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Martin Fuchs and Joachim Weickert

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Martin Fuchs

Universität des Saarlandes
Fachbereich 6.1 Mathematik
Postfach 15 11 50
D–66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Joachim Weickert

Mathematical Image Analysis Group
Faculty of Mathematics and Computer Science
Saarland University
Building E1.7
D–66041 Saarbrücken
Germany
weickert@mia.uni-saarland.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax:	+ 49 681 302 4443
e-Mail:	preprint@math.uni-sb.de
WWW:	http://www.math.uni-sb.de/

Dedicated to Professor Nina Uraltseva on the occasion of her 85th birthday.

Abstract

The TV-regularization method of Rudin, Osher and Fatemi [ROF] is widely used in mathematical image analysis. We consider a nonstationary and iterative variant of this approach and provide a mathematical theory that extends results of Radmoser et al. [RSW] to the BV setting. While existence and uniqueness, a maximum–minimum principle, and preservation of the average grey value are not hard to prove, we also establish convergence to a constant steady state and a large family of Lyapunov functionals. These properties allow to interpret iterated TV-regularization as a time-discrete scale-space representation of the original image.

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1 Introduction

The theory of regularisation methods for ill-posed problems goes back to Tikhonov [Ti2]. It has found numerous applications in inverse problems and mathematical image analysis, where it is common to model the solutions of ill-posed problems as minimizers of energy functionals. Typically the Euler–Lagrange equations of these functionals are elliptic PDEs. On the other hand, parabolic PDEs in image analysis often occur in the context of scale-space representations. Scale-spaces are embeddings of an image into a parameterized family of gradually smoothed or simplified versions of it, provided that this family satisfies a number of properties [Ii, AGLM]. They are useful intermediate representations for extracting semantic image information [Wi]. For nonlinear diffusion processes, scale-space properties have been established in [We] in the continuous, space-discrete and fully discrete setting, where the diffusion time serves a scale parameter. These properties include well-posedness results, preservation of the average grey value, maximum–minimum principles, Lyapunov functionals, and convergence to a constant steady state. It was possible to regard also regularisation methods as scale-spaces by interpreting their Euler-Lagrange equations as fully implicit discretizations of a parabolic PDE with a single time step [SW]. Radmoser et al. [RSW] have extended these results to nonstationary iterative regularization methods where the regularization parameter may vary from iteration to iteration. The energy functionals considered in [RSW] have densities of superlinear growth with respect to the quantity ∇u, as it is stated in formula (8) of [RSW]. Moreover, they have to satisfy five assumptions I.-V. that are stated at p. 100 of this reference. Hypothesis IV is violated for the total variation regularization in its original formulation by Rudin, Osher and Fatemi [ROF], which is of linear growth and belongs to the most widely used energy functionals for image restoration. It is also false for the variants of the TV-model studied e.g. in [BF1, BF2, BF3] and [BFW]. The analytical difficulties caused by the linear growth of the underlying energy densities are explained on p. 101 of [RSW], where the reader will also find further references.

In the present work we like to demonstrate that even in the linear growth case it is possible to show that at least a nonstationary iterative regularization process with adequate formulation in the space BV of functions of finite total variation has similar scale-space properties as the approaches considered by Radmoser et al. [RSW]. While it is not very difficult to prove existence, uniqueness, a maximum–minimum principle, and preservation of the average grey value, it is more demanding to establish a large family of Lyapunov functionals and to show convergence in a rather strong sense to a constant image as the regularization parameter tends to infinity.

To be precise, let us summarize our assumptions:

(A1) $\Omega \subset \mathbb{R}^d$ denotes a bounded Lipschitz domain.

(A2) f is a measurable function $\Omega \to \mathbb{R}$, which satisfies $0 \le f \le 1$ a.e.;

(A3) $F : \mathbb{R}^d \to [0, \infty)$ denotes a convex function (w.l.o.g. $F(0) = 0$) being of linear growth in the sense that $\nu_1|p| - \nu_2 \le F(p) \le \nu_3|p| + \nu_4$, $p \in \mathbb{R}^d$, holds with constants $\nu_1, \nu_3 > 0$, $\nu_2, \nu_4 \ge 0$.

