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A Liouville theorem for stationary incompressible fluids of von Mises type

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Abstract

We consider entire solutions u of the equations describing the stationary flow of a generalized Newtonian fluid in 2D concentrating on the question, if a Liouville-type result holds in the sense that the boundedness of u implies its constancy. A positive answer is true for p -fluids in the case $p > 1$ (including the classical Navier-Stokes system for the choice $p = 2$), and recently we established this Liouville property for the Prandtl-Eyring fluid model, for which the dissipative potential has nearly linear growth. Here we finally discuss the case of perfectly plastic fluids whose flow is governed by a von Mises-type stress-strain relation formally corresponding to the case $p = 1$. It turns out that, for dissipative potentials of linear growth, the condition of μ -ellipticity with exponent $\mu < 2$ is sufficient for proving the Liouville theorem.

In this note we look at entire solutions $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of the homogeneous equation

$$(1) \quad -\operatorname{div} [T^D(\varepsilon(u))] + u^k \partial_k u + \nabla \pi = 0$$

together with the incompressibility condition

$$(2) \quad \operatorname{div} u = 0.$$

Here u denotes the velocity field of a fluid and $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the a priori unknown pressure function. In equation (1) and in what follows we adopt the convention of summation with respect to indices repeated twice. By $\varepsilon(u)$ we denote the symmetric gradient of the field u and T^D represents the deviatoric part of the stress tensor being characteristic for the fluid

under consideration. For further mathematical and also physical explanations we refer to the monographs of Ladyzhenskaya [14], Galdi [8], [9] and of Málek, Necăs, Rokyta, Růžička [15] as well as to the book [6]. A case of particular interest arises, when T^D is of the type

$$(3) \quad T^D = \nabla H$$

for a potential H such that

$$(4) \quad H(\varepsilon) = h(|\varepsilon|)$$

holds with a given density $h : [0, \infty) \rightarrow [0, \infty)$ of class C^2 . Combining (3) and (4) we see that

$$(5) \quad T^D(\varepsilon) = \frac{h'(|\varepsilon|)}{|\varepsilon|} \varepsilon$$

holds, and equation (5) includes as particular cases

(i) power law models:

$$h(t) = t^p \text{ or } = (\delta + t^2)^{p/2}, \quad 1 < p < \infty, \quad \delta > 0,$$

(ii) Prandtl-Eyring fluids: $h(t) = t \ln(1 + t)$, $t \geq 0$.

As a matter of fact we recover the Navier-Stokes equation (NSE) just by letting $h(t) = t^2$, and in their fundamental paper [13], Koch, Nadirashvili, Seregin and Sverák obtained the following Liouville-type result as a byproduct of their investigations on the regularity of solutions to the instationary variant of (NSE).

THEOREM 1. *Suppose that $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a solution of (1) and (2). Let in addition (3) and (4) hold for the choice $h(t) = \nu t^2$ with some positive constant ν . Then, if*

$$(6) \quad \sup_{\mathbb{R}^2} |u| < \infty,$$

u must be a constant vector.

For (NSE), different types of Liouville theorems were studied. For example, Gilbarg and Weinberger showed in their paper [10] the constancy of finite energy solutions to (NSE) in the plane, more precisely, the conclusion of Theorem 1 remains valid if (6) is replaced by

$$(7) \quad \int_{\mathbb{R}^2} |\nabla u|^2 dx < \infty.$$

We wish to remark that the proofs of the above results use the linearity of the leading part of (NSE) in an essential way. However, referring to the results obtained in [2], [3], [7], [21] and [22], we could show by applying appropriate arguments:

THEOREM 2. *Suppose that $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ is a (weak) solution of equations (1) and (2) on the whole plane with T^D given by (3) and (4) for a function h being subject to (i) or (ii). Suppose that either (6) holds or that (7) is replaced by*

$$(8) \quad \int_{\mathbb{R}^2} h(|\nabla u|) \, dx < \infty.$$

Then u is a constant vector.

