the Drinfeld setting (with notation as in [Var16]; for the concept in general see for example [Gek86]) because of the canonical isomorphism

$$\overline{\mathbb{F}}_q[G] \underset{\overline{\mathbb{F}}_q}{\otimes} \mathcal{C}_\infty \xrightarrow{\cong} \mathcal{C}_\infty[G].$$

1.4 Proposition. *The group G is generated by the elements*

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where a and t pass through $\mathbb{F}_q^\times$.

Remark. When we speak of *the* generators of G we always refer to the matrices described in the above proposition. Furthermore, whenever a matrix of type $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ occurs in a formula, it is implied that $a \in \mathbb{F}_q^\times$ is arbitrary (analogously for $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$).

Through Maschke's theorem (for example [Alp86, I, 3, Theorem 1]) we know that the group algebra $\overline{\mathbb{F}}_q[G]$ is not semi-simple, since $\mathrm{char}(\overline{\mathbb{F}}_q)$ divides the order of G. Thus, the specific representation theory that we encounter here is fundamentally different from the classical representation theory over the complex numbers.

For instance, the most immediate difference is that not every module can be written as a direct sum of simple modules. Instead, we study composition series of G-modules (see for example [Ben91, Section 1.1]). Here the simple modules occur as composition factors.

Next, we are going to outline two methods by which a representation can be modified to obtain new representations:

1.5 Definition. Let M be a G-module. We call a module of shape

$$M \otimes (\det)^\sigma, \quad \sigma \in \mathbb{Z},$$

a *determinant twist* of M. Here, $(\det)^\sigma$ is the σ-th power of the *determinant character* and depends only on $\sigma \bmod q - 1$.

1.6 Definition. Let M be a G-module and let $\theta^j : \mathbb{F}_q \to \mathbb{F}_q$, $x \mapsto x^{p^j}$, $0 \leq j \leq r - 1$, be a power of the Frobenius automorphism. We equip the space M with a new group action by defining

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \underset{\theta^j}{\cdot} x := \begin{pmatrix} a^{p^j} & b^{p^j} \\ c^{p^j} & d^{p^j} \end{pmatrix} x \quad \text{for all } x \in M.$$

We denote the resulting module structure by M^{θ^j} and call it a *Frobenius twist* of M.

1.7 Notation. From now on, let V denote the tautological two-dimensional representation of G. That is, as a vector space $V = \overline{\mathbb{F}}_q^2$ and G acts from the left by matrix multiplication. We write (X, Y) for the standard basis of V. For a non-negative integer n let $\mathrm{Sym}^n(V)$ be the n-th symmetric power of V. If we consider the basis $(X^{n-i}Y^i \mid 0 \le i \le n)$ of $\mathrm{Sym}^n(V)$, the action of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} X^{n-i}Y^i = (aX + cY)^{n-i}(bX + dY)^i.$$

The theory of symmetric powers of V is closely connected to the description of the simple G-modules. The latter have been described by Steinberg in a more general setting [Ste67]. However, for applications in this paper we are going to use the following classification, which follows Bonnafé [Bon11] and Wack [Wac96].

1.8 Notation. Let $0 \le s \le q - 1$ have the p-adic expansion $s = \sum_{i=0}^{r-1} s_i p^i$, $0 \le s_i \le p - 1$. The G-module $\mathfrak{S}(s)$ is defined by

$$\mathfrak{S}(s) := \bigotimes_{i=0}^{r-1} \left(\mathrm{Sym}^{s_i}(V)\right)^{\theta^i}.$$

For $\sigma \in \mathbb{Z}$ let

$$\mathfrak{S}(s, \sigma) := \mathfrak{S}(s) \otimes (\det)^\sigma.$$

1.9 Theorem. *A complete system of representatives of the isomorphism classes of simple G-modules is given by the modules $\mathfrak{S}(s, \sigma)$ with $0 \le s \le q-1$ and $\sigma \in \mathbb{Z}/(q-1)\mathbb{Z}$.*

Proof. The corresponding result for the group $\mathrm{SL}(2, \mathbb{F}_q)$ is proven for example in [Bon11, Theorem 10.1.8] or [Wac96, Korollar 3.11]. The step from the special linear group to the general linear group is explained in [Wac96, Section 3.3]. $\qquad\square$

Remark. In contrast to the situation in characteristic 0, the symmetric powers of the tautological representation are in general not simple G-modules. This is due to the vanishing of certain binomial coefficients modulo p.
In fact, the module $\mathfrak{S}(n)$ is the unique simple submodule of $\mathrm{Sym}^n(V)$ (for $\mathrm{SL}(2, \mathbb{F}_q)$ this is also stated in [Bon11, Theorem 10.1.8], which implies the corresponding result for G).

2 The modules $N[\delta]$

Next, we are going to study a special class of G-modules. With help from results by Bardoe and Sin [BS00], the structure of these modules can be described in detail. Our focus on the two-dimensional case allows us to state their results in a more explicit way. At the same time, we adjust and expand their notation to make the presentation of the results more suitable to our desired applications.

2.1 Notation. Let $B \leq G$ be the standard Borel subgroup of upper triangular matrices. For $1 \leq \delta \leq q - 1$ define the induced G-module $N[\delta] :=$ $\mathrm{Ind}_B^G(\chi_\delta)$, where χ_δ is the character of B that acts by

$$\chi_\delta \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = d^\delta.$$

We have $\dim N[\delta] = q + 1$ for all δ.

Remark. 1. We choose the analytic implementation of the induced representation, as described for example in [Lan02, XVIII, §7]. That is, we define the induced representation as a space of functions $G \to \overline{\mathbb{F}}_q$ with certain transformation properties.

2. At first glance, our implementation of the modules $N[\delta]$ differs from the one of the modules $A[d]$ in [BS00]. However, the relation to certain induced modules is already mentioned in Remark (3) to [BS00, Theorem C]. One can show that

$$N[\delta] \cong A[\delta] \otimes (\det)^\delta.$$

In fact, the main difference between the respective settings is a question of duality. Where the authors of [BS00] base their definitions on the dual V^* of the tautological representation V, we are working with the representation V itself. While this necessitates careful attention to the technicalities when transferring results from one setting to the other, these differences cancel out in the end so that our version of [BS00, Theorem C], stated in Theorem 2.17, agrees essentially verbatim with the original (cf. [Var15, Anhang B] for a detailed examination).

3. For $\delta \in \mathbb{Z}$ we define $N[\delta]$ to be the module determined by $\delta \mod q - 1$.

According to the definition of the induced representation, each function in $N[\delta]$ is uniquely determined by its values on a set of representatives for $B \backslash G$.

Using the Bruhat decomposition of G, we fix the set of representatives

$$R := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & 1 \\ 1 & v \end{pmatrix} \mid v \in \mathbb{F}_q \right\}.$$

For our work with the modules $N[\delta]$ we need explicit bases.

2.2 Lemma. *Let $1 \leq \delta \leq q - 1$. For $u \in \mathbb{F}_q$ denote by $F_u^{(\delta)} \in N[\delta]$ the function that is determined for $\sigma \in R$ by*

$$F_u^{(\delta)}(\sigma) = \begin{cases} 1 & \sigma = \left(\begin{smallmatrix} 0 & 1 \\ 1 & u \end{smallmatrix} \right) \\ 0 & \sigma \neq \left(\begin{smallmatrix} 0 & 1 \\ 1 & u \end{smallmatrix} \right). \end{cases}$$

Further let $F_\infty^{(\delta)} \in N[\delta]$ be given by

$$F_\infty^{(\delta)}(\sigma) = \begin{cases} 1 & \sigma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \\ 0 & \sigma \neq \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right). \end{cases}$$

