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Abstract

We provide a version of the Hopf-Oleinik boundary point lemma for general elliptic equations in divergence form under the sharp requirements on the coefficients of equations and on the boundary of a domain.

1 Introduction

The Boundary Point Principle, known also as the “normal derivative lemma”, is one of the important tools in qualitative analysis of partial differential equations. This principle states that *a supersolution of a partial differential equation with a minimum value at a boundary point, must increase linearly away from its boundary minimum provided the boundary is smooth enough.*

The history of this famous principle begins with a pioneering paper of S. Zaremba [Zar10] where the above assertion was established for the Laplace equation in a three-dimensional domain Ω satisfying an interior touching ball condition. Notice that the major part of all known results on the normal derivative lemma concerns equations with nondivergence structure and strong solutions. A key contribution to the investigation of this problem for elliptic equations was made simultaneously and independently by E. Hopf [Hop52] and O.A. Oleinik [Ole52] (by this reason, all the statements of such type are often called the Hopf-Oleinik lemma). The corresponding comprehensive historical review can be found in [AN16].

The case of the divergence-type equations

$$-\operatorname{div}(A(x)\operatorname{grad}u(x)) = 0$$

is less studied. It is well known that the Boundary Point Principle fails for uniformly elliptic equations in divergence form with bounded and even continuous coefficients of the matrix $A(x)$ (see, for instance, [Gil60], [GT83], [PS07] and [Naz12]). However, the normal derivative lemma holds true if the leading coefficients are more smooth.

The sharp requirements on the regularity of the boundary of a domain, providing the validity of the Boundary Point Principle for the Laplace equation, were independently and simultaneously formulated in the papers [VM67] and [Wid67].

The first result for weak solutions of equations with divergence structure was proved by R. Finn and D. Gilbarg [FG57]. They considered a two-dimensional bounded domain with $\mathcal{C}^{1,\alpha}$-regular boundary, the Hölder continuous entries of the matrix $A(x)$ and continuous lower order coefficients. Recently, in [KK17] (see also [SdL15]) the normal derivative lemma was established in n-dimensional domains ($n \geqslant 3$) for equations with the lower-order coefficients from the Lebesgue space L^q, $q > n$, under the same assumptions on the leading coefficients and on the boundary as in [FG57].

The goal of this paper is to prove a version of the normal derivative lemma for the general divergence-type equations under strongly weakened assumptions closed to the necessary ones.

1.1 Notation and conventions

Throughout the paper we use the following notation:
$x = (x_1, \dots, x_{n-1}, x_n)$ is a point in $\mathbb{R}^n$;
$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$;
$|x|$ is the Euclidean norm in $\mathbb{R}^n$;
$B_r(x^0)$ is the open ball in $\mathbb{R}^n$ with center x^0 and radius r; $\quad B_r = B_r(0)$;

D_i denotes the operator of (weak) differentiation with respect to x_i;
$D = (D_1, \dots, D_{n-1}, D_n)$.

We use the letters C and N (with or without indices) to denote various constants. To indicate that, say, C depends on some parameters, we list them in parentheses: $C(\dots)$.

Definition 1. We say that a function $\sigma : [0, 1] \to \mathbb{R}_+$ belongs to the class $\mathcal{D}$ if

- σ is increasing, and $\sigma(0) = 0$;
- $\sigma(t)/t$ is summable and decreasing.

It should be noted that our assumption about the decay of $\sigma(t)/t$ is not restrictive (see Remark 1.2 [AN16] for more details).

For $\sigma \in \mathcal{D}$ we define the function $\mathcal{J}_\sigma$ as

$$\mathcal{J}_\sigma(s) := \int\limits_0^s \frac{\sigma(\tau)}{\tau} d\tau.$$

Definition 2. Let $\mathcal{E}$ be a bounded domain in $\mathbb{R}^n$. We say that a function $\zeta : \mathcal{E} \to \mathbb{R}$ belongs to the class $\mathcal{C}^{0,\mathcal{D}}(\mathcal{E})$, if

- $\zeta \in \mathcal{C}(\overline{\mathcal{E}})$;
- $|\zeta(x) - \zeta(y)| \leqslant \sigma(|x - y|),\ \forall x, y \in \mathcal{E}$, and σ belongs to the class $\mathcal{D}$.