REMARK 1. *In the case of (NSE) weak solutions of some local Sobolev class are automatically smooth. This is in general not clear for generalized Newtonian fluids and we therefore assume at least $u \in C^1$. A slightly weaker formulation can be found in [12].*

REMARK 2. *A comprehensive survey of Theorem 2 including further related results is given in the paper [4].*

Let us now turn to the von Mises flow (see, e.g., [20], [11], [16]) for which we formally have

$$(9) \quad h(t) = t, \quad t \geq 0,$$

which means (recall (4)) $H(\varepsilon(u)) = |\varepsilon(u)|$. Since this density is neither differentiable nor strictly convex, equation (3) and thereby identity (1) can only be interpreted in a very weak sense, and we have no idea, if in this setting a Liouville-type result can be expected. For this reason we follow the ideas of [1] and replace the density from (9) through a family $h_\mu, \mu > 1$, of more regular densities being still of linear growth and such that $h_\mu(|\varepsilon(u)|) \rightarrow |\varepsilon(u)|$ as $\mu \rightarrow \infty$. For example we may take ($\mu > 1$)

$$(10) \quad h_\mu(t) = (\mu - 1)\Phi_\mu(t), \quad t \geq 0,$$

where we have abbreviated

$$(11) \quad \begin{aligned} \Phi_\mu(t) &:= \int_0^t \int_0^s (1+r)^{-\mu} \, dr \, ds \\ &= \begin{cases} \frac{1}{\mu-1}t + \frac{1}{\mu-1} \frac{1}{\mu-2}(t+1)^{-\mu+2} - \frac{1}{\mu-1} \frac{1}{\mu-2}, & \mu \neq 2, \\ t - \ln(1+t), & \mu = 2. \end{cases} \end{aligned}$$

Note that actually

$$\lim_{\mu \rightarrow \infty} h_\mu(t) = t,$$

and if we formally let $\mu = 1$ in the first line of (11), then - up to negligible terms - we recover the Prandtl-Eyring model (ii). In the same spirit, the choice $\mu < 1$ leads to p -fluids with value $p := 2 - \mu$. Of course our considerations are not limited to the particular

density h_μ . More general, we can choose any function $h : [0, \infty) \rightarrow [0, \infty)$ of class C^2 such that

$$(12) \quad h(0) = h'(0) = 0,$$

$$(13) \quad c_1(1+t)^{-\mu} \leq \min \left\{ h''(t), \frac{h'(t)}{t} \right\},$$

$$(14) \quad \max \left\{ h''(t), \frac{h'(t)}{t} \right\} \leq \frac{c_2}{1+t}$$

for any $t \geq 0$ and with exponent

$$(15) \quad \mu \in (1, \infty),$$

c_1, c_2 denoting positive constants. It is immediate that the functions h_μ defined in (10) satisfy (12) - (14). Moreover, if we define H according to (4), then the conditions (12) - (14) imply the μ -ellipticity of H , i.e.

$$(16) \quad c_1(1+|\varepsilon|)^{-\mu}|\sigma|^2 \leq D^2H(\varepsilon)(\sigma, \sigma) \leq c_2(1+|\varepsilon|)^{-1}|\sigma|^2$$

being valid for (2×2) -matrices ε, σ . In addition, the potential H is of linear growth. More precisely we have (see Lemma 2.7 in [1])

$$(17) \quad c_1 2^{-\mu}(|\varepsilon| - 1) \leq H(\varepsilon) \leq c_2|\varepsilon|$$

with c_1 and c_2 from (13) and (14), respectively. Now we can state our main result:

THEOREM 3. *Let $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ denote a (weak) solution of (1) and (2) on the whole plane with deviatoric stress tensor defined according to (3) and (4), where h satisfies the conditions (12) - (15). In addition we assume that*

$$(18) \quad \mu < 2.$$

If then the velocity field u is bounded, it must be a constant vector.

REMARK 3. *The limitation (18) enters for two reasons. First, as it will become evident from the proof of Theorem 3, it plays the role of a technical restriction making it possible to manage certain quantities. Second, the results obtained in [1] suggest that there is some hope for the existence of regular weak solutions in case $\mu < 2$ motivating our assumption $u \in C^1$, whereas counterexamples taken from a slightly different setting (see [5]) indicate that for $\mu > 2$ equations of the form (1) may fail to have solutions even on bounded domains, which can be found in some Sobolev space. Due to the linear growth of H stated in (17) and with respect to the ellipticity condition (16), the space of functions with bounded deformation (see, e.g. [19], [17], [18]) seems to be the appropriate class for discussing (1) but it is not evident how to give a reasonable “very weak” formulation of equation (1) and to investigate the Liouville property in this setting.*

