Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint

Elliptic Variational Problems with Nonstandard Growth

Michael Bildhauer and Martin Fuchs

Preprint No. 49

Saarbrücken 2002

Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Elliptic Variational Problems with Nonstandard Growth

Michael Bildhauer

Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken
Germany
E-Mail: bibi@math.uni-sb.de

Martin Fuchs

Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken
Germany
E-Mail: fuchs@math.uni-sb.de

submitted: January 24, 2002

Preprint No. 49

Saarbrücken 2002

Edited by
FR 6.1 – Mathematik
Im Stadtwald
D–66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

1 Introduction

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and a variational integrand $f\colon \mathbb{R}^{nN} \to \mathbb{R}$ of class $C^2(\mathbb{R}^{nN})$ we consider the autonomous minimization problem

$$J[w] := \int_\Omega f(\nabla w)\, dx \to \min \tag{$\mathcal{P}$}$$

among mappings $w\colon \Omega \to \mathbb{R}^N$, $N \geq 1$, with prescribed Dirichlet boundary data u_0. Depending on f, the comparison functions are additionally assumed to be elements of a suitable energy class $\mathbb{K}$. In the following, the variational integrand is always assumed to be strictly convex (in the sense of definition), thus we do not touch the quasiconvex case (compare, for instance, [Ev], [FH], [EG], [AF1], [AF2], [CFM]).

The purpose of our studies is to establish regularity results for minimizers of problem $(\mathcal{P})$ under quite general growth and structure conditions imposed on f. Before going into details let us give a brief historical overview.

A.1 Power Growth

Having the standard example $f_p(Z) = (1+|Z|^2)^{p/2}$, $1 < p$, in mind, let us assume that the growth rates from above and below coincide, i.e. for some number $p > 1$ and with constants c_1, c_2, C, λ, $\Lambda > 0$ the integrand f satisfies for all Z, $Y \in \mathbb{R}^{nN}$ (note that the second line of (1) implies the first one)

$$\begin{array}{rcccl} c_1|Z|^p - c_2 & \leq & f(Z) & \leq & C(1+|Z|^p)\,, \\ \lambda(1+|Z|^2)^{\frac{p-2}{2}}|Y|^2 & \leq & D^2f(Z)(Y,Y) & \leq & \Lambda(1+|Z|^2)^{\frac{p-2}{2}}|Y|^2\,. \end{array} \tag{1}$$

With the pioneering work of De Giorgi, Moser, Nash as well as of Ladyzhenskaya and Ural'tseva, local $C^{1,\alpha}$-regularity of minimizers of problem $(\mathcal{P})$ in the scalar case is well known in this setting, and of course many other authors could be mentioned (see [DG1], [Mos], [Na] and [LU] for a complete overview and a detailled list of references).

In the vectorvalued situation $N > 1$, the twodimensional case $n = 2$ differs substantially from the situation in higher dimensions: a classical result of Morrey ensures full regularity if $n = 2$ (here we like to refer to [Mor1], the first monograph on multiple integrals in the calculus of variations, where again detailled references can be found).

AMS Subject Classification: 49N60, 49N99, 35J45
Keywords: regularity, minimizers, nonstandard growth

However, according to an example of De Giorgi (see [DG3], compare also [GiM2], [Ne] and the recent example [SY]), there is no hope to prove an analogous result of this strength if $n \geq 3$ and $N > 1$. Here we merely expect partially regular solutions, i.e. there is an open set $\Omega_0 \subset \Omega$ of full Lebesgue measure such that $C^{1,\alpha}$-regularity holds on Ω_0. A theorem of this type is proved in any dimension and in a quite general setting by Anzellotti and Giaquinta ([AG]), where the whole scale of integrands up to the limit case of linear growth is covered (with some suitable notion of relaxation). In addition, the assumptions on the second derivatives are much weaker than stated above, i.e. their partial regularity result is true whenever $D^2 f(Z) > 0$ holds for any matrix Z.