Typical examples of densities F satisfying (A3) are the TV-density introduced in [ROF]

$$F(p) := |p| \tag{1}$$

and its approximations (see e.g. [BF1, BF2] and [BFW])

$$F(p) := \sqrt{\varepsilon^2 + |p|^2} - \varepsilon,\ \varepsilon > 0\,, \tag{2}$$

$$F(p) := \Phi_\mu(|p|),\ \mu > 1\,, \tag{3}$$

with function $\Phi_\mu : [0,\infty) \to [0,\infty)$ given by

$$\Phi_\mu(t) := \int_0^t \int_0^s (1+r)^{-\mu}\, dr\, ds \tag{4}$$

or in more explicit form

$$\begin{cases} \Phi_\mu(t) = \dfrac{1}{\mu-1} t + \dfrac{1}{\mu-1}\dfrac{1}{\mu-2} t^{-\mu+2} - \dfrac{1}{\mu-1}\dfrac{1}{\mu-2}\,,\ \mu \neq 2\,, \\ \\ \Phi_2(t) = t - \ln(1+t)\,. \end{cases} \tag{5}$$

Note, that (5) immediately implies

$$(\mu-1)\Phi_\mu(|p|) \to |p| \text{ as } \mu \to \infty,\ p \in \mathbb{R}^d\,, \tag{6}$$

and relation (6) shows in which way the density introduced in (3) approximates the TV-case from (1).

(A4) Let (h_k) denote a sequence of positive numbers satisfying $\lim_{k\to\infty} h_k = \infty$. We further let $t_k := \sum_{i=1}^k h_i,\ k \in \mathbb{N}$, and define $t_0 := 0$ as well as $u(t_0) = u(0) := f$ with f from (A2).

For $n \in \mathbb{N}$ and $t \in (t_{n-1}, t_n]$ we then consider the functional

$$J[t,w] := \int_\Omega \left|w - u\left(t_{n-1}\right)\right|^2\, dx + (t - t_{n-1}) \int_\Omega F(\nabla w)\, dx\,, \tag{7}$$

and clearly formula (7) needs some comments:

i) For the moment $u(t_{n-1})$ denotes a measurable function $\Omega \to \mathbb{R}$ satisfying the minimal requirement (see (A2), (A4))

$$0 \leq u(t_{n-1}) \leq 1 \ \text{ a.e. on } \Omega\,. \tag{8}$$

In what follows we will give a precise meaning of $u(t)$ for $t = t_k$, $k \in \mathbb{N}_0$, and also for arbitrary numbers $t \geq 0$.

ii) Under the hypothesis (8) the energy $J[t,\cdot]$ introduced in (7) makes sense for functions w from the Sobolev space $W^{1,1}(\Omega)$ (compare, e.g. [Ad]) with finite values, if $d \leq 2$, which in case $d \geq 3$ is true, if we additionally require $w \in L^2(\Omega)$.

iii) However, due to the non-reflexivity of the class $W^{1,1}(\Omega)$, the problem $J[t,\cdot] \to \min$ has to be replaced by its relaxed variant, i.e. for functions $w \in BV(\Omega)$ (:= the space of $L^1(\Omega)$-functions with finite total variation, see for example [Gi] or [AFP]) we let

$$(9)\qquad \widetilde{J}[t,w] := \int_\Omega |w - u(t_{n-1})|^2 \, dx + (t - t_{n-1}) \int_\Omega F\left(\nabla^a w\right) \, dx + (t - t_{n-1}) \int_\Omega F^\infty \left(\frac{\nabla^s w}{|\nabla^s w|} \right) d\,|\nabla^s w| \,,$$

where $\nabla w = \nabla^a w \mathcal{L}^d + \nabla^s w$ is the Lebesgue decomposition of the measure ∇w and F^∞ denotes the recession function of F, i.e.

$$F^\infty(p) := \lim_{t \to \infty} \frac{1}{t} \, F(tp), \;\; p \in \mathbb{R}^d \,.$$

Now we can state our first result. It guarantees existence, uniqueness, a maximum–minimum principle, and preservation of the average grey value. Maximum–minimum principles are important in image analysis, since the grey values of a digital image are bounded due to their bytewise encoding. Thus, one is interested in avoiding filtered results that lie outside these bounds.

THEOREM 1. *Let (A1–A4) hold and let $u(t_{n-1})$ satisfy (8) for some $n \in \mathbb{N}$. Consider $t \in (t_{n-1}, t_n]$. Then it holds:*

a) The variational problem

$$(10)\qquad \widetilde{J}[t,\cdot] \to \min \;\; \text{in } BV(\Omega)$$

with $\widetilde{J}$ from (9) admits a unique solution $u(t) \in BV(\Omega)$.

b) $u(t)$ satisfies $0 \leq u(t) \leq 1$ a.e. on Ω.

c) We have $M(u(t)) := \fint_\Omega u(t)\, dx = M(u(t_{n-1}))$.