REMARK 4. We emphasize that our arguments for the proof of Theorem 3 fail in dimension $n \geq 3$. However, for the three dimensional Stokes-type problem,

$$\begin{cases} -\operatorname{div} [\nabla H(\varepsilon(u))] + \nabla \pi = 0 \\ \operatorname{div} u = 0 \end{cases}$$

we have a variant of Theorem 1.3 from [7], i.e. under the assumptions of Theorem 3 on the function H and the parameter μ , every entire solution $u \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ of the above system, for which $|x|^{-\alpha}|u(x)|$ is bounded with some $\alpha \in [0, 1/2)$, must be constant. The proof follows along the lines of [7]. In the case $n = 2$, this result holds true even for the optimal parameter range $\alpha \in [0, 1)$, cf. Theorem 1.1 in [7].

Proof of Theorem 3. We follow the arguments outlined in Section 3 of [7] keeping the notation introduced in Theorem 3 and assuming that all the hypothesis of Theorem 3 are valid. We start with

Lemma 1. *There exists a constant $c = c(\|u\|_{L^\infty(\mathbb{R}^2)}) < \infty$ such that*

$$(19) \quad \int_{Q_R(x_0)} H(\varepsilon(u)) \, dx \leq cR$$

holds for any square $Q_R(x_0) \subset \mathbb{R}^2$, $Q_R(x_0) := \{x \in \mathbb{R}^2 : |x_i - x_{0i}| < R, i = 1, 2\}$, $R > 0$, $x_0 \in \mathbb{R}^2$.

Proof of Lemma 1. From equations (1) and (2) we infer (recalling also (3)) after an integration by parts

$$(20) \quad 0 = \int_{Q_{2R}(x_0)} DH(\varepsilon(u)) : \varepsilon(\varphi) \, dx + \int_{Q_{2R}(x_0)} u^k \partial_k u^i \varphi^i \, dx$$

for any test vector φ such that $\operatorname{div} \varphi = 0$ on $Q_{2R}(x_0)$ and $\varphi = 0$ on $\partial Q_{2R}(x_0)$. Let $\eta \in C_0^\infty(Q_{2R}(x_0))$ be a cut-off function such that $0 \leq \eta(x) \leq 1$, $\eta \equiv 1$ on $Q_R(x_0)$ and $|\nabla \eta| \leq cR^{-1}$, with c denoting a generic positive constant. Observing that the scalar function $f = \operatorname{div}(\eta^2 u)$ fulfills the assumptions of Lemma 3.2 in [7], we infer that there exists $w \in \dot{W}^{1,2}(Q_{2R}(x_0), \mathbb{R}^2)$ such that

$$\begin{aligned} \operatorname{div} w &= f \quad \text{together with} \\ \|\nabla w\|_{L^2(Q_{2R}(x_0))} &\leq c\|f\|_{L^2(Q_{2R}(x_0))} \end{aligned}$$

for a constant $c > 0$ which is independent of R and x_0 . Now choosing $\varphi = \eta^2 u - w$ in (20), we arrive at the identity

$$(21) \quad \begin{aligned} &\int_{Q_{2R}(x_0)} DH(\varepsilon(u)) : \varepsilon(u) \eta^2 \, dx + 2 \int_{Q_{2R}(x_0)} \frac{\partial H}{\partial \varepsilon_{i\alpha}}(\varepsilon(u)) \partial_\alpha \eta u^i \eta \, dx \\ &- \int_{Q_{2R}(x_0)} DH(\varepsilon(u)) : \varepsilon(w) \, dx \\ &= \int_{Q_{2R}(x_0)} u^k \partial_k u^i w^i \, dx - \int_{Q_{2R}(x_0)} u^k \partial_k u^i u^i \eta^2 \, dx. \end{aligned}$$

Let us look at the quantities on the l.h.s. of (21): for any tensor ε it holds (recall (4) and (5))

$$DH(\varepsilon) : \varepsilon = \frac{h'(|\varepsilon|)}{|\varepsilon|} |\varepsilon|^2,$$

and the convexity of h (see (13)) together with (12) implies

$$0 = h(0) \geq h(t) - th'(t),$$

thus $th'(t) \geq h(t)$ and therefore

$$(22) \quad \int_{Q_{2R}(x_0)} DH(\varepsilon(u)) : \varepsilon(u) \eta^2 dx \geq \int_{Q_{2R}(x_0)} \eta^2 H(\varepsilon(u)) dx.$$