In what follows, Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. We suppose that $\partial\Omega \in \mathcal{C}^{1,\mathcal{D}}$, which means that $\partial\Omega$ is locally the graph of a $\mathcal{C}^1$-function F satisfying $DF \in \mathcal{C}^{0,\mathcal{D}}$.
$d(x)$ denotes the distance between x and $\partial\Omega$.

$\mathcal{L}$ is a uniformly elliptic operator with measurable coefficients:

$$\mathcal{L}u \equiv -D_i(a^{ij}(x)D_j u) + b^i(x)D_i u. \tag{1}$$

We adopt the convention that the indices i and j run from 1 to n. We also adopt the convention regarding summation with respect to repeated indices.

The coefficients of $\mathcal{L}$ should satisfy the following conditions:

$$\begin{gathered} a^{ij} \in \mathcal{C}^{0,\mathcal{D}}(\Omega) \qquad \text{for all} \quad i,j = 1,\dots,n, \\ \nu\mathcal{I}_n \leq (a^{ij}(x)) \leq \nu^{-1}\mathcal{I}_n, \end{gathered} \tag{2}$$

and

$$\omega(r) := \sup_{x\in\Omega} \int\limits_{B_r(x)\cap\Omega} \frac{|\mathbf{b}(y)|}{|x-y|^{n-1}} \cdot \frac{d(y)}{d(y)+|x-y|}\, dy \to 0 \qquad \text{as} \quad r \to 0. \tag{3}$$

Here ν is a positive constant, $\mathcal{I}_n$ is identity $(n \times n)$-matrix, while $\mathbf{b}(y) = (b^1(y),\dots,b^n(y))$.

Notice that condition (3) says that the function $\dfrac{|\mathbf{b}(y)|}{|x-y|^{n-1}} \cdot \dfrac{d(y)}{d(y)+|x-y|}$ is integrable uniformly with respect to x. Moreover, in any strict interior subset of Ω condition (3) means that $\mathbf{b}$ is an element of the Kato class $K_{n,1}$. (For the definition of the scale of the Kato classes $K_{n,\alpha}$ with $\alpha < n$ the reader is referred to the paper [DH98]).

2 Main result

Our main result is stated as follows.

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega \in \mathcal{C}^{1,\mathcal{D}}$, let $\mathcal{L}$ be defined by (1), and let assumptions (2)-(3) be fulfilled.*

In addition, assume that a nonconstant function $u \in \mathcal{C}^1(\overline{\Omega})$ satisfies, in the weak sense, the inequality

$$\mathcal{L}u \geqslant 0 \qquad \textit{in} \quad \Omega.$$

Then, if u attends its minimum at point $x^0 \in \partial\Omega$, we have

$$\frac{\partial u}{\partial \mathbf{n}}(x^0) < 0.$$

Here $\frac{\partial}{\partial \mathbf{n}}$ is the derivative with respect to the exterior normal on $\partial\Omega$.

Remark 2.2. *Notice that all the assumptions on a^{ij} and* b *are invariant under the $\mathcal{C}^{1,\mathcal{D}}$-regular change of variables. So, without loss of generality, we may consider $\partial\Omega$ locally as a flat boundary $x_n = 0$. In addition we may assume without restriction that $x^0 = 0$ and $B_R \cap \mathbb{R}^n_+ \subset \Omega$ for some $R > 0$.*

Consider for $0 < \rho < R/2$ the point $x^\rho = (0, \dots, 0, \rho)$ and the annulus

$$A_\rho := \{x : \rho/2 < |x - x^\rho| < \rho\} \subset \Omega.$$

Let x^* be an arbitrary point in $\overline{A}_\rho$. Following [FG57] (see also [SdL15]) we define the auxiliary functions z and ψ_{x^*} as solutions of the problems