We apply Theorem 1 choosing $n = 1$ and recalling (see (A4)) that $u(t_0) := f$. According to (A2) we have the validity of (8), and by letting $t = t_1$ in Theorem 1 we obtain the BV-solution $u(t_1)$ of $J[t_1, w] \to \min$ in $\mathrm{BV}(\Omega)$, and $u(t_1)$ satisfies (8) as well as $M(u(t_1)) = M(u(t_0)) = M(f)$. Proceeding by induction we obtain a sequence $(u(t_n))_{n\in\mathbb{N}}$ in $\mathrm{BV}(\Omega)$ with $u(t_n)$ being the solution of (10) for the choice $t = t_n$.

Our second result deals with convergence to a constant steady state. Since scale-spaces aim at gradually simpler image representations, it is desirable that the simplest representation consists of a flat image which has the same average grey value as the original image.

THEOREM 2. *Under the assumptions of Theorem 1 and with the notation introduced above it holds:*

a) $\|u(t_n) - M(f)\|_{L^p(\Omega)} \to 0$ *as* $n \to \infty$ *for any finite* p, *if* $d = 1$, *and for any* $p < \frac{d}{d-1}$, *if the case* $d \geq 2$ *is considered. Moreover, if* $d \geq 2$, *then* $u(t_n) \rightharpoonup M(f)$ *as* $n \to \infty$ *in* $L^{d/d-1}(\Omega)$.

b) Suppose that F *is of type (3) for some* $\mu \in (1, 2)$. *Then we have* $u(t_n) \to M(f)$ *as* $n \to \infty$ *locally uniformly on* Ω .

REMARK 1. *The results of Theorem 2 can be seen as TV-variants of the statements given in Theorem 4 (b) of [RSW].*

REMARK 2. *The result of Theorem 2 b) is not limited to the particular density (3), we can consider any* μ*-elliptic integrand with* $\mu \in (1, 2)$. *The interested reader is referred to [BF1-3], [BFMT] and [BFW].*

As outlined in [SW] or [RSW] we can introduce Lyapunov functionals for the nonstationary regularization methods considered here leading to an appropriate version of Theorem 4.1 (a) from [RSW] with some adjustments resulting from the linear growth of F stated in (A3). In what follows let $r : [0, 1] \to \mathbb{R}$ denote a function such that $r \in C^2([0, 1])$ and $r'' \geq 0$. With $u(t)$ from Theorem 1 we define the Lyapunov functional

$$V(t) := \phi(u(t)) := \int_\Omega r(u(t))\, dx\,.$$

THEOREM 3. *Let (A1–A4) hold. Referring to the notation introduced before we have the following properties of the Lyapunov functional:*

a) V is bounded from below in the sense that

$$V(t) = \phi(u(t)) \geq \phi(M(f))$$

holds, $M(\cdot)$ denoting the mean value.

Suppose that F is of type (3) for some $\mu \in (1,2)$ or even more general a μ-elliptic density of linear growth again with $1 < \mu < 2$. Then the next assertions are true:

b) V is a continuous function on the interval $[0,\infty)$.

c) For $t \in (t_{n-1}, t_n]$ we have

$$\int_\Omega r'(u(t))(u(t) - u(t_{n-1}))\, dx \leq 0\,.$$

d) If $t \in (t_{n-1}, t_n]$, then

$$V(t) - V(t_{n-1}) \leq 0\,.$$

REMARK 3. *These Lyapunov functionals allow to interpret $\{u(t_n)\,|\, n \in \mathbb{N}\}$ as a family of representations that are simpler in many aspects [We]: Choosing e.g. $r(s) := |s|^2$, $r(s) := (s - M(f))^2$, and $r(s) := s \ln s$ for $s > 0$ implies that the image energy $\|u(t_n)\|^2_{L^2(\Omega)}$ and the variance are decreasing in n, and the entropy $-\int_\Omega u(t_n)\, \ln(u(t_n))\, dx$, a measure of missing information, is increasing in n.*

From the proof of Theorem 3 we will deduce

Corollary 1. *The continuity of the Lyapunov functional V stated in b) of the theorem holds just under the assumptions (A1–A4) and does not require any further properties of the density F. In particular we have the continuity of V for the TV-density (1).*

REMARK 4. *For references explaining the notion of μ-ellipticity the reader should consult Remark 2, the most general discussion can be found in [BFMT], formulas (1.5) - (1.8).*

REMARK 5. *If we impose the stronger hypothesis that $r'' > 0$ on $[0,1]$, then V is a strict Lyapunov functional in the sense of [RSW], Theorem 4.1 (a), 3. - 5. We leave the details to the reader.*

With respect to Corollary 1 the question arises, if parts c) and d) of Theorem 3 actually can be established under weaker hypotheses on the density F. We have the following result.