Using the boundedness of DH (compare (14)) we see

$$\left| 2 \int_{Q_{2R}(x_0)} \frac{\partial H}{\partial \varepsilon_{i\alpha}}(\varepsilon(u)) \partial_\alpha \eta u^i \eta dx \right| \leq c \int_{Q_{2R}(x_0)} |\nabla \eta| |u| dx \leq cR$$

on account of $|\nabla \eta| \leq c/R$ by the choice of η and due to the boundedness of u . Again by the boundedness of DH it follows from the properties of w

$$\begin{aligned} \left| \int_{Q_{2R}(x_0)} DH(\varepsilon(u)) : \varepsilon(w) dx \right| &\leq c \int_{Q_{2R}(x_0)} |\varepsilon(w)| dx \\ &\leq cR \|\varepsilon(w)\|_{L^2(Q_{2R}(x_0))} \leq cR \|\operatorname{div}(\eta^2 u)\|_{L^2(Q_{2R}(x_0))} \\ &\leq cR \|u \cdot \nabla \eta\|_{L^2(Q_{2R}(x_0))} \leq cR. \end{aligned}$$

Let us note that during our calculations c is a generic constant independent of R and x_0 . Combining the above estimates with (22) and returning to (21), we get

$$(23) \quad \int_{Q_{2R}(x_0)} \eta^2 H(\varepsilon(u)) dx \leq cR + |\text{r.h.s. of (21)}|.$$

For the quantities occurring on the r.h.s. of (21) we can quote (3.6) and (3.7) in [7], hence

$$(24) \quad |\text{r.h.s. of (21)}| \leq cR,$$

and by inserting (24) into (23) the claim of Lemma 1 follows. \square

Up to now neither the condition of μ -ellipticity (see (13) and (16)) nor the bound (18) on the parameter μ have entered our discussion, which means that estimate (19) is valid under much weaker hypotheses as required in Theorem 3. The full strength of our assumptions is needed in the next step. We return to equation (20) and replace φ by $\partial_\alpha \varphi$ for $\varphi \in C_0^\infty(Q_{\frac{3}{2}R}(x_0), \mathbb{R}^2)$, $\operatorname{div} \varphi = 0$. After an integration by parts we obtain

$$(25) \quad \begin{aligned} 0 &= \int_{Q_{\frac{3}{2}R}(x_0)} D^2 H(\varepsilon(u)) (\partial_\alpha \varepsilon(u), \varepsilon(\varphi)) \, dx \\ &\quad - \int_{Q_{\frac{3}{2}R}(x_0)} u^k \partial_k u^i \partial_\alpha \varphi^i \, dx, \quad \alpha = 1, 2. \end{aligned}$$

At this stage we recall our assumption $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, which enables us with the help of the difference-quotient technique to deduce $u \in W_{\text{loc}}^{2,2}(\mathbb{R}^2, \mathbb{R}^2)$ from equation (20) and to justify (25). Note also that (25) corresponds exactly to formula (3.10) in [7] and with the choice of φ as in this reference, we deduce the identity (3.12). Without any changes we can pass to inequality (3.18) from this reference, i.e. setting $\omega := D^2 H(\varepsilon(u)) (\partial_\alpha \varepsilon(u), \partial_\alpha \varepsilon(u))$, we have

$$(26) \quad \int_{Q_{\frac{3}{2}R}(x_0)} \varphi^2 \omega \, dx \leq \delta \int_{Q_{\frac{3}{2}R}(x_0)} \omega \, dx + c(\delta) \left[\frac{1}{R^2} \int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx + \frac{1}{R} \int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx \right]$$

for any $\delta > 0$ and all squares $Q_{2R}(x_0)$. Specifying φ as $0 \leq \varphi \leq 1$, $\varphi = 1$ on $Q_{\frac{3}{2}R}(x_0)$ and $|\nabla \varphi| \leq c/R$, we furthermore pass to inequality (3.19) from [7] which reads as

$$(27) \quad \begin{aligned} \int_{Q_R(x_0)} \omega \, dx &\leq \delta \int_{Q_{2R}(x_0)} \omega \, dx \\ &+ c(\delta) \left[R^{-4} \int_{Q_{2R}(x_0)} |u|^2 \, dx + R^{-3} \int_{Q_{2R}(x_0)} |u|^2 \, dx \right. \\ &\left. + R^{-2} \int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 \, dx + R^{-1} \int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 \, dx \right]. \end{aligned}$$