$$\begin{cases} \mathcal{L}_0 z = 0 & \text{in } A_\rho, \\ z = 1 & \text{on } \partial B_{\rho/2}(x^\rho), \\ z = 0 & \text{on } \partial B_\rho(x^\rho), \end{cases} \qquad \begin{cases} \mathcal{L}_0^{x^*} \psi_{x^*} = 0 & \text{in } A_\rho, \\ \psi_{x^*} = 1 & \text{on } \partial B_{\rho/2}(x^\rho), \\ \psi_{x^*} = 0 & \text{on } \partial B_\rho(x^\rho), \end{cases} \tag{4}$$

where the operators $\mathcal{L}_0$ and $\mathcal{L}_0^{x^*}$ are determined by the formulas

$$\mathcal{L}_0 z := -D_i(a^{ij}(x) D_j z) \qquad \text{and} \qquad \mathcal{L}_0^{x^*} \psi_{x^*} := -D_i(a^{ij}(x^*) D_j \psi_{x^*}),$$

respectively. It is well known that $\psi_{x^*} \in \mathcal{C}^\infty(\overline{A}_\rho)$, and the existence of (unique) weak solution z follows from the general elliptic theory.

Lemma 2.3. *There exists $C_1 = C_1(n, \nu, \sigma) > 0$ such that the inequality*

$$|Dz(x^*) - D\psi_{x^*}(x^*)| \leqslant C_1 \frac{\mathcal{J}_\sigma(2\rho)}{\rho} \tag{5}$$

holds true for all $\rho \leqslant R/2$.

Proof. Setting $w^{(1)} = z - \psi_{x^*}$ we observe that $w^{(1)}$ vanishes on ∂A_ρ. Hence, $w^{(1)}$ can be represented in A_ρ as

$$w^{(1)}(x) = \int\limits_{A_\rho} G_\rho^{x^*}(x, y) \mathcal{L}_0^{x^*} w^{(1)}(y) dy \overset{(\star)}{=} \int\limits_{A_\rho} G_\rho^{x^*}(x, y) \left(\mathcal{L}_0^{x^*} z(y) - \mathcal{L}_0 z(y) \right) dy,$$

where $G_\rho^{x^*}$ stands for the Green function of the operator $\mathcal{L}_0^{x^*}$ in A_ρ. The equality $(\star)$ follows from the relation $\mathcal{L}_0^{x^*} \psi_{x^*} = \mathcal{L}_0 z = 0$, see (4).

Applying integration by parts we get another version of the representation formula:

$$w^{(1)}(x) = \int\limits_{A_\rho} D_{y_i} G_\rho^{x^*}(x, y) \left(a^{ij}(x^*) - a^{ij}(y) \right) D_j z(y) dy. \tag{6}$$

Differentiating both sides of equality (6) with respect to x_k we get the system of equations

$$D_k w^{(1)}(x^*) = \int\limits_{A_\rho} D_{x_k} D_{y_i} G_\rho^{x^*}(x^*, y) \left(a^{ij}(x^*) - a^{ij}(y)\right) D_j z(y) dy, \qquad k = 1, \dots, n. \tag{7}$$

According to Lemma 3.2 [GW82], $z \in \mathcal{C}^1(\overline{A}_\rho)$, and the following estimate holds for $y \in \overline{A}_\rho$:

$$|Dz(y)| \leqslant \frac{N_1}{\rho}, \tag{8}$$

where N_1 depends only on n, ν, and σ. Moreover, due to Theorem 3.3 [GW82] we have also the estimate for the Green function $G_\rho^{x^*}(x, y)$:

$$|D_x D_y G_\rho^{x^*}(x, y)| \leqslant \frac{N_2}{|x - y|^n} \qquad \forall x, y \in A_\rho, \tag{9}$$

where N_2 is completely determined by n, ν, and σ.