Then these functions comprise a basis of $N[\delta]$. $\qquad\square$

2.3 Lemma. *Let $1 \leq \delta \leq q - 1$. The transformation properties of the functions $F_\nu^{(\delta)} \in N[\delta]$ with $\nu \in \mathbb{F}_q \cup \{\infty\}$ under the generators of G are determined as follows:*

$$\begin{aligned}
\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix} \right) F_u^{(\delta)} &= a^\delta F_{ua}^{(\delta)}, \quad u \in \mathbb{F}_q, \\
\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix} \right) F_\infty^{(\delta)} &= F_\infty^{(\delta)}, \\
\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix} \right) F_u^{(\delta)} &= F_{u-t}^{(\delta)}, \quad u \in \mathbb{F}_q, \\
\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix} \right) F_\infty^{(\delta)} &= F_\infty^{(\delta)}, \\
\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) F_u^{(\delta)} &= u^{-\delta} F_{u^{-1}}^{(\delta)}, \quad u \in \mathbb{F}_q^\times, \\
\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) F_0^{(\delta)} &= F_\infty^{(\delta)}, \\
\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) F_\infty^{(\delta)} &= F_0^{(\delta)}.
\end{aligned}$$

Proof. By means of a straightforward computation we see that in each case both functions are equal on R and thus identical. $\qquad\square$

2.4 Lemma. *Let $1 \leq \delta \leq q - 1$. For $0 \leq i \leq q - 1$ put*

$$f_i^{(\delta)} := \sum_{u \in \mathbb{F}_q} u^i F_u^{(\delta)}$$

with the convention $0^0 = 1$. Further let

$$f_\infty^{(\delta)} := \sum_{u \in \mathbb{F}_q} u^\delta F_u^{(\delta)} + F_\infty^{(\delta)}.$$

Then these functions form another basis of $N[\delta]$. $\qquad\square$

2.5 Lemma. *Let* $1 \leq \delta \leq q - 1$*. The generators of* G *act on the basis given in Lemma 2.4 by:*

$$\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) f_i^{(\delta)} = a^{\delta - i} f_i^{(\delta)}, \quad 0 \leq i \leq q - 1,$$

$$\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) f_\infty^{(\delta)} = f_\infty^{(\delta)},$$

$$\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) f_i^{(\delta)} = \sum_{j=0}^{i} \binom{i}{j} t^{i-j} f_j^{(\delta)}, \quad 0 \leq i \leq q - 1,$$

$$\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) f_\infty^{(\delta)} = \sum_{j=0}^{\delta - 1} \binom{\delta}{j} t^{\delta - j} f_j^{(\delta)} + f_\infty^{(\delta)},$$

$$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) f_i^{(\delta)} = f_{\delta - i}^{(\delta)}, \quad 1 \leq i \leq \delta - 1,$$

$$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) f_i^{(\delta)} = f_{q - 1 + \delta - i}^{(\delta)}, \quad \delta \leq i \leq q - 1,$$

$$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) f_0^{(\delta)} = f_\infty^{(\delta)},$$

$$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) f_\infty^{(\delta)} = f_0^{(\delta)}.$$

Proof. This follows from the transformation properties given in Lemma 2.3. $\qquad\square$

2.6 Proposition. *Let* $1 \leq \delta, \delta' \leq q - 1$*. Then*

$$\dim \mathrm{Hom}_G(N[\delta], N[\delta']) = \begin{cases} 1 & 1 \leq \delta = \delta' < q - 1 \\ 2 & \delta = \delta' = q - 1 \\ 0 & else. \end{cases}$$

Proof. One verifies that as a B-module $N[\delta]$ has precisely two eigenvectors (up to scaling), namely $f_0^{(\delta)}$ and $F_\infty^{(\delta)}$. Then the statement is obtained immediately by comparing the corresponding characters and applying Frobenius reciprocity. $\qquad\square$

In order to state results from [BS00] for this class of modules, we need to introduce the following two interrelated concepts. Both concepts and their relation are already described in the cited paper, see for example [BS00, Theorem C]. However, we make some adjustments with regard to the desired applications.

2.7 Notation. Let $1 \leq \delta \leq q - 1$ with p-adic expansion $\delta = \sum_{i=0}^{r-1} \delta_i p^i$.

1. The set $\mathcal{P}[\delta]$ of *parameters* for δ consists of all tuples $\mathbf{t} = (t_0, \dots, t_{r-1}) \in \{0, 1\}^r$ that satisfy

$$0 \leq \delta_j + t_{j+1} p - t_j \leq 2(p - 1) \quad \text{for } 0 \leq j \leq r - 1.$$

Here, we understand $t_r = t_0$.

We define a partial order on $\mathcal{P}[\delta]$ by $(t'_0, \ldots, t'_{r-1}) \leq (t_0, \ldots, t_{r-1})$ if and only if $t'_j \leq t_j$ for all j.

2. We map each parameter $\mathbf{t} \in \mathcal{P}[\delta]$ to a unique tuple $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{r-1})$ in $\{0, \ldots, 2(p-1)\}^r$ by defining

$$\alpha_j = \delta_j + t_{j+1}p - t_j \quad \text{for } 0 \leq j \leq r-1, \tag{1}$$

where $t_r = t_0$. This induces an injective map

$$\mathrm{typ}_\delta : \mathcal{P}[\delta] \to \mathcal{T} := \{0, \ldots, 2(p-1)\}^r \setminus \{(0, \ldots, 0)\}.$$

We write

$$\mathcal{T}[\delta] := \mathrm{typ}_\delta\left(\mathcal{P}[\delta]\right) \subseteq \mathcal{T}$$

and call $\mathcal{T}[\delta]$ the set of *types* for δ.

Remark. 1. In contrast to the corresponding definitions in [BS00], we allow the case $\delta = q - 1$ in the above.

Another difference lies in our approach that considers different values of δ simultaneously and thus leads to a general preference of types $\boldsymbol{\alpha}$ over parameters $\mathbf{t}$ as we will see in the following.

2. As with the modules $N[\delta]$, we also define $\mathcal{P}[\delta]$ and $\mathcal{T}[\delta]$ for $\delta \in \mathbb{Z}$ in the natural way.

3. The identification $t_r = t_0$ (and accordingly $\alpha_r = \alpha_0$) allows us to interpret these tuples as sequences with index set $\mathbb{Z}$ and a cyclic structure. For example, we say that t_{r-1} is the left neighbor of t_0.

A straightforward calculation provides the following connection between a type and the corresponding δ.

2.8 Lemma. *Let $1 \leq \delta \leq q - 1$ and let $\mathbf{t} \in \mathcal{P}[\delta]$. For $\boldsymbol{\alpha} = \mathrm{typ}_\delta(\mathbf{t}) \in \mathcal{T}$ we have*

$$\sum_{i=0}^{r-1} \alpha_i p^i = \delta + t_0(q - 1).$$

$\square$

2.9 Notation. We define the map

$$\mathfrak{d} : \mathcal{T} \to \{1, \ldots, q-1\}$$

$$\boldsymbol{\alpha} \mapsto \left[\sum_{i=0}^{r-1} \alpha_i p^i\right].$$

Here, "$[\cdot]$" denotes the representative modulo $q - 1$ in $\{1, \ldots, q-1\}$.

Using the preceding, we obtain the following alternative description of the types for δ.

2.10 Proposition. *Let $1 \leq \delta \leq q - 1$. Then*

$$\mathcal{T}[\delta] = \{\boldsymbol{\alpha} \in \mathcal{T} \mid \mathfrak{d}(\boldsymbol{\alpha}) = \delta\}.$$

In particular, the set $\mathcal{T}$ is the disjoint union of the sets $\mathcal{T}[\delta]$ with $1 \leq \delta \leq q - 1$.

Proof. The inclusion "$\subseteq$" follows immediately from Lemma 2.8.
For the inverse inclusion the p-adic coefficients of $\mathfrak{d}(\boldsymbol{\alpha})$ and the unique preimage of $\boldsymbol{\alpha}$ under $\mathrm{typ}_{\mathfrak{d}(\boldsymbol{\alpha})}$ can be determined simultaneously by a simple algorithm.
The basic idea is to cover the tuple $\boldsymbol{\alpha}$ by sections $(\alpha_m, \ldots, \alpha_{m+n})$ such that only the entries α_m and α_{m+n} differ from $p - 1$. On each such segment, equation (1) can be solved for the corresponding t_i and δ_i, which provides the desired solutions.
The fact that $\mathcal{T}$ is the disjoint union of the individual $\mathcal{T}[\delta]$ is a direct consequence of the first part of the proposition. $\square$

2.11 Notation. The *parametrizing maps $e, \eta : \mathcal{T} \to \{0, \ldots, q-1\}$* are defined by

$$e(\boldsymbol{\alpha}) = \sum_{i=0}^{r-1} e^*(\alpha_i) p^i \quad \text{with } e^*(\alpha) = \begin{cases} \alpha & 0 \leq \alpha \leq p - 1 \\ 2(p-1) - \alpha & p - 1 < \alpha \leq 2(p-1), \end{cases}$$

and

$$\eta(\boldsymbol{\alpha}) = \sum_{\substack{i=0 \\ \alpha_i > p-1}}^{r-1} (\alpha_i - (p-1)) p^i.$$

As usual, the empty sum is 0.