We see that it remains to discuss the quantity $\int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 \, dx$. We have

$$\begin{aligned} \int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 \, dx &= - \int_{Q_{2R}(x_0)} u^i \partial_j (\varepsilon_{ij}(u) \varphi^2) \, dx \\ &= - \int_{Q_{2R}(x_0)} u^i \partial_j \varepsilon_{ij}(u) \varphi^2 \, dx - \int_{Q_{2R}(x_0)} u^i \varepsilon_{ij}(u) \partial_j \varphi^2 \, dx \\ &\leq c \left[\int_{Q_{2R}(x_0)} |\nabla \varepsilon(u)| \varphi^2 \, dx + R^{-1} \int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx \right] \end{aligned}$$

and (for any $\tau > 0$)

$$(28) \quad \begin{aligned} \int_{Q_{2R}(x_0)} |\nabla \varepsilon(u)| \varphi^2 \, dx &= \int_{Q_{2R}(x_0)} \varphi |\nabla \varepsilon(u)| (1 + |\varepsilon(u)|)^{-\mu/2} (1 + |\varepsilon(u)|)^{\mu/2} \varphi \, dx \\ &\leq \tau \int_{Q_{2R}(x_0)} \omega \, dx + c\tau^{-1} \int_{Q_{2R}(x_0)} \varphi^2 (1 + |\varepsilon(u)|)^\mu \, dx, \end{aligned}$$

where we have used Hölder's inequality and condition (16). Let us choose $\tau := \delta c(\delta)^{-1} R^2$ in (28) with $c(\delta)$ from (27). Then it follows with a new constant $\tilde{c}(\delta)$

$$(29) \quad c(\delta)R^{-2} \int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 dx \leq \delta \int_{Q_{2R}(x_0)} \omega dx + \\ \tilde{c}(\delta) \left[R^{-4} \int_{Q_{2R}(x_0)} \varphi^2 (1 + |\varepsilon(u)|)^\mu dx + R^{-3} \int_{Q_{2R}(x_0)} |\varepsilon(u)| dx \right],$$

and if we select $\tau := \delta c(\delta)^{-1} R$ in (28), we find

$$(30) \quad c(\delta)R^{-1} \int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 dx \leq \delta \int_{Q_{2R}(x_0)} \omega dx + \\ \tilde{c}(\delta) \left[R^{-2} \int_{Q_{2R}(x_0)} \varphi^2 (1 + |\varepsilon(u)|)^\mu dx + R^{-2} \int_{Q_{2R}(x_0)} |\varepsilon(u)| dx \right].$$

In (29) and (30) we have to get rid of the quantity $\int_{Q_{2R}(x_0)} \varphi^2 (1 + |\varepsilon(u)|)^\mu dx$ by absorbing it into the left-hand sides. It holds for any $\lambda > 0$

$$(31) \quad \int_{Q_{2R}(x_0)} (1 + |\varepsilon(u)|)^\mu \varphi^2 dx \leq c \left[R^2 + \int_{Q_{2R}(x_0)} |\varepsilon(u)|^\mu \varphi^2 dx \right] \\ \leq c \left[R^2 + \lambda \int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 dx + R^2 \lambda^{\frac{\mu}{\mu-2}} \right]$$

on account of Young's inequality and due to our assumption $\mu < 2$. By choosing λ proportional to R^2 we infer from (29)

$$(32) \quad c(\delta)R^{-2} \int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 dx \leq \delta \int_{Q_{2R}(x_0)} \omega dx \\ + \tilde{c}(\delta) \left[R^{-2} + R^{-2+\frac{2\mu}{\mu-2}} + R^{-3} \int_{Q_{2R}(x_0)} |\varepsilon(u)| dx \right],$$

whereas $\lambda \sim R$ in combination with (30) yields

$$(33) \quad c(\delta)R^{-1} \int_{Q_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 dx \leq \delta \int_{Q_{2R}(x_0)} \omega dx + \\ \tilde{c}(\delta) \left[1 + R^{\frac{\mu}{\mu-2}} + R^{-2} \int_{Q_{2R}(x_0)} |\varepsilon(u)| dx \right].$$