Finally, combination of (7)-(9) with condition (2) implies

$$|Dw^{(1)}(x^*)| \leqslant \frac{N_1 N_2}{\rho} \int\limits_{B_{2\rho}(x^*)} \frac{\sigma(|x^* - y|)}{|x^* - y|^n} \, dy,$$

and (5) follows. $\square$

Further, we introduce the barrier function v defined as the weak solution of the Dirichlet problem

$$\begin{cases} \mathcal{L}v = 0 & \text{in} \quad A_\rho, \\ \;\; v = 1 & \text{on} \quad \partial B_{\rho/2}(x^\rho), \\ \;\; v = 0 & \text{on} \quad \partial B_\rho(x^\rho). \end{cases} \tag{10}$$

Theorem 2.4. *There exists $\rho_0 > 0$ such that for all $\rho \leqslant \rho_0$ the problem (10) admits a unique solution $v \in \mathcal{C}^1(\overline{A}_\rho)$. Moreover, the inequality*

$$|Dv(x) - Dz(x)| \leqslant C_2 \, \frac{\omega(2\rho)}{\rho} \tag{11}$$

holds true for any $x \in A_\rho$. Here $C_2 = C_2(n, \nu, \sigma) > 0$, ρ_0 is completely defined by n, ν, σ, and ω, while $z \in \mathcal{C}^1(\overline{A}_\rho)$ is defined in (4).

Proof. Consider in A_ρ the auxiliary function $w^{(2)} = v - z$. We observe that $w^{(2)}$ vanishes on ∂A_ρ, and

$$\mathcal{L}_0 w^{(2)} = \mathcal{L}_0 v = \mathcal{L} v - b^i D_i v = -b^i \left(D_i w^{(2)} + D_i z \right) \quad \text{in} \quad A_\rho.$$

Hence, $w^{(2)}$ can be represented in A_ρ via corresponding Green function $G_{0,\rho}(x,y)$ as

$$w^{(2)}(x) = -\int\limits_{A_\rho} G_{0,\rho}(x,y) b^i(y) \left(D_i w^{(2)}(y) + D_i z(y) \right) dy.$$

Differentiation with respect to x_k gives

$$D_k w^{(2)}(x) = -\int\limits_{A_\rho} D_{x_k} G_{0,\rho}(x,y) b^i(y) \left(D_i w^{(2)}(y) + D_i z(y) \right) dy.$$

Therefore, we get the relation

$$(\mathbb{I} + \mathbb{T})\, D w^{(2)} = -\mathbb{T} D z, \tag{12}$$

where $\mathbb{I}$ stands for the identity operator, while $\mathbb{T}$ denotes the matrix operator whose (k,i) entries are integral operators with kernels $D_{x_k} G_{0,\rho}(x,y) b^i(y)$.

The statement of Theorem follows from the next assertion.

Lemma 2.5. *The operator $\mathbb{T}$ is bounded in $\mathcal{C}(\overline{A}_\rho)$, and*

$$\|\mathbb{T}\|_{\mathcal{C}\to\mathcal{C}} \leqslant C_3\, \omega(2\rho),$$

where C_3 depends only on n, ν, and σ.

Proof. Theorem 3.3 [GW82] provides the estimate

$$|D_x G_{0,\rho}(x,y)| \leqslant N_3 \min\left\{ |x-y|^{1-n}; \operatorname{dist}\{y, \partial A_\rho\} |x-y|^{-n} \right\} \tag{13}$$

for any $x, y \in A_\rho$. Here N_3 is the constant depending only on n, ν, and σ.

Since $\operatorname{dist}\{y, \partial A_\rho\} \leqslant d(y)$ for any $y \in A_\rho$, the combination of estimate (13) with condition (3) gives

$$\int\limits_{B_r(x)\cap A_\rho} |D_x G_{0,\rho}(x,y)|\, |\mathbf{b}(y)|\, dy \leqslant N_3 \omega(r), \qquad \forall x \in A_\rho, \quad r \leqslant 2\rho. \tag{14}$$

For arbitrary vector function $\mathbf{f} \in \mathcal{C}(\overline{A}_\rho)$ we have

$$|\mathbb{T}\mathbf{f}(x)| \leqslant \|\mathbf{f}\|_{\mathcal{C}(\overline{A}_\rho)} \cdot \int\limits_{A_\rho} |D_x G_{0,\rho}(x,y)|\, |\mathbf{b}(y)|\, dy \leqslant N_3\, \omega(2\rho) \cdot \|\mathbf{f}\|_{\mathcal{C}(\overline{A}_\rho)}, \quad x \in \overline{A}_\rho.$$