Since we can immediately read off the p-adic coefficients of images of the parametrizing maps, we obtain:

2.12 Lemma. *Let $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathcal{T}$ such that*

$$e(\boldsymbol{\alpha}) = e(\boldsymbol{\alpha}') \quad and \quad \eta(\boldsymbol{\alpha}) \equiv \eta(\boldsymbol{\alpha}') \mod q - 1.$$

Then $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$. $\square$

A short calculation gives the following relation between the functions defined so far:

2.13 Lemma. *Let $\boldsymbol{\alpha} \in \mathcal{T}$ and $\mathbf{t} \in \mathcal{P}[\mathfrak{d}(\boldsymbol{\alpha})]$ such that $\mathrm{typ}_{\mathfrak{d}(\boldsymbol{\alpha})}(\mathbf{t}) = \boldsymbol{\alpha}$. Then*

$$e(\boldsymbol{\alpha}) + 2\eta(\boldsymbol{\alpha}) = \mathfrak{d}(\boldsymbol{\alpha}) + t_0(q-1).$$

$\square$

For the desired applications of these concepts in the final section of this paper, we require a description of the fibers of the map e.

The preimage of a singleton $\{m\}$ with $0 \leq m \leq q-1$ strongly depends on the set of indices at which the p-adic coefficients of m differ from $p-1$. Therefore, we define:

2.14 Definition. Let $0 \leq m \leq q-1$ with p-adic expansion $m = \sum_{j=0}^{r-1} m_j p^j$. The *dual support* of m is defined to be the set

$$\mathrm{dsupp}(m) = \{0 \leq j \leq r-1 \mid m_j < p-1\}.$$

The concept of the dual support allows for the following explicit description of the fibers of e:

2.15 Proposition. *Let $0 \leq m \leq q-1$. For $U \subseteq \mathrm{dsupp}(m)$ (including the empty set!) define the tuple*

$$\boldsymbol{\alpha}(m, U) = (\alpha_0(m, U), \ldots, \alpha_{r-1}(m, U)) \in \{0, \ldots, 2(p-1)\}^r,$$

where

$$\alpha_j(m, U) = \begin{cases} 2(p-1) - m_j & j \in U \\ m_j & j \notin U. \end{cases}$$

Then:

1. *If m is strictly greater than 0, we have*

$$\{\boldsymbol{\alpha} \in \mathcal{T} \mid e(\boldsymbol{\alpha}) = m\} = \{\boldsymbol{\alpha}(m, U) \mid U \subseteq \mathrm{dsupp}(m)\}.$$

   The fiber contains $2^{\# \mathrm{dsupp}(m)}$ elements.

2. *We get*

$$\{\boldsymbol{\alpha} \in \mathcal{T} \mid e(\boldsymbol{\alpha}) = 0\} = \{\boldsymbol{\alpha}(0, U) \mid \emptyset \neq U \subseteq \{0, \ldots, r-1\}\}.$$

   The fiber contains $2^r - 1$ elements.

In particular, we conclude that the map e is surjective.

Proof. According to the definition of e, we have to find all $\boldsymbol{\alpha} \in \mathcal{T}$ such that

$$e^*(\alpha_j) = m_j \quad \text{for all } 0 \le j \le r - 1.$$

In case m_j equals $p-1$ this is uniquely solved by $\alpha_j = p-1$. For $0 \le m_j < p-1$ both $\alpha_j = m_j < p-1$ and $\alpha_j = 2(p-1) - m_j > p-1$ are admissible.

Thus we see that, by construction, each element of the fiber of m must be of shape $\boldsymbol{\alpha}(m, U)$ for a suitable $U \subseteq \operatorname{dsupp}(m)$. On the other hand, all $\boldsymbol{\alpha}(m, U)$ are contained in the preimage of $\{m\}$ with the sole exception of $\boldsymbol{\alpha}(0, \emptyset) = (0, \ldots, 0)$.

Since U consists precisely of those indices j such that $\alpha(m, U)_j > p - 1$, we find that $\boldsymbol{\alpha}(m, U) \ne \boldsymbol{\alpha}(m, U')$ for $U \ne U'$. $\qquad\square$

As a direct consequence we observe the following behavior of η:

2.16 Corollary. *The values of η on a fiber of e are pairwise distinct modulo $q - 1$. Using the above parametization for the elements of the fiber of a singleton $\{m\}$, we find*

$$\eta(\boldsymbol{\alpha}(m, U)) = \sum_{j \in U} (p - 1 - m_j) p^j.$$

We can now state the principal result from [BS00] that is applicable for the present situation. Some adjustments to the notation have been made; in particular, as far as types and the newly introduced parametrizing maps e and η are concerned.

2.17 Theorem (Bardoe-Sin)**.** *Let $1 \le \delta \le q - 2$. Then:*

1. *The module $N[\delta]$ is multiplicity free, that is, all composition factors in a composition series of $N[\delta]$ occur with multiplicity one.*

2. *The composition factors of $N[\delta]$ can be parametrized by $\mathcal{P}[\delta]$ as well as by $\mathcal{T}[\delta]$. For $\mathbf{t} \in \mathcal{P}[\delta]$ with type $\boldsymbol{\alpha} = \operatorname{typ}_\delta(\mathbf{t}) \in \mathcal{T}[\delta]$ the corresponding composition factor is*

$$\mathfrak{S}(e(\boldsymbol{\alpha}), \eta(\boldsymbol{\alpha})). \tag{2}$$

3. *For a submodule $U \subseteq N[\delta]$ let $\mathcal{P}[\delta]_U \subseteq \mathcal{P}[\delta]$ be the set of parameters of its composition factors. Then $\mathcal{P}[\delta]_U$ is an ideal of $(\mathcal{P}[\delta], \le)$, i.e. a subset of $\mathcal{P}[\delta]$ that is closed under the relation "$\le$" from Notation 2.7.*

4. *The map $U \mapsto \mathcal{P}[\delta]_U$ describes an isomorphism from the submodule lattice of $N[\delta]$ onto the ideal lattice of $(\mathcal{P}[\delta], \le)$, ordered by inclusion.*

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Proof. All claims are proven in [BS00, Theorem C]. For a detailed examination of the change of notation that is involved, see [Var15, Anhang B]. □

The structure of the module $N[q-1]$ can be read off from the transformation properties described in Lemma 2.5.

2.18 Proposition. *The module $N[q-1]$ admits a decomposition as a direct sum of simple G-modules*

$$N[q-1] = \underbrace{\left\langle f_0^{(q-1)}, \dots, f_{q-2}^{(q-1)}, f_\infty^{(q-1)} \right\rangle}_{\cong \mathrm{Sym}^{q-1}(V)} \oplus \underbrace{\left\langle f_0^{(q-1)} - f_{q-1}^{(q-1)} + f_\infty^{(q-1)} \right\rangle}_{\cong \overline{\mathbb{F}}_q}.$$

The set $\mathcal{P}[q-1]$ contains the two parameters

$$(0, \dots, 0) \ \text{with type } (p-1, \dots, p-1),$$
$$(1, \dots, 1) \ \text{with type } (2(p-1), \dots, 2(p-1)).$$

The simple submodules of $N[q-1]$ are parametrized by the types in $\mathcal{T}[q-1]$ according to (2). The type $(p-1, \dots, p-1)$ belongs to the submodule that is isomorphic to $\mathrm{Sym}^{q-1}(V)$.

Remark.   1. In the notation from [BS00], our module $N[q-1]$ corresponds with the module k^P, which is studied in Theorem A of the cited paper. Despite its semi-simplicity, we study this module alongside the other $N[\delta]$, since they are defined uniformly and occur side by side in our applications.

    For $q = 2$, the only module of this class is semi-simple, namely $N[1] = \overline{\mathbb{F}}_2 \oplus V$.