We insert (32) and (33) into (27), replacing δ by $\delta/3$ and get for any $\delta > 0$ and all squares

$Q_{2R}(x_0) \subset \mathbb{R}^2$

$$\begin{aligned}
(34) \quad & \int_{Q_R(x_0)} \omega \, dx \leq \delta \int_{Q_{2R}(x_0)} \omega \, dx \\
& + c(\delta) \left[R^{-4} \int_{Q_{2R}(x_0)} |u|^2 \, dx + R^{-3} \int_{Q_{2R}(x_0)} |u|^2 \, dx \right. \\
& + R^{-4} \int_{Q_{2R}(x_0)} 1 \, dx + R^{-4+\frac{2\mu}{\mu-2}} \int_{Q_{2R}(x_0)} 1 \, dx \\
& + R^{-2} \int_{Q_{2R}(x_0)} 1 \, dx + R^{-2+\frac{\mu}{\mu-2}} \int_{Q_{2R}(x_0)} 1 \, dx \\
& \left. + R^{-3} \int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx + R^{-2} \int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx \right] \\
& =: \delta \int_{Q_{2R}(x_0)} \omega \, dx + c(\delta)\Theta(R).
\end{aligned}$$

To inequality (34) we can apply Lemma 3.1 from [7] to get

$$(35) \quad \int_{Q_R(x_0)} \omega \, dx \leq c\Theta(R)$$

for arbitrary squares $Q_R(x_0)$. Clearly, $\Theta(R)$ is bounded and hence

$$(36) \quad \int_{\mathbb{R}^2} \omega \, dx =: \omega_\infty < \infty.$$

Our goal is to show $\Theta(R) \rightarrow 0$ as $R \rightarrow \infty$ since, together with inequality (35), this implies $\varepsilon(\nabla u) \equiv 0$, hence $\nabla^2 u \equiv 0$ which means that u is an affine function. The assumption on the boundedness of u then gives the assertion of Theorem 3.

We start with the term $R^{-2} \int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx$ by noting that due to $h'(0) = h(0) = 0$ we have (for $\beta > 0$ arbitrarily small)

$$\begin{aligned}
h(x) &= \int_0^1 (1-t)h''(tx) \, dt x^2 \stackrel{(13)}{\geq} c_1 x^2 \int_0^1 (1-t)(1+tx)^{-\mu} \, dt \geq c_1 x^2 \int_0^{\beta/x} \frac{1-t}{(1+tx)^\mu} \, dt \\
&\geq c_1 x^2 \int_0^{\beta/x} (1-t) \, dt (1+\beta)^{-\mu} = c_1 x^2 (1+\beta)^{-\mu} \left(\frac{\beta}{x} - \underbrace{\frac{1}{2} \frac{\beta^2}{x^2}}_{\leq \frac{1}{2} \beta \frac{1}{x}} \right) \geq \frac{1}{2} \beta x c_1 (1+\beta)^{-\mu},
\end{aligned}$$

which means that

$$(37) \quad h(x) \geq c(\beta)x \quad \text{for all } x \geq \beta.$$

Consequently, it holds

$$R^{-2} \int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx = R^{-2} \int_{\substack{Q_{2R}(x_0) \\ \cap \{|\varepsilon(u)| \geq \beta\}}} |\varepsilon(u)| \, dx + R^{-2} \int_{\substack{Q_{2R}(x_0) \\ \cap \{|\varepsilon(u)| < \beta\}}} |\varepsilon(u)| \, dx =: T_1 + T_2,$$

with $\lim_{R \rightarrow \infty} T_1 = 0$ by (37) and (19). Moreover, $\limsup_{R \rightarrow \infty} T_2 \leq c\beta$ and since β may be chosen arbitrarily small, we infer

$$(38) \quad \lim_{R \rightarrow \infty} \frac{1}{R^2} \int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx = 0.$$