It remains to show that $\mathbb{T}\mathbf{f} \in \mathcal{C}(\overline{A}_\rho)$. For $x, \tilde{x} \in \overline{A}_\rho$ and any small $\delta > 0$ we have

$$(\mathbb{T}\mathbf{f})(x) - (\mathbb{T}\mathbf{f})(\tilde{x}) = J_1 + J_2$$
$$:= \Big(\int\limits_{A_\rho \cap B_\delta(\tilde{x})} + \int\limits_{A_\rho \setminus B_\delta(\tilde{x})} \Big) \big(D_x G_{0,\rho}(x,y) - D_x G_{0,\rho}(\tilde{x},y)\big) \otimes [\mathbf{b}(y) \cdot \mathbf{f}(y)]\, dy.$$

If $|x - \tilde{x}| \leqslant \delta/2$ then (14) gives

$$|J_1| \leqslant \|\mathbf{f}\|_{\mathcal{C}(\overline{A}_\rho)} \cdot \int\limits_{B_\delta(\tilde{x}) \cap A_\rho} \big(|D_x G_{0,\rho}(x,y)| + |D_x G_{0,\rho}(\tilde{x},y)|\big)\, |\mathbf{b}(y)|\, dy$$
$$\leqslant 2N_3\, \omega(3\delta/2) \cdot \|\mathbf{f}\|_{\mathcal{C}(\overline{A}_\rho)}.$$

Thus, given ε we can choose δ such that $|J_1| \leqslant \varepsilon$.

On the other hand, $D_x G_{0,\rho}(x,y)$ is continuous w.r.t. x for $x \neq y$. Thus, it is equicontinuous on the compact set

$$\{(x,y) :\ x \in \overline{B}_{\delta/2}(\tilde{x}) \cap \overline{A}_\rho,\ y \in \overline{A}_\rho \setminus B_\delta(\tilde{x})\}.$$

Therefore, for chosen δ we obtain, as $|x - \tilde{x}| \to 0$,

$$|J_2| \leqslant \|\mathbf{f}\|_{\mathcal{C}(\overline{A}_\rho)} \cdot \int\limits_{A_\rho} |\mathbf{b}(y)|\, dy \cdot \max_{y \in \overline{A}_\rho \setminus B_\delta(\tilde{x})} |D_x G_{0,\rho}(x,y) - D_x G_{0,\rho}(\tilde{x},y)| \to 0,$$

and the Lemma follows. □

We continue the proof of Theorem 2.4. Choose the value of ρ_0 so small that $\omega(2\rho_0) \leqslant (2C_3)^{-1}$, where C_3 is the constant from Lemma 2.5. Then by the Banach theorem the operator $(\mathbb{I} + \mathbb{T})$ in (12) is invertible. This gives the existence and uniqueness of $w^{(2)} \in \mathcal{C}^1(\overline{A}_\rho)$, and thus, the unique solvability of the problem (10). Moreover, Lemma 2.5 and inequality (8) provide (11). The proof is complete. □

To prove Theorem 2.1 we need the following maximum principle.

Lemma 2.6. *Let $\mathcal{L}$ be defined by (1), and let assumptions (2)-(3) be satisfied in a domain $\mathcal{E}$. Suppose that a function $w \in \mathcal{C}^1(\mathcal{E})$ satisfies*

$$\mathcal{L}w \geqslant 0 \quad in\ \ \mathcal{E}; \qquad w \geqslant 0 \quad on\ \ \partial\mathcal{E}.$$

Then $w \geqslant 0$ in $\mathcal{E}$.