    2. From our previous results for types, in particular Lemma 2.12, we observe that the module $\bigoplus_{\delta=1}^{q-1} N[\delta]$ is multiplicity free. Note that this already follows from a more general statement, see [BS00, Lemma 2.1]. However, we will encounter a situation that is similar to the first argument when we introduce the pattern of n in section 4.

For $\delta = 1$ we observe the following interesting case:

2.19 Proposition. *Let $q > 2$. There is an isomorphism of G-modules*

$$N[1] \cong \mathrm{Sym}^q(V).$$

In particular, $\mathrm{Sym}^q(V)$ is uniserial, that is, the set of its submodules is totally ordered under inclusion.

Proof. The isomorphism can be read off from Lemma 2.5. The uniqueness of the composition series of $N[1]$ follows from Theorem 2.17. $\qquad\square$

The theory of the modules $N[\delta]$ can also be applied to describe some other symmetric powers.

2.20 Proposition. *For $1 \leq \delta \leq q-1$ we identify the module $\mathrm{Sym}^\delta(V)$ with its image under the embedding of G-modules $\mathrm{Sym}^\delta(V) \hookrightarrow N[\delta]$, given by*

$$X^{\delta-i}Y^i \mapsto f_i^{(\delta)}, \quad 0 \leq i \leq \delta - 1,$$
$$Y^\delta \mapsto f_\infty^{(\delta)}$$

and linear extension. As a submodule of $N[\delta]$, $\mathrm{Sym}^\delta(V)$ corresponds with the ideal $\mathcal{P}_0[\delta] := \{\mathbf{t} \in \mathcal{P}[\delta] \mid t_0 = 0\}$ of $(\mathcal{P}[\delta], \leq)$.

Proof. The embedding is obvious from the transformation properties described in Lemma 2.5.
For $1 \leq \delta \leq q-2$ the corresponding ideal is described in [BS00, Section 10]. In the case $\delta = q-1$ the statement follows directly from Proposition 2.18. $\quad\square$

The G-module structure of these symmetric powers can now be easily derived from Theorem 2.17 (for $\delta = q-1$ the symmetric power is a simple module).

3   A filtration of $\mathrm{Sym}^n(V)$

In this section we study a certain submodule of $\mathrm{Sym}^n(V)$ for $n \geq q+1$ and a related submodule filtration. The central element in these constructions is an eigenvector of G, whose transformation properties can be verified by a straightforward computation.

3.1 Lemma. *Let $n \geq q+1$. Multiplication with $XY^q - X^qY$ induces an injective G-homomorphism*

$$\mathrm{Sym}^{n-(q+1)}(V) \otimes (\det)^1 \to \mathrm{Sym}^n(V).$$

Proof. The map is obviously injective. The determinant twist on the left-hand side ensures the G-equivariance due to the fact that $XY^q - X^qY \in \mathrm{Sym}^{q+1}(V)$ is an eigenvector with character $(\det)^1$. $\qquad\square$

We study the image of this G-homomorphism in more detail.

3.2 Notation. Let $n \geq q + 1$. Let $L(n)$ be the submodule of $\mathrm{Sym}^n(V)$ that is isomorphic to $\mathrm{Sym}^{n-(q+1)}(V) \otimes (\det)^1$ by means of the multiplication map described in Lemma 3.1.

For ease of notation, we denote the monomials in $\mathrm{Sym}^n(V)$ by

$$Z_j := X^{n-j}Y^j, \quad 0 \leq j \leq n,$$

as long as $n \geq q + 1$ is fixed.

The following lemma allows us to describe congruences of monomials modulo $L(n)$.

3.3 Lemma. *Let $n \geq q + 1$. A basis of the submodule $L(n) \subseteq \mathrm{Sym}^n(V)$ is given by the elements*

$$Z_{j+q} - Z_{j+1} \qquad \text{with } 0 \leq j \leq n - (q+1).$$

In particular, two monomials $Z_i, Z_j \in \mathrm{Sym}^n(V)$, $0 \leq i, j \leq n$, are congruent modulo $L(n)$ if and only if

$$1 \leq i, j \leq n - 1 \quad \text{and} \quad i \equiv j \mod q - 1.$$

A basis of the quotient module $\mathrm{Sym}^n(V)/L(n)$ is given by

$$\{Z_j \mid 0 \leq j \leq q - 1\} \cup \{Z_n\}.$$

Proof. Obviously, the elements $Z_{j+q} - Z_{j+1}$ are the images of the monomials in $\mathrm{Sym}^{n-(q+1)}(V) \otimes (\det)^1$ under the given multiplication map. The remaining statements follow immediately. $\qquad\square$

3.4 Notation. In the following, we fix the unique decomposition

$$n = \nu + \widehat{\nu}(q - 1) \quad \text{with } 1 \leq \nu \leq q - 1.$$

3.5 Proposition. *Let $n \geq q + 1$. Then there is a G-isomorphism*

$$\mathrm{Sym}^n(V)/L(n) \xrightarrow{\cong} N[\nu] = N[n]$$

given by

$$Z_j \mapsto f_j^{(\nu)}, \quad 0 \leq j \leq q - 1,$$
$$Z_n \mapsto f_\infty^{(\nu)}$$

and linear extension.

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Proof. Obviously, the map is an isomorphism of vector spaces.
In order to verify its G-equivariance, we study the basis of $L(n)$ determined in Lemma 3.3 under the action of the generators of G.
We obtain immediately

$$\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) Z_j = a^{n-j} Z_j = a^{\nu-j} Z_j, \quad 0 \le j \le q-1,$$
$$\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) Z_n = Z_n, \quad \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) Z_0 = Z_n, \quad \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) Z_n = Z_0.$$

Also, we observe

$$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) Z_j = Z_{n-j} \equiv Z_{[\nu-j]} \mod L(n), \quad 1 \le j \le q-1,$$

where "$[\cdot]$" denotes the representative modulo $q-1$ in $\{1,\ldots,q\text{-}1\}$.
For $0 \le j \le q-1$ we have

$$\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) Z_j = \sum_{l=0}^{j} \binom{j}{l} t^{j-l} Z_l.$$

Finally, we must give a presentation of

$$\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) Z_n = \sum_{l=0}^{n} \binom{n}{l} t^{n-l} Z_l$$

with respect to the chosen basis of $\mathrm{Sym}^n(V)/L(n)$.
According to Lemma 3.3 we have

$$\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) Z_n \equiv Z_0 + \sum_{b=1}^{q-1} \underbrace{\left(\sum_{\substack{l=1 \\ l \equiv b \bmod q-1}}^{n-1} \binom{n}{l} \right)}_{=: \lambda_b} t^{\nu-b} Z_b + Z_n \mod L(n).$$

By means of the decomposition of n fixed in Notation 3.4 we obtain for $1 \le b \le q-1$

$$\lambda_b = \sum_{m=0}^{\hat{\nu}} \binom{\nu + \hat{\nu}(q-1)}{b + m(q-1)} - \delta_{\nu,b}$$

with Kronecker delta. Here we make use of the fact that

$$\binom{\nu + \hat{\nu}(q-1)}{b + \hat{\nu}(q-1)} = \begin{cases} 1 & b = \nu \\ 0 & \nu < b \le q-1 \end{cases}$$

to achieve a uniform upper bound of the sum over m.

Using a general result from the theory of binomial coefficients modulo p (see [Var15, Proposition A.16]) the sum can be simplified as follows:

$$\lambda_b = \begin{cases} \binom{\nu}{b} & 1 \leq b \leq \nu - 1 \\ 0 & \nu \leq b \leq q - 1. \end{cases}$$

Hence,

$$\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) Z_n \equiv \sum_{l=0}^{\nu-1} \binom{\nu}{l} t^{\nu-l} Z_l + Z_n \mod L(n).$$

The G-equivariance of the given isomorphism now follows by means of comparison with Lemma 2.5. $\qquad\square$

We can now use the preceding to construct the following G-module filtration of symmetric powers.