Next we note that, choosing a test function $\varphi \in C_0^\infty(Q_{2R}(x_0))$ with $\varphi \equiv 1$ on $T_{\frac{3}{2}R}(x_0) := Q_{\frac{3}{2}R}(x_0) - \overline{Q_R(x_0)}$ and such that $\text{spt } \varphi \subset T_{2R}(x_0) := Q_{2R}(x_0) - \overline{Q_{\frac{R}{2}}(x_0)}$, we can derive the following more refined version of inequality (26), which is identical to inequality (3.27) from [7]:

$$(39) \quad \int_{Q_R(x_0)} \omega \, dx \leq \delta \int_{Q_{\frac{3}{2}R}(x_0)} \omega \, dx + c(\delta) \left[\frac{1}{R^2} \int_{T_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx + \frac{1}{R} \int_{T_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx + \frac{1}{R} \left(\int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx \right)^{1/2} \left(\int_{T_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx \right)^{1/2} \right].$$

With the same arguments as in [7], we derive inequalities (3.28), (3.29) for the term $\int_{T_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx$:

$$(40) \quad \int_{T_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx \leq c \left[\int_{T_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 \, dx + \frac{1}{R^2} \int_{T_{2R}(x_0)} |u|^2 \, dx \right] \leq c \left[\Phi(R) + \frac{1}{R^2} \int_{T_{2R}(x_0)} |u|^2 \, dx \right],$$

where

$$\Phi(R) := \left(\int_{T_{2R}(x_0)} \omega \, dx \right)^{1/2} \left(\int_{T_{2R}(x_0)} \varphi^2 (1 + |\varepsilon(u)|)^\mu \, dx \right)^{1/2} + \frac{1}{R} \int_{T_{2R}(x_0)} |\varepsilon(u)| \, dx.$$

With the estimate (31) we infer

$$\Phi(R) \leq c \left(\int_{T_{2R}(x_0)} \omega \, dx \right)^{1/2} \left[R + \left(\int_{T_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 \, dx \right)^{1/2} \right] + \frac{1}{R} \int_{T_{2R}(x_0)} |\varepsilon(u)| \, dx,$$

and an application of Young's inequality yields ($\alpha > 0$ may be arbitrarily small)

$$\Phi(R) \leq c \left(\int_{T_{2R}(x_0)} \omega \, dx \right)^{1/2} \left[R + \alpha^{-1} + \alpha \int_{T_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 \, dx \right] + \frac{1}{R} \int_{T_{2R}(x_0)} |\varepsilon(u)| \, dx.$$

Since

$$(41) \quad \int_{T_{2R}(x_0)} \omega \, dx \rightarrow 0 \quad \text{for } R \rightarrow \infty$$

by (36), we may absorb the term $\alpha \int_{T_{2R}(x_0)} \varphi^2 |\varepsilon(u)|^2 \, dx$ in the middle-term of the estimate (40) for α sufficiently small, thus

$$\begin{aligned} & \int_{T_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx \\ & \leq c \left[\left(\int_{T_{2R}(x_0)} \omega \, dx \right)^{1/2} (R + \alpha^{-1}) + \frac{1}{R} \int_{T_{2R}(x_0)} |\varepsilon(u)| \, dx + \frac{1}{R^2} \int_{T_{2R}(x_0)} |u|^2 \, dx \right] \end{aligned}$$

and the boundedness of u together with (41) and (38) therefore implies

$$\frac{1}{R} \int_{T_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx \rightarrow 0 \quad \text{for } R \rightarrow \infty.$$

It remains to discuss the quantity

$$\Psi(R) := \left(\frac{1}{R} \int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx \right)^{1/2} \left(\frac{1}{R} \int_{T_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx \right)^{1/2}.$$

We have already established that the second factor goes to 0 for $R \rightarrow \infty$ and it therefore suffices to show that the first factor is bounded. Arguing as above, we see that

$$\begin{aligned} & \int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx \\ & \leq c \left[\left(\int_{Q_{2R}(x_0)} \omega \, dx \right)^{1/2} (R + \alpha^{-1}) + \frac{1}{R} \int_{Q_{2R}(x_0)} |\varepsilon(u)| \, dx + \frac{1}{R^2} \int_{Q_{2R}(x_0)} |u|^2 \, dx \right]. \end{aligned}$$

Now, the boundedness of $\int_{Q_{2R}(x_0)} \omega \, dx$ together with (38) clearly implies the boundedness of $R^{-1} \int_{Q_{\frac{3}{2}R}(x_0)} |\nabla u|^2 \, dx$, which finishes the proof of Theorem 3. \square

References

- [1] M. Bildhauer, M. Fuchs, and J. Müller. Existence and regularity for stationary incompressible flows with dissipative potentials of linear growth. To appear in *Journal of Mathematical Fluid Mechanics*. Preprint: <http://www.math.uni-sb.de/service/preprints/preprint398.pdf>.