To keep the historical line, we like to mention the earlier contributions on partial regularity given in [Mor2], [GiM1], [Giu] (compare also [DG2], [Al], a detailled overview can be found in [Gia1]).

Finally, if we impose some additional structure like $f(Z) = g(|Z|^2)$ in the vectorvalued setting, then partial regularity is improved to full $C^{1,\alpha}_{loc}$-regularity of solutions. Results of this kind are mainly connected with the name of Uhlenbeck (see [Uh], where the full strength of (1) is not needed which means that also degenerate ellipticity can be considered).

A.2 Anisotropic Power Growth

The study of anisotropic variational problems was forced by Marcellini [Ma1]–[Ma6] and is a natural extension of (1). To give some motivation we may consider the case $n = 2$, $2 \leq p \leq q$ and replace f_p by

$$f_{p,q}(Z) = (1 + |Z_1|^2)^{\frac{p}{2}} + (1 + |Z_2|^2)^{\frac{q}{2}}\,, \qquad Z = (Z_1, Z_2) \in \mathbb{R}^{2N}\,,$$

hence f is allowed to have different growth rates from above and from below. The natural generalization of the structure condition (1) is the requirement that f satisfies (again the growth conditions on the second derivatives imply the corresponding growth rates of f)

$$\begin{array}{rcccl} c_1|Z|^p - c_2 & \leq & f(Z) & \leq & C(1 + |Z|^q)\,, \\ \lambda(1 + |Z|^2)^{\frac{p-2}{2}}|Y|^2 & \leq & D^2 f(Z)(Y,Y) & \leq & \Lambda(1 + |Z|^2)^{\frac{q-2}{2}}|Y|^2 \end{array} \tag{2}$$

for all Z, $Y \in \mathbb{R}^{nN}$, where as usual c_1, c_2, C, λ, Λ denote some positive constants and $1 < p \leq q$.

If p and q differ too much, then it turns out that even in the scalar case singularities may occur (we mention only one famous example given in [Gia2]). However, following the work of Marcellini, suitable assumptions on

p and q yield regular solutions. Note that [Ma4] also covers the case $N > 1$ with some additional structure condition.

In the general vectorial setting only a few contributions are available, we like to refer to the papers of Acerbi and Fusco ([AF3]) and Passarelli Di Napoli and Siepe ([PS]), where partial regularity theorems are obtained under quite restrictive assumptions on p and q excluding any subquadratic growth.

If some additional boundedness condition is imposed, then the above results are improved by Esposito, Leonetti and Mingione ([ELM]) and Choe ([Ch]). In [ELM] higher integrability (up to a certain extend) is established ($N \geq 1$, $2 \leq p$) under a quite weak relation between p and q. A theorem for energy densities $f(Z) = g(|Z|^2)$ is found in [Ch].

B.1 Growth Conditions Involving N-Functions

Studying the monograph of Seregin and the second author ([FS2]), it is obvious that many problems in mathematical physics are not within the reach of power growth models – the theories of Prandtl-Eyring fluids and of plastic materials with logarithmic hardening serve as typical examples. The variational integrands under consideration are of nearly linear growth, for example we have to study the logarithmic integrand

$$f(Z) = |Z| \ln(1 + |Z|)$$

which satisfies none of the conditions (1) or (2).

The main results on integrands with logarithmic structure are proved by Frehse and Seregin ([FrS]: full regularity if $n = 2$), Seregin and the second author ([FS1]: partial regularity if $n \leq 4$), Esposito and Mingione ([EM]: partial regularity in any dimension) and finally by Mingione and Siepe ([MS]: full regularity in any dimension).