Proof. In the paper [Zha96] the Harnack inequality was established for the divergence-type operators with the Hölder continuous coefficients a^{ij} and b^i belonging to the Kato class $K_{n,1}$. However, it is mentioned in [Zha96] that the assumption of the Hölder continuity of leading coefficients is needed only for the pointwise gradient estimate of the Green function for the operator $\mathcal{L}_0$. Since by [GW82, Theorem 3.3] this estimate holds for operators with Dini coefficients (see (13)), the Harnack inequality is valid for the operator $\mathcal{L}$. This implies the assertion of Lemma. $\square$

Proof of Theorem 2.1. It is well known that the Boundary Point Principle holds true for the operator with constant coefficients. Using this statement for the operator $\mathcal{L}_0^{x^*}$ (see (4)) with $x^* = 0$ in the annulus A_1 and rescaling A_1 into A_ρ we get the estimate

$$D_n\psi_0(0) \geqslant \frac{N_4(n,\nu)}{\rho} > 0.$$

Furthermore, the inequalities (5) and (11) imply for sufficiently small ρ

$$\begin{aligned} D_n v(0) &\geqslant D_n\psi_0(0) - |Dz(0) - D\psi_0(0)| - |Dv(0) - Dz(0)| \\ &\geqslant \frac{N_4}{\rho} - C_1\,\frac{\mathcal{J}_\sigma(2\rho)}{\rho} - C_2\,\frac{\omega(2\rho)}{\rho} \geqslant \frac{N_4}{2\rho}. \end{aligned}$$

Given ρ, we have for sufficiently small ε

$$\mathcal{L}(u - u(0) - \varepsilon v) \geqslant 0 \quad \text{in } \ A_\rho; \qquad u - u(0) - \varepsilon v \geqslant 0 \quad \text{on } \ \partial A_\rho.$$

By Lemma 2.6 the estimate $u - u(0) \geqslant \varepsilon v$ holds true in A_ρ, with equality at the origin. This gives

$$\frac{\partial u}{\partial \mathbf{n}}(0) = -D_n u(0) \leqslant -\varepsilon D_n v(0),$$

which completes the proof. $\square$

3 Some sufficient conditions for validity of (3)

Throughout this section we will denote various constants depending on n only by the letter N without indices.

In the paper [Naz12], where equations *in non-divergence form* were studied, the following restrictions on the lower order coefficients b^i were imposed: either

$$\mathbf{b} \in L^n(\Omega); \qquad \sup_{x\in\Omega} \|\mathbf{b}\|_{n,B_\rho(x)\cap\Omega} \leqslant C\sigma_1(\rho), \tag{15}$$

or

$$|\mathbf{b}(y)| \leqslant C\,\frac{\sigma_1(d(y))}{d(y)} \tag{16}$$

(here $\sigma_1 \in \mathcal{D}$).

Lemma 3.1. *The restriction (15) implies the validity of condition (3). The same statement is true for (16), if $\partial\Omega \in \mathcal{C}^{1,\mathcal{D}}$.*

Proof. To simplify the notation, assume that $\mathbf{b}$ is extended by zero outside of Ω.

Let (15) hold. Then one can estimate $\omega(r)$ as

$$\omega(r) \leqslant \sup_{x\in\Omega} \sum_{k=0}^{\infty} \int\limits_{r/2^{k+1}}^{r/2^k} \int\limits_{\mathcal{S}_1} |\mathbf{b}(x+\rho\Theta)|\, d\Theta d\rho$$

$$\leqslant \sup_{x\in\Omega} \sum_{k=0}^{\infty} \Bigg(\int\limits_{r/2^{k+1}}^{r/2^k} \int\limits_{\mathcal{S}_1} |\mathbf{b}(x+\rho\Theta)|^n \rho^{n-1}\, d\Theta d\rho \Bigg)^{\frac{1}{n}} \cdot \Bigg(\int\limits_{r/2^{k+1}}^{r/2^k} \int\limits_{\mathcal{S}_1} \frac{d\Theta d\rho}{\rho} \Bigg)^{\frac{n-1}{n}}$$

$$\leqslant NC \sum_{k=0}^{\infty} \sigma_1(r/2^k) \leqslant NC\mathcal{J}_{\sigma_1}(2r),$$

and (3) follows.