THEOREM 4. *The statements c) and d) of Theorem 3 hold under the assumptions (A1–A4) provided we additionally require that F from (A3) is of class C^1 with $DF(0) = 0$.*

REMARK 6. *According to Theorem 4 we have the inequalities stated in c) and d) of Theorem 3 for smooth variants of the TV-case (1), i.e. we can consider any density of type (2) or (3).*

2 Proofs of the results

Proof of Theorem 1.

a) The existence of at least one solution $u \in \mathrm{BV}(\Omega)$ to problem (10) is an immediate consequence of the lower semicontinuity results stated in e.g. [AFP], Theorem 5.47 (and subsequent remarks) applied to a $\widetilde{J}[t, \cdot]$-minimizing sequence. The unique solvability of problem (10) follows from the strict convexity of $w \mapsto \int_\Omega |w - u(t_{n-1})|^2\, dx$.

b) As outlined in e.g. [BF2] we can combine (8) with the $\widetilde{J}[t, \cdot]$-minimality of $u(t)$ to get

$$\widetilde{J}[t, u(t)] \le \widetilde{J}\left[t, \max\left\{u(t), 0\right\}\right],$$
$$\widetilde{J}[t, u(t)] \le \widetilde{J}\left[t, \min\left\{u(t), 1\right\}\right],$$

thus $0 \le u(t) \le 1$ a.e. by uniqueness.

c) We have for any $\varepsilon \in \mathbb{R}$

$$\widetilde{J}\left[t, u(t) + \varepsilon \cdot 1\right] \ge \widetilde{J}\left[t, u(t)\right], \tag{11}$$

and since $\nabla\left(u(t) + \varepsilon \cdot 1\right) = \nabla u(t)$ we find from (11)

$$0 = \frac{d}{d\varepsilon|0} \int_\Omega \left|u(t) + \varepsilon \cdot 1 - u(t_{n-1})\right|^2\, dx,$$

which immediately shows $M\left(u(t)\right) = M\left(u(t_{n-1})\right)$.

$\square$

Proof of Theorem 2. We first recall the properties of the sequences h_k and t_k from (A4) and observe that by definition of $u(t_n)$ we have $\widetilde{J}[t_n, u(t_n)] \leq \widetilde{J}[t_n, w]$ for any $w \in \mathrm{BV}(\Omega)$. Choosing $w = 0$ we obtain

$$\begin{aligned}
&h_n \int_\Omega F\left(\nabla^a u(t_n)\right)\, dx + h_n \int_\Omega F^\infty\left(\frac{\nabla^s u(t_n)}{|\nabla^s u(t_n)|}\right) d\left|\nabla^s u(t_n)\right| \\
&+ \int_\Omega \left|u(t_n) - u(t_{n-1})\right|^2\, dx \\
&\leq h_n \int_\Omega F(0)\, dx + \int_\Omega \left|0 - u(t_{n-1})\right|^2\, dx \leq \mathcal{L}^d(\Omega)\,,
\end{aligned}$$

since we assume $F(0) = 0$ and $0 \leq u(t_{n-1}) \leq 1$ holds a.e. on Ω for our sequence $u(t_k)$. The linear growth of F stated in (A3) therefore yields for the total variation of $u(t_n)$

$$\int_\Omega |\nabla u(t_n)| \to 0 \text{ as } n \to \infty\,, \tag{12}$$

in particular, we find that $u(t_n)$ is a bounded sequence in $\mathrm{BV}(\Omega)$ and by BV-compactness there exists $u \in \mathrm{BV}(\Omega)$ such that for a subsequence $u(t_{n_k})$

$$\begin{cases} \|u(t_{n_k}) - u\|_{L^p(\Omega)} \to 0 \text{ as } k \to \infty \text{ for } p < \dfrac{d}{d-1}\ (:= \infty,\ \text{if } d = 1) \\ \text{and } u(t_{n_k}) \rightharpoondown u \text{ in } L^{d/d-1}(\Omega),\ \text{if } d \geq 2\,. \end{cases} \tag{13}$$

From (12) we deduce $\nabla u = 0$, thus u is a constant function. But then we have (by the convergences (13))

$$u = \int_\Omega u\, dx = \lim_{k\to\infty} \fint_\Omega u(t_{n_k})\, dx\,.$$

At the same time (see Theorem 1.c)) $M(u(t_n)) = M(f)$, thus $u = M(f)$ and the convergences (13) are true not only for a subsequence. This proves part a) of Theorem 2.