3.6 Theorem. *Let* $n \in \mathbb{N}$ *with unique decomposition* $n = \mathfrak{n} + \widehat{\mathfrak{n}}(q+1)$, $0 \leq \mathfrak{n} \leq q$. *Then* $\mathrm{Sym}^n(V)$ *admits a filtration of G-submodules*

$$\{0\} \subsetneq L^{(\widehat{\mathfrak{n}},n)} \subsetneq L^{(\widehat{\mathfrak{n}}-1,n)} \subsetneq \cdots \subsetneq L^{(1,n)} \subsetneq L^{(0,n)} = \mathrm{Sym}^n(V),$$

where

$$L^{(i,n)} \cong \mathrm{Sym}^{n-i(q+1)}(V) \otimes (\det)^i \quad \text{for } 0 \leq i \leq \widehat{\mathfrak{n}}.$$

For $0 \leq i \leq \widehat{\mathfrak{n}} - 1$ *the successive quotients satisfy*

$$L^{(i,n)}/L^{(i+1,n)} \cong N[n - 2i, i].$$

Proof. The existence of the filtration follows immediately if we define $L^{(i,n)}$ to be the image in $\mathrm{Sym}^n(V)$ of $\mathrm{Sym}^{n-i(q+1)}(V) \otimes (\det)^i$ under multiplication with $(XY^q - X^qY)^i$, cf. Lemma 3.1.
Using the notation previously established in this section, this definition gives

$$L^{(i,n)} \cong \mathrm{Sym}^{n-i(q+1)}(V) \otimes (\det)^i$$

and

$$L^{(i+1,n)} \cong L(n - i(q+1)) \otimes (\det)^i.$$

The structure of the successive quotients can now be read off directly by means of Proposition 3.5. $\qquad\square$

Finally, one may ask: Is there a G-module complement of $L(n)$ in $\mathrm{Sym}^n(V)$? The answer is negative if n is not a multiple of q.

3.7 Theorem. *Let* $n \geq q+1$ *be not divisible by q. Then there is no G-module complement of $L(n)$ in* $\mathrm{Sym}^n(V)$.

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Remark. If, on the other hand, n is a multiple of q then the answer is in fact positive. However, the proof of this statement in the general context of symmetric powers is cumbersome due to the required calculations. It is much simpler to make use of the connection between symmetric powers and Drinfeld modular forms. We will discuss this result in a future publication.

The basic idea for our proof of Theorem 3.7 is to show that there is no submodule of $\mathrm{Sym}^n(V)$ that is both a complement to $L(n)$ and G-isomorphic to $N[\nu]$. This variant is equivalent to the theorem according to Proposition 3.5. The actual steps vary depending on q and n as described in the following. First, let us assume that q is not 2. In this situation we may use the following lemma, which describes a necessary (but in general not sufficient) condition for elements of $\mathrm{Sym}^n(V)$ with special transformation properties:

3.8 Lemma. *Let q be strictly larger than 2 and let $n \geq q+1$ be not divisible by q. If an element $P \in \mathrm{Sym}^n(V)$ is invariant under matrices of types $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, then P is a linear combination of monomials Z_j such that*

$$0 \leq j \leq n - (q-1) \quad and \quad j \equiv n \mod q - 1.$$

Proof. Assume $P \in \mathrm{Sym}^n(V)$ has the stated transformation properties. Since

$$\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) Z_j = a^{n-j} Z_j \quad \text{for } 0 \leq j \leq n$$

and P is invariant under the matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we can write

$$P = \lambda Z_n + P'$$

with $\lambda \in \overline{\mathbb{F}}_q$ and P' a linear combination of Z_j such that $j \leq n - (q-1)$ and $j \equiv n \mod q - 1$.
Due to the invariance of P under $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ we have

$$\begin{aligned}
\lambda Z_n + P' = P &= \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) P \\
&= \lambda \sum_{j=0}^{n} \binom{n}{j} t^{n-j} Z_j + \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) P',
\end{aligned}$$

where $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) P'$ is again a linear combination of Z_j with $j \leq n - (q-1)$. In order to obtain that λ must equal 0 under the initial assumption, we have to find an index l such that

$$n - (q-1) < l < n \tag{3}$$

and

$$\binom{n}{l} \neq 0, \tag{4}$$

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(that is, such that Z_l occurs in $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) Z_n$ with a non-trivial coefficient).
Indeed, we find such an index as follows: According to the prerequisites we have the unique decomposition

$$n = m + \widehat{m}q \quad \text{mit } 1 \le m \le q - 1.$$

If, in addition, $m < q - 1$ holds, then $l := n - m$ obviously satisfies condition (3). Furthermore, the Lucas congruence for binomial coefficients implies

$$\binom{n}{n-m} = \binom{m + \widehat{m}q}{\widehat{m}q} \equiv \binom{m}{0}\binom{\widehat{m}}{\widehat{m}} \equiv 1 \mod p,$$

and thus condition (4) is satisfied as well.
For $m = q - 1$ it is trivial to show that $l := n - 1$ satisfies both conditions. $\square$

If we additionally assume that n is not divisible by $q - 1$, Theorem 3.7 is equivalent to the following statement:

3.9 Proposition. *Let $q > 2$ and let $n \ge q + 1$ be divisible neither by q nor by $q - 1$. Then there is no G-homomorphism $N[\nu] \to \mathrm{Sym}^n(V)$ whose image is a G-module complement to $L(n)$.*

Proof. Let $\varphi : N[\nu] \to \mathrm{Sym}^n(V)$ be a G-equivariant homomorphism. In Lemma 3.3 we have seen that the monomial Z_n does not occur in any element of $L(n)$ with a non-trivial coefficient. Therefore the proof is complete if we show that no element in the image of φ contains Z_n.
Consider the basis $\{f_b^{(\nu)} \mid 0 \le b \le q - 1\} \cup \{f_\infty^{(\nu)}\}$ of $N[\nu]$. Using Lemma 2.5 we can derive necessary conditions for the images of the basis elements under φ.
Since φ is G-equivariant, the transformation properties of the basis elements and the monomials Z_j under $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ imply

$$\varphi(f_b^{(\nu)}) = \sum_{\substack{j=0 \\ j \equiv b \bmod q-1}}^{n} \lambda_j^{(b)} Z_j, \quad 0 \le b \le q - 1,$$

$$\varphi(f_\infty^{(\nu)}) = \sum_{\substack{j=0 \\ j \equiv \nu \bmod q-1}}^{n} \lambda_j^{(\infty)} Z_j$$

with coefficients in $\overline{\mathbb{F}}_q$.
Thus all $\varphi(f_b^{(\nu)})$ with $b \ne \nu$ can be written as linear combinations of monomials Z_j with $0 \le j < n$. Since we assume that $q - 1$ does not divide n, this is true for $\varphi(f_{q-1}^{(\nu)})$ in particular.

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This also means that $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \varphi(f_{q-1}^{(\nu)})$ does not contain the monomial Z_n. By means of a straightforward calculation we obtain

$$\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \varphi(f_{q-1}^{(\nu)}) = \sum_{l=0}^{q-1} \binom{q-1}{l} t^{q-1-l} \varphi(f_l^{(\nu)}).$$

On the one hand, we have seen previously that none of the summands for $l \neq \nu$ contains Z_n. Since on the other hand the coefficient of $\varphi(f_\nu^{(\nu)})$ in the remaining summand is

$$\binom{q-1}{\nu} t^{-\nu} = (-1)^\nu t^{-\nu} \neq 0,$$

we see that Z_n does not occur in $\varphi(f_\nu^{(\nu)})$.