Now let $\partial\Omega \in \mathcal{C}^{1,\mathcal{D}}$, and let (16) hold. If $d(x) \geq 2r$ then we use the decay of $\sigma_1(t)/t$ and estimate the integral in (3) via

$$C\,\frac{\sigma_1(r)}{r} \int\limits_{B_r(x)} \frac{dy}{|x-y|^{n-1}} \leqslant NC\sigma_1(r).$$

If $d(x) < 2r$ then, as in Remark 2.2, we can assume that $\partial\Omega$ is locally the plane $x_n = 0$, and $x = (0,\dots,0,\rho)$. We split the integral in (3) into two parts

$$I_1 + I_2 := \Bigg(\int\limits_{B_r(x)\setminus A} + \int\limits_{B_r(x)\cap A} \Bigg) \frac{|\mathbf{b}(y)|}{|x-y|^{n-1}} \cdot \frac{y_n}{y_n + |x-y|}\, dy,$$

where $A = \{y : |y_n - \rho| < \rho/2\}$.

On the set $B_r(x)\setminus A$ we have $|x_n - \rho| > 3y_n$, and therefore

$$I_1 \leqslant NC \int\limits_0^{3r} \int\limits_0^{r} \frac{\sigma(y_n)t^{n-2}}{(y_n+t)^n}\, dt dy_n \leqslant NC\mathcal{J}_{\sigma_1}(3r).$$

Next,

$$I_2 \leqslant NC\sigma(3\rho/2) \int\limits_0^{r} \int\limits_{\rho/2}^{3\rho/2} \frac{dy_n}{(y_n+t)^{n-1}}\, \frac{t^{n-2}}{\rho+t}\, dt \leqslant NC\sigma(3r),$$

and (3) again follows. $\square$

References

[AN16] D. E. Apushkinskaya and A. I. Nazarov. A counterexample to the Hopf-Oleinik lemma (elliptic case). *Anal. PDE*, 9(2):439–458, 2016.

[DH98] E. B. Davies and A. M. Hinz. Kato class potentials for higher order elliptic operators. *J. London Math. Soc. (2)*, 58(3):669–678, 1998.

[FG57] R. Finn and D. Gilbarg. Asymptotic behavior and uniquenes of plane subsonic flows. *Comm. Pure Appl. Math.*, 10:23–63, 1957.

[Gil60] D. Gilbarg. Some hydrodynamic applications of function theoretic properties of elliptic equations. *Math. Z.*, 72:165–174, 1959/1960.

[GT83] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[GW82] M. Grüter and K.-O. Widman. The Green function for uniformly elliptic equations. *Manuscripta Math.*, 37(3):303–342, 1982.

[Hop52] E. Hopf. A remark on linear elliptic differential equations of second order. *Proc. Amer. Math. Soc.*, 3:791–793, 1952.

[KK17] V. Kozlov and N. Kuznetsov. A comparison theorem for super- and subsolutions of $\nabla^2 u + f(u) = 0$ and its application to water waves with vorticity. preprint, arXiv: 1710.032251.v1 [math.AP], 2017.

[Naz12] A. I. Nazarov. A centennial of the Zaremba-Hopf-Oleinik lemma. *SIAM J. Math. Anal.*, 44(1):437–453, 2012.

[Ole52] O. A. Oleĭnik. On properties of solutions of certain boundary problems for equations of elliptic type. *Mat. Sb. (N.S.)*, 30 (72):695–702, 1952.

[PS07] P. Pucci and J. Serrin. *The maximum principle*, volume 73 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 2007.

[SdL15] J. C. Sabina de Lis. Hopf maximum principle revisited. *Electron. J. Differential Equations*, 2015(115):1–9, 2015.

[VM67] G. M. Veržbinskiĭ and V. G. Maz'ya. The asymptotics of solutions of the Dirichlet problem near a non-regular frontier. *Dokl. Akad. Nauk SSSR*, 176:498–501, 1967. [Russian]; English transl. in Soviet Math. Dokl. **8** (1967), 1110–1113.

[Wid67] K.-O. Widman. Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations. *Math. Scand.*, 21:17–37, 1967.

[Zar10] S. Zaremba. Sur un problème mixte relatif à l' équation de Laplace. *Bull. Acad. Sci. Cracovie. Cl. Sci. Math. Nat. Ser. A*, pages 313–344, 1910.

[Zha96] Q. Zhang. A Harnack inequality for the equation $\nabla(a\nabla u) + b\nabla u = 0$, when $|b| \in K_{n+1}$. *Manuscripta Math.*, 89(1):61–77, 1996.