Next we discuss b) for $d = 2$. The case $d = 1$ is analyzed in [FMT], for $d \geq 3$ we refer to [Ti1], which means that in these references one will find the adjustments of the following arguments. Now, for $d = 2$, it has been shown in [BFT] that $u(t_n) \in C^1(\Omega)$, which obviously improves the $W^{2,s}_{\mathrm{loc}}(\Omega)$-regularity of $u(t_n)$ established for $s < 2$ in (3.13) of [BF2] to any $s < \infty$. We therefore deduce from the $\widetilde{J}[t_n, \cdot]$-minimality of $u(t_n)$ (observing that $\widetilde{J} = J$ on $W^{1,1}(\Omega)$ and letting $u_k := u(t_k)$)

$$h_n \int_\Omega D^2F\,(\nabla u_n)\,(\partial_\alpha \nabla u_n, \nabla\varphi)\;dx = 2\int_\Omega (u_n - u_{n-1})\,\partial_\alpha\varphi\,dx,\;\; \alpha = 1, 2\,,$$

for φ with compact support in Ω. Letting $\varphi := \eta^2\partial_\alpha u_n$ with $\eta \in C^1_0(\Omega)$, $0 \leq \eta \leq 1$, we get (from now on summation with respect to α)

$$\begin{aligned} &h_n \int_\Omega D^2F\,(\nabla u_n)\,(\partial_\alpha\nabla u_n, \partial_\alpha\nabla u_n)\,\eta^2\,dx \\ &= -2h_n \int_\Omega D^2F(\nabla u_n)\,(\partial_\alpha\nabla u_n, \nabla\eta)\,\eta\,\partial_\alpha u_n\,dx \\ &+ 2\int_\Omega (u_n - u_{n-1})\,\partial_\alpha\left(\eta^2\partial_\alpha u_n\right)\,dx\,. \end{aligned} \tag{14}$$

Applying the Cauchy-Schwarz inequality to the bilinear form $D^2F(\nabla u_n)$ and then using Young's inequality we easily deduce from (14) (c denoting a constant > 0 independent of n)

$$\begin{aligned} &h_n\frac{1}{2}\int_\Omega D^2F\,(\nabla u_n)\,(\partial_\alpha\nabla u_n, \partial_\alpha\nabla u_n)\,\eta^2\,dx \\ &\leq 2\int_\Omega (u_n - u_{n-1})\partial_\alpha\left(\eta^2\partial_\alpha u_n\right)\,dx \\ &+ ch_n\int_\Omega |\nabla\eta|^2\left|D^2F\,(\nabla u_n)\right|\,|\nabla u_n|^2\;dx\,. \end{aligned} \tag{15}$$

In order to proceed we observe that for F as in (3)

$$\nu_5\,(1 + |p|)^{-\mu}\,|q|^2 \leq D^2F(p)(q,q) \leq \nu_6\,(1 + |p|)^{-1}\,|q|^2,\;\; p, q \in \mathbb{R}^2\,, \tag{16}$$

holds with constants $\nu_5, \nu_6 > 0$. Applying (16) on both sides of (15) and observing (after integration by parts)

$$
\begin{aligned}
&\int_\Omega (u_n - u_{n-1}) \partial_\alpha \left(\eta^2 \partial_\alpha u_n\right) \, dx \\
&= - \int_\Omega \partial_\alpha u_n \partial_\alpha u_n \eta^2 \, dx - \int_\Omega u_{n-1} \partial_\alpha \left(\eta^2 \partial_\alpha u_n\right) \, dx \\
&\overset{(0 \le u_n \le 1)}{\le} - \int_\Omega \partial_\alpha u_n \partial_\alpha u_n \eta^2 \, dx + \int_\Omega |\nabla \eta^2| |\nabla u_n| \, dx + \int_\Omega \eta^2 |\nabla^2 u_n| \, dx
\end{aligned}
$$

we obtain

$$
\begin{aligned}
(17) \quad & h_n \int_\Omega (1 + |\nabla u_n|)^{-\mu} \left|\nabla^2 u_n\right|^2 \eta^2 \, dx + \int_\Omega \eta^2 \left|\nabla u_n\right|^2 \, dx \\
& \le c \left\{ h_n \int_\Omega |\nabla \eta|^2 \left|\nabla u_n\right| \, dx + \int_\Omega \left|\nabla \eta^2\right| \left|\nabla u_n\right| \, dx + \int_\Omega \eta^2 \left|\nabla^2 u_n\right| \, dx \right\} .
\end{aligned}
$$