- [2] M. Bildhauer, M. Fuchs, and G. Zhang. Liouville-type theorems for steady flows of degenerate power law fluids in the plane. *J. Math. Fluid Mech.*, 15(3):583–616, 2013.
- [3] M. Fuchs. Liouville theorems for stationary flows of shear thickening fluids in the plane. *J. Math. Fluid Mech.*, 14(3):421–444, 2012.
- [4] M. Fuchs. Variations on Liouville’s theorem in the setting of stationary flows of generalized Newtonian fluids in the plane. In *Proceedings of the St. Petersburg Mathematical Society. Vol. XV. Advances in mathematical analysis of partial differential equations*, volume 232 of *Amer. Math. Soc. Transl. Ser. 2*, pages 79–98. Amer. Math. Soc., Providence, RI, 2014.
- [5] M. Fuchs, J. Müller, and C. Tietz. Signal recovery via TV-type energies. *Algebra i Analiz*, 29(4):159–195, 2017.
- [6] M. Fuchs and G. Seregin. *Variational methods for problems from plasticity theory and for generalized Newtonian fluids*, volume 1749 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.
- [7] M. Fuchs and G. Zhang. Liouville theorems for entire local minimizers of energies defined on the class $L \log L$ and for entire solutions of the stationary Prandtl-Eyring fluid model. *Calc. Var. Partial Differential Equations*, 44(1-2):271–295, 2012.
- [8] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I. Linearized steady problems*, volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994.
- [9] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. II. Nonlinear steady problems*, volume 39 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994.
- [10] D. Gilbarg and H. F. Weinberger. Asymptotic properties of steady plane solutions of the Navier-Stokes equations with bounded Dirichlet integral. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5(2):381–404, 1978.
- [11] K. Hohenemser and W. Prager. Über die Ansätze der Mechanik isotroper Continua. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 12(4):216–226, 1932.
- [12] B. Ja Jin and K. Kang. Liouville Theorem for the Steady-State Non-Newtonian Navier-Stokes Equations in Two Dimensions. *Journal of Mathematical Fluid Mechanics*, 16(2):275–292, 2014.
- [13] G. Koch, N. Nadirashvili, G. Seregin, and V. Sverák. Liouville theorems for the Navier-Stokes equations and applications. *Acta Math.*, 203(1):83–105, 2009.

- [14] O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow*. Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2. Gordon and Breach, Science Publishers, New York-London-Paris, 1969.
- [15] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDEs*, volume 13 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall, London, 1996.
- [16] W. Prager. *Einführung in die Kontinuumsmechanik*. Lehr- und Handbücher der Ingenieurwissenschaften, Bd. 20. Birkhäuser Verlag, Basel-Stuttgart, 1961.
- [17] P.-M. Suquet. Sur une nouveau cadre fonctionnel pour les équations de la plasticité. *C. R. Acad. Sc. Paris (A)*, 286:1129–1132, 1978.
- [18] P.-M. Suquet. Un espace fonctionnel pour les équations de la plasticité. *Ann. Fac. Sci. Toulouse Math. (5)*, 1(1):77–87, 1979.
- [19] R. Temam and G. Strang. Functions of bounded deformation. *Arch. Rational Mech. Anal.*, 75(1):7–21, 1980/81.
- [20] R. von Mises. Mechanik der festen Körper im plastisch- deformablen Zustand. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, pages 582–592, 1913.
- [21] G. Zhang. A note on Liouville theorem for stationary flows of shear thickening fluids in the plane. *J. Math. Fluid Mech.*, 15(4):771–782, 2013.
- [22] G. Zhang. Liouville theorems for stationary flows of shear thickening fluids in 2D. *Ann. Acad. Sci. Fenn. Math.*, 40(2):889–905, 2015.

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