To conclude that $\varphi(f_\infty^{(\nu)})$ does not contain Z_n either, consider

$$f_\infty^{(\nu)} - f_\nu^{(\nu)} = F_\infty^{(\nu)} \in N[\nu].$$

According to Lemma 2.3 this element is invariant under matrices of types $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. Hence, its image under the G-equivariant map φ has the same invariances. Thus, Lemma 3.8 implies that Z_n does not occur in $\varphi(f_\infty^{(\nu)} - f_\nu^{(\nu)})$ and therefore neither in $\varphi(f_\infty^{(\nu)})$. $\qquad\square$

If, on the other hand, $q - 1$ divides n, we may use that $N[q-1] \cong \overline{\mathbb{F}}_q \oplus \mathrm{Sym}^{q-1}(V)$ to simplify the proof of the theorem. It is now sufficient to show:

3.10 Proposition. *Let q be strictly larger than 2 and let $n \geq q + 1$ be a multiple of $q - 1$ but not of q. If an element $P \in \mathrm{Sym}^n(V)$ is invariant under the action of G, then $P \in L(n)$.*

Proof. Let $P \in \mathrm{Sym}^n(V)$ be G-invariant. Lemma 3.8 together with the invariance of P under $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ implies that P can be written uniquely as a linear combination of monomials Z_j such that

$$q - 1 \leq j \leq n - (q-1) \quad \text{and} \quad j \equiv n \equiv 0 \mod q - 1.$$

By means of Lemma 3.3 we know that all such monomials are congruent to Z_{q-1} modulo $L(n)$. We can thus write

$$P = \lambda Z_{q-1} + P_L$$

with $\lambda \in \overline{\mathbb{F}}_q$ and $P_L \in L(n)$. Consider the action of $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ on both sides of this equation and compare the respective coefficients of Z_0, taking into account the invariance of P under $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. We obtain $P = P_L \in L(n)$ and the proof is complete. $\qquad\square$

For $q = 2$ the situation is similar, since it is again sufficient to prove that there is no G-invariant element in $\mathrm{Sym}^n(V) \setminus L(n)$. However, we cannot make use of Lemma 3.8 since $q - 1 = 1$.

3.11 Proposition. *Let q equal 2 and let $n \geq 3$ be odd. Then there is no element in $\mathrm{Sym}^n(V) \setminus L(n)$ that is invariant under the action of G.*

Proof. In this situation one can verify directly that any G-invariant element of $\mathrm{Sym}^n(V)$ is of shape

$$
P = \sum_{j=1}^{\frac{n-1}{2}} \lambda_j (Z_j - Z_{n-j})
$$

with coefficients in $\overline{\mathbb{F}}_q$ which implies together with Lemma 3.3 that $P \in L(n)$. $\qquad\square$

Taken together, Propositions 3.9, 3.10 and 3.11 prove Theorem 3.7.

4   Multiplicities of composition factors

The fact that each of the successive quotients of the filtration described in Theorem 3.6 is a determinant twist of some $N[\delta]$ opens the possibility to determine the composition factors of $\mathrm{Sym}^n(V)$ for $n \geq q + 1$. (As mentioned in section 2, the situation is well-known for $n \leq q$.)

4.1 Proposition. *Let $n \geq q + 1$ with unique decomposition $n = \mathfrak{n} + \widehat{\mathfrak{n}}(q+1)$, $0 \leq \mathfrak{n} \leq q$. Then $\mathrm{Sym}^n(V)$ has the same composition factors (counting multiplicities) as the module*

$$
\bigoplus_{i=0}^{\widehat{\mathfrak{n}}-1} N[n - 2i, i] \oplus (\mathrm{Sym}^{\mathfrak{n}}(V) \otimes (\det)^{\widehat{\mathfrak{n}}}).
$$

$\qquad\square$

By definition, the modules $N[n - 2i, i]$ in the above direct sum depend only on the classes of n and i modulo $q - 1$. Therefore as a first step, we study the case where i passes through a full system of representatives modulo $q - 1$.

4.2 Notation. In the remainder of this section let $\mathcal{R}$ be a fixed but arbitrary full system of representatives modulo $q - 1$. For $x \in \mathbb{Z}$ denote by $[x]_\mathcal{R}$ its representative modulo $q - 1$ in $\mathcal{R}$.
For $n \in \mathbb{N}$ the module

$$
\bigoplus_{i \in \mathcal{R}} N[n - 2i, i]
$$

is independent of the choice of $\mathcal{R}$ and depends only on n modulo $q-1$. We call this module the *pattern* of n.

In order to determine all parameters δ such that determinant twists of $N[\delta]$ occur in the pattern of n, we have to solve the congruence

$$n - 2i \equiv \delta \mod q - 1 \tag{5}$$

for $i \in \mathcal{R}$. The structure of the set of solutions depends on whether $q = p^r$ is an odd or even prime power and is an easy exercise in elementary number theory.

4.3 Lemma. *Let $n \in \mathbb{N}$.*

1. *If p equals 2, then for each $1 \leq \delta \leq q-1$ there is precisely one determinant twist of $N[\delta]$ among the direct summands of the pattern of n. The exact twist is determined by*

$$i \equiv 2^{r-1}(n - \delta) \mod q - 1.$$

2. *For $p > 2$ only determinant twists of modules $N[\delta]$ with $\delta \equiv n \bmod 2$ occur as direct summands in the pattern. There are two twists in each such case, given by*

$$i \equiv \frac{n - \delta}{2} \mod q - 1 \quad and \quad i \equiv \frac{n - \delta}{2} + \frac{q - 1}{2} \mod q - 1.$$

The above result immediately implies the following parametrization of the composition factors of the pattern.

4.4 Lemma. *Let $n \in \mathbb{N}$. The composition factors of the pattern of n can be parametrized by the set*

$$\mathcal{K} := \mathcal{K}(n) := \{(\boldsymbol{\alpha}, i) \in \mathcal{T} \times \mathcal{R} \mid \mathfrak{d}(\boldsymbol{\alpha}) \equiv n - 2i \bmod q - 1\}.$$

The composition factor for one such pair $(\boldsymbol{\alpha}, i) \in \mathcal{K}$ is isomorphic to

$$\mathfrak{S}(e(\boldsymbol{\alpha}), \eta(\boldsymbol{\alpha}) + i)$$

and occurs in the summand $N[n - 2i, i]$ of the pattern.
In case p equals 2, the projection $(\boldsymbol{\alpha}, i) \mapsto \boldsymbol{\alpha}$ defines a bijection

$$\mathcal{K} \to \mathcal{T}.$$

For $p > 2$ the map $(\boldsymbol{\alpha}, i) \mapsto \boldsymbol{\alpha}$ describes a surjective $2 : 1$-map

$$\mathcal{K} \to \bigcup_{\substack{1 \leq \delta \leq q-1 \\ \delta \equiv n \bmod 2}} \mathcal{T}[\delta].$$

In general the pattern is not multiplicity free. This difference compared to the module $\bigoplus_{\delta=1}^{q-1} N[\delta]$ (see the remark following Proposition 2.18) is caused by the additional determinant twists in the summands of the pattern.

In order to count the multiplicity of a given simple G-module as a composition factor of the pattern, we have to count the summands $N[n-2i,i]$ which admit this particular simple module as a composition factor. We use the classification of the simple G-modules given in Theorem 1.9.

4.5 Notation. Let $n \in \mathbb{N}$. For $0 \le m \le q-1$ and $\mu \in \mathcal{R}$ define $I_{m,\mu}(n)$ to be the set of all $i \in \mathcal{R}$ such that $\mathfrak{S}(m,\mu)$ is isomorphic to a composition factor of $N[n-2i,i]$.

4.6 Theorem. *Let $n \in \mathbb{N}$. Let $0 \le m \le q-1$ and $\mu \in \mathcal{R}$. Then the simple module $\mathfrak{S}(m,\mu)$ is isomorphic to a composition factor of the pattern of n if and only if m and μ satisfy the congruence*

$$m \equiv n - 2\mu \mod q - 1. \tag{6}$$

In this case:

1. *For $m > 0$ with p-adic expansion $\sum_{j=0}^{r-1} m_j p^j$ one has*

$$I_{m,\mu}(n) = \left\{ \left[\mu - \sum_{j \in U}(p-1-m_j)p^j \right]_{\mathcal{R}} \mid U \subseteq \mathrm{dsupp}(m) \right\}.$$

   The multiplicity of $\mathfrak{S}(m,\mu)$ as a composition factor of the pattern of n is $2^{\# \, \mathrm{dsupp}(m)}$.

2. *For $m = 0$ one has*

$$I_{0,\mu}(n) = \left\{ \left[\mu - \sum_{j \in U}(p-1)p^j \right]_{\mathcal{R}} \mid \emptyset \neq U \subseteq \{0,\ldots,r-1\} \right\}.$$

   The simple module $\mathfrak{S}(0,\mu)$ has multiplicity $2^r - 1$ as a composition factor of the pattern of n.