To the last term on the right-hand side of (17) we apply Young's inequality:

$$
\begin{aligned}
(18) \quad & \int_\Omega \eta^2 \left|\nabla^2 u_n\right| \, dx = \int_\Omega (1 + |\nabla u_n|)^{-\mu/2} \left|\nabla^2 u_n\right| (1 + |\nabla u_n|)^{+\mu/2} \eta^2 \, dx \\
& \le \frac{1}{2} h_n \int_\Omega \eta^2 (1 + |\nabla u_n|)^{-\mu} \left|\nabla^2 u_n\right|^2 \, dx + c h_n^{-1} \int_\Omega \eta^2 (1 + |\nabla u_n|)^{\mu} \, dx \, ,
\end{aligned}
$$

and since $\mu < 2$, we can control $\int_\Omega \eta^2 (1 + |\nabla u_n|)^\mu \, dx$ through the quantity $\int_\Omega \eta^2 |\nabla u_n|^2 \, dx$ occurring on the left-hand side of (17). Therefore, using estimate (18), we finally arrive at

$$
\begin{aligned}
(19) \qquad & h_n \int_\Omega (1 + |\nabla u_n|)^{-\mu} \left|\nabla^2 u_n\right|^2 \eta^2 \, dx + \int_\Omega \eta^2 \left|\nabla u_n\right|^2 \, dx \\
& \le c \left\{ h_n \int_\Omega |\nabla \eta|^2 \left|\nabla u_n\right| \, dx + \int_\Omega \left|\nabla \eta^2\right| \left|\nabla u_n\right| \, dx \right\} .
\end{aligned}
$$

Dividing both sides of (19) by h_n and quoting (12), it follows

$$
(20) \qquad \int_\Omega \eta^2 (1 + |\nabla u_n|)^{-\mu} \left|\nabla^2 u_n\right|^2 \, dx \to 0 \text{ as } n \to \infty \, .
$$

Next let $\varphi_n := (1 + |\nabla u_n|)^{1-\mu/2}$. Combining (12) and (20) we infer

$$
\sup_n \|\varphi_n\|_{W^{1,2}(\Omega^*)} \le c(\Omega^*) < \infty
$$

for any subdomain Ω^*, thus by Sobolev's embedding theorem

$$\sup_n \|\varphi_n\|_{L^q(\Omega^*)} \le c(\Omega^*, q) < \infty$$

for any finite q and in conclusion

$$\sup_n \|u_n\|_{W^{1,q}(\Omega^*)} \le c(\Omega^*, q) < \infty$$

again for any $q < \infty$, hence (by Sobolev's embedding)

$$\sup_n \|u_n\|_{C^{0,\beta}(\overline{\Omega^*})} \le c(\Omega^*, \beta) < \infty$$

for arbitrary $\beta \in (0,1)$. Quoting Arcela's theorem and using part a) of Theorem 2, our claim follows. □

Proof of Theorem 3.

a) Let $t \in (t_{n-1}, t_n]$ for some $n \in \mathbb{N}$. From Jensen's inequality we get

$$V(t) = \mathcal{L}^d(\Omega) \fint_\Omega r(u(t))\, dx \ge \mathcal{L}^d(\Omega) r \left(\fint_\Omega u(t)\, dx \right) = \int_\Omega r\left(M(u(t))\right) dx$$

and Theorem 1 c) implies

$$V(t) \ge \phi\left(M\left(u\left(t_{n-1}\right)\right)\right) .$$

From the comments after Theorem 1 concerning the definition of the sequence $u(t_n)$ our claim follows.

Suppose now that we are in the μ-elliptic case (3) for some $\mu \in (1,2)$. Quoting Theorem 1 a) and Theorem 2 from [BFMT] covering even the case of general μ-elliptic densities, we find that the solution $u(t)$ of problem (10) actually is of class $W^{1,1}(\Omega)$ and thereby minimizes the functional J defined in (7) within the space $W^{1,1}(\Omega)$, in particular we have for $t \in (t_{n-1}, t_n]$ and $\varphi \in W^{1,1}(\Omega)$

$$0 = (t - t_{n-1}) \int_\Omega DF\left(\nabla u\left(t\right)\right) \cdot \nabla\varphi\, dx + 2 \int_\Omega \varphi\left(u\left(t\right) - u\left(t_{n-1}\right)\right) dx\,. \tag{21}$$

b) We fix some $s \in (t_{n-1}, t_n)$ and choose a sequence (s_k) such that $s_k \in (t_{n-1}, t_n)$ and $s_k \to s$. The case $s = t_n$ is left to the reader. Letting $t = s$ and $t = s_k$ in (21), choosing $\varphi = u(s) - u(s_k)$ in both cases and subtracting the results we get

$$\begin{aligned}
0 &= 2 \int_\Omega (u(s) - u(s_k))^2 \, dx \\
&+ \int_\Omega [(s - t_{n-1})\, DF(\nabla u(s)) - (s_k - t_{n-1})\, DF(\nabla u(s_k))] \cdot \\
&\qquad (\nabla u(s) - \nabla u(s_k)) \, dx \\
&= 2 \int_\Omega (u(s) - u(s_k))^2 \, dx \\
&+ (s - t_{n-1}) \int_\Omega (DF(\nabla u(s)) - DF(\nabla u(s_k))) \cdot (\nabla u(s) - \nabla u(s_k)) \, dx \\
&+ (s - s_k) \int_\Omega DF(\nabla u(s_k)) \cdot (\nabla u(s) - \nabla u(s_k)) \, dx \\
&=: T_1^{(k)} + T_2^{(k)} + T_3^{(k)}
\end{aligned}$$

with $T_2^{(k)} \geq 0$ by the convexity of F. From the $J[s, \cdot]$-minimality of $u(s)$ it follows $J[s, u(s)] \leq J[s, 0]$, hence account of (A3) and (8) combined with b) of Theorem 1

$$\int_\Omega |\nabla u(s)| \, dx \leq c/s - t_{n-1}\,, \tag{22}$$

and from $J\,[s_k, u(s_k)] \leq J\,[s_k, 0]$ we get

$$\int_\Omega |\nabla u(s_k)| \, dx \leq c/s_k - t_{n-1}\,. \tag{23}$$

The boundedness of DF together with (22) and (23) yields

$$0 = \lim_{k\to\infty} T_3^{(k)}\,,$$

and we end up with

$$\lim_{k\to\infty} T_1^{(k)} = \lim_{k\to\infty} 2 \int_\Omega (u(s) - u(s_k))^2 \, dx = 0\,. \tag{24}$$

Equation (24) states that $\lim_{k\to\infty} \|u(s_k) - u(s)\|_{L^2(\Omega)} \to 0$, thus we obtain

$$u(\widetilde{s}_k) \to u(s) \text{ a.e. on } \Omega \tag{25}$$

at least for a subsequence $(\widetilde{s}_k) \subset (s_k)$. Recalling $r \in C^2([0,1])$ and $0 \le u(\cdot) \le 1$ a.e. on Ω, Lebesgue's theorem on dominated convergence implies on account of (25)

$$\lim_{k\to\infty} \int_\Omega r(u(\widetilde{s}_k))\,dx = \int_\Omega r(u(s))\,dx\,. \tag{26}$$

Suppose now that (26) does not hold for the sequence (s_k). Then, for some $\varepsilon > 0$ and a subsequence $(s_k') \subset (s_k)$ we have

$$\left| \int_\Omega r\left(u(s_k')\right)\,dx - \int_\Omega r\left((s)\right)\,dx \right| \ge \varepsilon\,. \tag{27}$$

But by the same reasoning as above we can extract a subsequence $(s_k'') \subset (s_k')$ with (25) and in conclusion (26) contradicting (27), hence (26) is true for (s_k) and the continuity of V at s follows.

c) We choose $\varphi = r'(u(t))$ in (21), hence

$$\begin{aligned} 0 = 2 &\int_\Omega r'(u(t))(u(t) - u(t_{n-1}))\,dx \\ &+ (t - t_{n-1}) \int_\Omega DF(\nabla u(t)) \cdot \nabla(r'(u(t)))\,dx\,, \end{aligned}$$

and $DF(\nabla u(t)) \cdot \nabla(r'(u(t)) = DF(\nabla u(t)) \cdot \nabla u(t) r''(u(t))$ is non-negative on account of $r'' \ge 0$ and $DF(\xi) \cdot \xi \ge 0$.

d) We just oberve that convexity of r implies

$$r\left(u(t_{n-1})\right) \ge r\left(u(t)\right) + r'\left(u(t)\right)\left(u(t_{n-1})\right) - u(t))$$

and from c) we obtain our claim by integrating the above inequality.