Proof. First we show the necessity of condition (6). Let therefore $\mathfrak{S}(m,\mu)$ be isomorphic to a composition factor of the pattern. Lemma 4.4 implies that there are $\boldsymbol{\alpha} \in \mathcal{T}$ and $i \in \mathcal{R}$ such that

$$\begin{aligned} m &= e(\boldsymbol{\alpha}), \\ \mu &\equiv \eta(\boldsymbol{\alpha}) + i \mod q - 1, \\ \mathfrak{d}(\boldsymbol{\alpha}) &\equiv n - 2i \mod q - 1. \end{aligned} \tag{7}$$

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After a short calculation, in which we use Lemma 2.13, we find

$$m \equiv n - 2\mu \mod q - 1,$$

as desired.

Now let m and μ satisfy condition (6). We determine the modules $N[n-2i, i]$ that have a composition factor isomorphic to $\mathfrak{S}(m, \mu)$. That is, we have to determine all pairs $(\boldsymbol{\alpha}, i) \in \mathcal{T} \times \mathcal{R}$ that satisfy the set of conditions (7). Then $I_{m,\mu}(n)$ consists of all i obtained this way.

The first condition in (7) implies that only the types $\boldsymbol{\alpha}$ in the preimage of m under the map e must be considered. For each such $\boldsymbol{\alpha}$ the second condition is only satisfied by the unique representative

$$i_{\boldsymbol{\alpha}} = [\mu - \eta(\boldsymbol{\alpha})]_{\mathcal{R}}.$$

Another short calculation involving Lemma 2.13 verifies that all pairs constructed this way satisfy the third condition as well.

The stated shape of the elemens of $I_{m,\mu}(n)$ follows from our prior results for the maps e and η, specifically Lemma 2.16. In particular, the representatives $i_{\boldsymbol{\alpha}}$ are pairwise distinct. $\qquad\square$

Remark.   1. As we have described in Lemma 4.3, we can easily determine explicitly all m and μ that satisfy condition (6).

    2. In addition to the multiplicities in the entire pattern, we can easily count multiplicities of composition factors for submodules of shape $\bigoplus_{i \in J} N[n - 2i, i]$ for a subset $J \subseteq \mathcal{R}$ by considering $J \cap I_{m,\mu}(n)$.

4.7 Theorem. *Let $n \in \mathbb{N}$. Counting multiplicities, the pattern of n has $(2p-1)^r - 1$ composition factors.*

Proof. This theorem can be proven combinatorially by counting all numbers in $\{0, \ldots, q-1\}$ with dual support of a fixed size and adding up the results. Alternatively, for even q the statement follows trivially from Lemma 4.4. $\quad\square$

In addition to the above results for the pattern, we need to determine the composition factors of $L^{(\hat{\mathfrak{n}},n)}$, the smallest non-trivial module of the filtration defined in Theorem 3.6. For brevity's sake we simply call it the *smallest filtration module*.

4.8 Proposition. *Let $n \in \mathbb{N}$ have the unique decomposition $n = \mathfrak{n} + \hat{\mathfrak{n}}(q+1)$, $0 \leq \mathfrak{n} \leq q$. The smallest filtration module is isomorphic to a submodule of $N[n - 2\hat{\mathfrak{n}}, \hat{\mathfrak{n}}]$ and is thus multiplicity free.*
To be more precise, one finds:

1. *For $\mathfrak{n} = 0$*

$$L^{(\hat{\mathfrak{n}},n)} \cong (\det)^{\hat{\mathfrak{n}}}$$

   is itself a simple module and isomorphic to the one-dimensional direct summand of $N[q - 1, \hat{\mathfrak{n}}] = N[n - 2\hat{\mathfrak{n}}, \hat{\mathfrak{n}}]$.

2. *For $1 \leq \mathfrak{n} \leq q - 1$ the module $L^{(\hat{\mathfrak{n}},n)}$ is isomorphic to the submodule*

$$\mathrm{Sym}^{[n - 2\hat{\mathfrak{n}}]}(V) \otimes (\det)^{\hat{\mathfrak{n}}} \subseteq N[n - 2\hat{\mathfrak{n}}, \hat{\mathfrak{n}}],$$

   where "$[\,\cdot\,]$" is the representative modulo $q - 1$ in $\{1, \ldots, q - 1\}$.

   The composition factors of $L^{(\hat{\mathfrak{n}},n)}$ are given by a subset of the composition factors of $N[n - 2\hat{\mathfrak{n}}, \hat{\mathfrak{n}}]$, cf. Proposition 2.20.

3. *For $\mathfrak{n} = q$ we get*

$$L^{(\hat{\mathfrak{n}},n)} \cong N[1, \hat{\mathfrak{n}}] = N[n - 2\hat{\mathfrak{n}}, \hat{\mathfrak{n}}].$$

Proof. The first and second statement follow immediately from the isomorphism $L^{(\hat{\mathfrak{n}},n)} \cong \mathrm{Sym}^{\mathfrak{n}}(V) \otimes (\det)^{\hat{\mathfrak{n}}}$ and the fact that $n - 2\hat{\mathfrak{n}} \equiv \mathfrak{n} \bmod q - 1$. In the third statement we additionally use our previous observation that $\mathrm{Sym}^q(V) \cong N[1]$. $\square$

In particular we see:

4.9 Corollary. *Let $n \in \mathbb{N}$. A simple module may only occur as a composition factor of $\mathrm{Sym}^n(V)$ if it is a composition factor of the pattern of n.*

We already know: To check whether or not the simple module $\mathfrak{S}(m, \mu)$ occurs among the composition factors of $N[n - 2\hat{\mathfrak{n}}, \hat{\mathfrak{n}}]$, we only have to determine if

$$[\hat{\mathfrak{n}}]_{\mathcal{R}} \in I_{m,\mu}(n)$$

holds. However, if the smallest filtration module is a strict submodule of $N[n - 2\hat{\mathfrak{n}}, \hat{\mathfrak{n}}]$, we must restrict to a suitable subset of $I_{m,\mu}(n)$.

4.10 Notation. For $0 \leq m \leq q - 1$ and $\mu \in \mathcal{R}$ we define the subset

$$I^0_{m,\mu}(n) \subseteq I_{m,\mu}(n)$$

to consist of those representatives $i \in \mathcal{R}$ such that $\mathfrak{S}(m, \mu)$ is isomorphic to a composition factor of the submodule $\mathrm{Sym}^{[n - 2i]}(V) \otimes (\det)^i$ of $N[n - 2i, i]$.

As before, this specific set of representatives can be determined explicitly.

4.11 Proposition. *Let $n \in \mathbb{N}$. Further let $0 \le m \le q - 1$ and $\mu \in \mathcal{R}$. Then*

$$m \equiv n - 2\mu \mod q - 1$$

is a necessary condition for $I^0_{m,\mu}(n)$ to be non-empty. If m and μ satisfy the congruence one finds:

1. *For $m > 0$ with p-adic expansion $\sum_{j=0}^{r-1} m_j p^j$ we have*

$$I^0_{m,\mu}(n) = \left\{ \left[\mu - \sum_{j \in U}(p - 1 - m_j)p^j \right]_{\mathcal{R}} \mid U \subseteq \operatorname{dsupp}(m) \setminus \{\max \operatorname{dsupp}(m)\} \right\}.$$

2. *For $m = 0$ we get*

$$I^0_{0,\mu}(n) = \left\{ \left[\mu - \sum_{j \in U}(p - 1)p^j \right]_{\mathcal{R}} \mid \emptyset \ne U \subseteq \{0, \ldots, r - 2\} \right\}.$$

In particular, $I^0_{0,\mu}(n)$ is empty for $r = 1$.

Proof. The necessity of the stated congruence follows from Theorem 4.6, since $I^0_{m,\mu}(n)$ is a subset of $I_{m,\mu}(n)$.

The structure of $I^0_{m,\mu}(n)$ can be determined by closer inspection of the proof of the cited theorem. There, we constructed a representative $i_{\boldsymbol{\alpha}}$ for each $\boldsymbol{\alpha}$ in the preimage of m under e. In the present situation we only consider those $\boldsymbol{\alpha}$ that satisfy an additional constraint: One can show that the simple module associated to a type $\boldsymbol{\alpha}$ is a composition factor of the submodule $\operatorname{Sym}^\delta(V) \subseteq N[\delta]$ if and only if either $\boldsymbol{\alpha} = (p - 1, \ldots, p - 1)$ or if $\alpha_j < p - 1$ holds, where $0 \le j \le r - 1$ is the largest index such that $\alpha_j \ne p - 1$.