□

Proof of Corollary 1. We consider s_k and s as specified after (21) and observe that the inequalities (22) and (23) continue to hold now for the total variations of the measures $\nabla u(s_k), \nabla u(s)$. From (23) we deduce the existence of $\widetilde{u} \in \mathrm{BV}(\Omega)$ such that

$$u(\widetilde{s}_k) \to \widetilde{u} \quad \text{in } L^1(\Omega) \tag{28}$$

holds, and by passing to another subsequence, if necessary, we may assume that

$$u(\widetilde{s}_k) \to \widetilde{u} \quad \text{a.e.} \quad \text{on } \Omega\,. \tag{29}$$

Let us fix a function $w \in \mathrm{BV}(\Omega)$. From the $\widetilde{J}[\widetilde{s}_k, \cdot]$-minimality of $u(\widetilde{s}_k)$ it follows

$$\widetilde{J}\left[\widetilde{s}_k, u(\widetilde{s}_k)\right] \le \widetilde{J}\left[\widetilde{s}_k, w\right]\,, \tag{30}$$

and clearly

$$\lim_{k\to\infty} \widetilde{J}[\widetilde{s}_k, w] = \widetilde{J}[s, w]\,. \tag{31}$$

Writing

$$\begin{aligned} &\widetilde{J}[\widetilde{s}_k, u(\widetilde{s}_k)] = \int_\Omega |u(\widetilde{s}_k) - u(t_{n-1})|^2\, dx \\ &+ (s - t_{n-1}) \int_\Omega F(\nabla u(\widetilde{s}_k)) + (\widetilde{s}_k - s) \int_\Omega F(\nabla u(\widetilde{s}_k)) =: U_1^{(k)} + U_2^{(k)} + U_3^{(k)} \end{aligned}$$

with an obvious meaning of $\int_\Omega F(\ldots)$, we deduce from (29) (recall $0 \le u(\ldots) \le 1$ a.e.)

$$\lim_{k\to\infty} U_1^{(k)} = \int_\Omega |\widetilde{u} - u(t_{n-1})|^2\, dx\,. \tag{32}$$

Recalling (23) (valid for the total variations of the measures $\nabla u(s_k)$) it is immediate that

$$\lim_{k\to\infty} U_3^{(k)} = 0\,. \tag{33}$$

Finally, the lower semicontinuity of $\int_\Omega F(\ldots)$ w.r.t. the convergence (28) (compare, e.g., [AFP], Theorem 5.47 and the subsequent remarks) implies

$$\int_\Omega F(\nabla \widetilde{u}) \le \liminf_{k\to\infty} \int_\Omega F(\nabla u(\widetilde{s}_k))\,. \tag{34}$$

Putting together (30) - (34), it follows $\widetilde{J}[s,\widetilde{u}] \le \widetilde{J}[s,w]$, thus $\widetilde{u} = u(s)$ by the uniqueness of the $\widetilde{J}[s,\cdot]$-minimizer. This means that we actually have (25), and we can proceed as done in the proof of b) from Theorem 3. $\square$

Proof of Theorem 4. As in [BF1] we consider the δ-regularization of the variational problem (7), i.e. for $\delta > 0$ we let $F_\delta(p) := \frac{\delta}{2}|p|^2 + (t - t_{n-1})F(p), p \in \mathbb{R}^d$, and denote by $u_\delta \in W^{1,2}(\Omega)$ the unique minimizer of

$$J_\delta[t,w] := \int_\Omega |w - u(t_{n-1})|^2\, dx + \int_\Omega F_\delta(\nabla w)\, dx \to \min \text{ in } W^{1,2}(\Omega)\,. \tag{35}$$

As discussed in [BF1-3] the structure of problem (35) together with the bound (8) yield

$$0 \le u_\delta \le 1 \quad \text{a.e. on } \Omega\,, \tag{36}$$

$$u_\delta \to u(t) \quad \text{as } \delta \to 0 \quad \text{in } L^1(\Omega)\,, \tag{37}$$

where $u(t)$ is the unique solution of problem (10) defined in Theorem 1. Moreover, on account of (37), we may assume

$$u_\delta \to u(t) \quad \text{a.e. on } \Omega \text{ as } \delta \to 0\,. \tag{38}$$

Recalling our assumptions concerning F, we deduce from (35)

$$0 = 2\int_\Omega \varphi\left(u_\delta - u(t_{n-1})\right)\, dx + \int_\Omega DF_\delta(\nabla u_\delta)\cdot\nabla\varphi\, dx \tag{39}$$

valid for any $\varphi \in W^{1,2}(\Omega)$, in particular the choice $\varphi := r'(u_\delta)$ is admissible (recall that $r \in C^2([0,1])$). Observing the inequality

$$r''(u_\delta)DF_\delta(\nabla u_\delta)\cdot\nabla u_\delta \ge 0 \quad \text{a.e. on } \Omega\,,$$

we obtain from (39)

$$\int_\Omega r'(u_\delta)(u_\delta - u(t_{n-1}))\, dx \le 0\,, \tag{40}$$

and by combining (40) with (36) and (38), we see the validity of claim c) of Theorem 3 for the situation at hand. Once having established c), part d) is obvious. $\square$

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