In terms of the parametrization of the preimage by subsets of $\operatorname{dsupp}(m)$ (see Proposition 2.15) this means that we allow precisely those subsets that do not include the maximum of $\operatorname{dsupp}(m)$ (with the further exception of the empty set if m equals 0).

The remaining statements now follow as in the proof of Theorem 4.6. $\qquad\square$

In the remaining cases from Proposition 4.8, that is $\mathfrak{n} = 0$ and $\mathfrak{n} = q$, the conditions for $\mathfrak{S}(m, \mu)$ to be a composition factor of the smallest filtration module are straightforward. We combine these with Proposition 4.11 to obtain the following exhaustive answer to this question:

4.12 Theorem. *Let $n \in \mathbb{N}$. Further let $0 \le m \le q - 1$ and $\mu \in \mathcal{R}$. Then:*

1. *For $\mathfrak{n} = 0$ the simple module $\mathfrak{S}(m, \mu)$ is a composition factor of $L^{(\hat{\mathfrak{n}}, n)}$ if and only if*

$$m = 0 \quad and \quad \mu \equiv \widehat{\mathfrak{n}} \mod q - 1.$$

2. *For $1 \leq \mathfrak{n} \leq q - 1$ the simple module $\mathfrak{S}(m, \mu)$ is a composition factor of $L^{(\hat{\mathfrak{n}}, n)}$ if and only if*

$$[\hat{\mathfrak{n}}]_{\mathcal{R}} \in I^0_{m,\mu}(n).$$

3. *For $\mathfrak{n} = q$ the simple module $\mathfrak{S}(m, \mu)$ is a composition factor of $L^{(\hat{\mathfrak{n}}, n)}$ if and only if*

$$[\hat{\mathfrak{n}}]_{\mathcal{R}} \in I_{m,\mu}(n).$$

For our final result, we describe algorithmically the methods obtained previously in this section. Specifically, we show how the individual steps can be combined to count multiplicities of composition factors for a given symmetric power $\mathrm{Sym}^n(V)$.

We begin by determining some data of the pattern. This step may be reused for different symmetric powers, since the output depends only on $n \bmod q - 1$.

4.13 Algorithm (Multiplicities in the pattern). Input: $n \in \mathbb{N}$.
Pass over all pairs (m, μ) with $0 \leq m \leq q - 1$ and $\mu \in \mathcal{R}$ such that

$$m \equiv n - 2\mu \mod q - 1.$$

For each such pair (m, μ) apply the following procedure:

1. Initialize the set

$$I^0 := \begin{cases} \{[\mu]_{\mathcal{R}}\} & m > 0 \\ \emptyset & m = 0. \end{cases}$$

2. If $m = q - 1$, then put

$$I_{q-1,\mu}(n) := I^0_{q-1,\mu}(n) := I^0$$

and terminate the procedure for the current pair.

3. Else determine the p-adic coefficients m_j of m, $0 \leq j \leq r - 1$, and read off the dual support $\mathrm{dsupp}(m)$. Put

$$l := \max \mathrm{dsupp}(m)$$

and initialize the set

$$I^1 := \left\{ [\mu - (p - 1 - m_l)p^l]_{\mathcal{R}} \right\}.$$

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4. Let U pass through all non-empty subsets of $\operatorname{dsupp}(m) \setminus \{l\}$. Define

$$i_U := \mu - \sum_{j \in U}(p - 1 - m_j)p^j$$

and put

$$I^0 := I^0 \cup \{[i_U]_{\mathcal{R}}\},$$
$$I^1 := I^1 \cup \left\{[i_U - (p - 1 - m_l)p^l]_{\mathcal{R}}\right\}.$$

5. Display the result for the current pair:

$$I^0_{m,\mu}(n) := I^0,$$
$$I_{m,\mu}(n) := I^0 \cup I^1.$$

Proof (Correctness). The correctness of the algorithm follows immediately from the explicit descriptions of $I_{m,\mu}(n)$ and $I^0_{m,\mu}(n)$ in Theorem 4.6 and Proposition 4.11, respectively. $\qquad\square$

For each simple module $\mathfrak{S}(m, \mu)$ we can now determine its multiplicity $\lambda(n; m, \mu)$ as a composition factor of $\operatorname{Sym}^n(V)$.

4.14 Algorithm (Multiplicity in symmetric powers). Input: $n \in \mathbb{N}$.

(Initialization) Pass through all pairs (m, μ) with $0 \le m \le q - 1$ and $\mu \in \mathcal{R}$. Put
$$\lambda(n; m, \mu) := 0.$$

(Precomputation) Compute the unique decomposition

$$\widehat{\mathfrak{n}} = u + v(q - 1) \quad \text{with } 0 \le u \le q - 2 \text{ and } v \in \mathbb{N}_0.$$

Define
$$J := \{[\widehat{\mathfrak{n}} - l]_{\mathcal{R}} \mid 1 \le l \le u\}.$$

If $\mathfrak{n} = q$, then add the representative $[\widehat{\mathfrak{n}}]_{\mathcal{R}}$ to the set J.

(Main loop) Pass through the pairs (m, μ) with $0 \le m \le q - 1$ and $\mu \in \mathcal{R}$ that satisfy

$$m \equiv n - 2\mu \mod q - 1.$$

For each such pair perform the following procedure:

1. (Data of the pattern) Look up the sets $I_{m,\mu}(n)$ and $I^0_{m,\mu}(n)$ in the output of Algorithm 4.13.

2. (Smallest filtration module) If $\mathfrak{n} = q$, put

$$\lambda_E(n; m, \mu) := 0.$$

   Else if $\mathfrak{n} = 0$, define

$$\lambda_E(n; m, \mu) := \begin{cases} 1 & m = 0 \text{ and } \mu \equiv \widehat{\mathfrak{n}} \bmod q - 1 \\ 0 & \text{else.} \end{cases}$$

   Else if $1 \leq \mathfrak{n} \leq q - 1$, put

$$\lambda_E(n; m, \mu) := \begin{cases} 1 & [\widehat{\mathfrak{n}}]_{\mathcal{R}} \in I^0_{m,\mu}(n) \\ 0 & \text{else.} \end{cases}$$

3. (Result) The result for the current pair (m, μ) is

$$\lambda(n; m, \mu) := v\,(\#I_{m,\mu}(n)) + \#\,(J \cap I_{m,\mu}(n)) + \lambda_E(n; m, \mu).$$

Proof (Correctness). In Proposition 4.1 we have already established that $\mathrm{Sym}^n(V)$ has the same composition factors as

$$\bigoplus_{i=0}^{\widehat{\mathfrak{n}}-1} N[n - 2i, i] \oplus \mathrm{Sym}^{\mathfrak{n}}(V) \otimes (\det)^{\widehat{\mathfrak{n}}}.$$

Let us first consider the partial sum over $N[n - 2i, i]$ for i between 0 and $\widehat{\mathfrak{n}} - u - 1$. It consists precisely of v copies of the pattern of n, since $\widehat{\mathfrak{n}} - u = v(q-1)$ according to the precomputation. The contribution to the multiplicity of the composition factor $\mathfrak{S}(m, \mu)$ is $v\#I_{m,\mu}(n)$.

By construction, the set J contains all the remaining indices i between $\widehat{\mathfrak{n}} - u - 1$ and $\widehat{\mathfrak{n}} - 1$. Furthermore, it contains the additional element $\widehat{\mathfrak{n}}$ if and only if the smallest filtration module is isomorphic to $N[n - 2\widehat{\mathfrak{n}}, \widehat{\mathfrak{n}}]$ (that is, if and only if $\mathfrak{n}$ equals q).

We know from the remark following Theorem 4.6 that the multiplicity of $\mathfrak{S}(m, \mu)$ as a composition factor of $\bigoplus_{i \in J} N[n - 2i, i]$ is $\#\,(J \cap I_{m,\mu}(n))$.

Finally, for $\mathfrak{n} \leq q - 1$ the contribution of the smallest filtration module has to be checked as described in Theorem 4.12 and is encoded in the variable $\lambda_E(n; m, \mu)$. $\qquad\square$

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