B.2 The First Extension of the Logarithm

As a first natural extension one may think of integrands which are bounded from above and below by the same quantity $A(|Z|)$, where A: $[0, \infty) \to [0, \infty)$ denotes some arbitrary N-function satisfying a Δ_2-condition (see [Ad] for precise definitions). Although this does not imply some natural bounds (in terms of A) on the second derivatives, (1) and (2) suggest the following model: given a N-function A as above and positive constants c, C, λ and Λ, we assume that our integrand f satisfies

$$\begin{array}{rcccl} cA(|Z|) & \leq & f(Z) & \leq & CA(|Z|)\,, \\ \lambda(1+|Z|^2)^{-\frac{\mu}{2}}|Y|^2 & \leq & D^2f(Z)(Y,Y) & \leq & \Lambda(1+|Z|^2)^{\frac{q-2}{2}}|Y|^2 \end{array} \tag{3}$$

for all $Z,\ Y \in \mathbb{R}^{nN}$ and for some real numbers $1 \leq \mu,\ 1 < q$, this choice being adapted to the logarithmic integrand which satifies (3) with $\mu = 1$ and $q = 1 + \varepsilon$ for any $\varepsilon > 0$. Note that the correspondence to (1) and (2) is only of formal nature: since we require $\mu \geq 1$, the μ-ellipticity condition, i.e. the first inequality in the second line of (3), does not give any information on the lower growth rate of f in terms of a power function with exponent $p > 1$.

A first investigation of variational problems with the structure (3) under some additional balancing conditions is due to Osmolovskii and the second author ([FO]), where partial regularity in the vectorvalued case is shown to be true in the case that $\mu < 4/n$.

Full regularity if $N = 1$ or if $N > 1$ and $f(Z) = g(|Z|^2)$ is established by Mingione and the second author (see [FM]) whenever $\mu < 1 + 2/n$.

2 Notation and statements of the results

Now let us give a precise formulation of our assumptions and results: for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, we consider the minimization problem

$$J[w] = \int_\Omega f(\nabla w)\,dx \to \min \quad \text{in } u_0 + \overset{\circ}{W}{}^1_A(\Omega;\mathbb{R}^N) \tag{$\mathcal{P}$}$$

where $\overset{\circ}{W}{}^1_A(\Omega;\mathbb{R}^N)$ denotes the subclass of the Orlicz-Sobolev space $W^1_A(\Omega;\mathbb{R}^N)$ generated by the N-function A (having the Δ_2-property) consisting of all mappings $\Omega \to \mathbb{R}^N$ with zero trace, and u_0 is a given function of class $W^1_A(\Omega;\mathbb{R}^N)$ with finite energy, i.e. $J[u_0] < \infty$. The energy density f is a function in $C^2(\mathbb{R}^{nN})$ which satisfies

$$c_1 A(|Z|) - c_2 \ \leq\ f(Z)\,, \tag{4}$$

$$\lambda(1+|Z|^2)^{-\frac{\mu}{2}}|Y|^2 \ \leq\ D^2 f(Z)(Y,Y) \ \leq\ \Lambda(1+|Z|^2)^{\frac{q-2}{2}}|Y|^2 \tag{5}$$

for all $Y,\ Z \in \mathbb{R}^{nN}$. Here c_1, c_2, λ and Λ denote positive constants, and $\mu \in \mathbb{R}$, $q > 1$ are fixed real numbers. Finally, we choose $s \geq 1$ according to

$$A(t) \geq const \cdot t^s \quad \text{for all } t \gg 1\,. \tag{6}$$

Let is look at some *Examples*, for which it is immediate how to choose the parameters s, μ and q.

i) $f(\nabla u) = |\nabla u| \ln(1 + |\nabla u|)$: $\mu = 1$, $s = 1$, $q = 1 + \varepsilon$;

ii) $f(\nabla u) = (1 + |\nabla u|^2)^{\frac{p}{2}} + (1 + |\partial_1 u|^2)^{\frac{t}{2}}$ for $1 < p < t$: $\mu = 2 - p$, $s = p$, $q = t$;

iii) $f(\nabla u) = \Phi_\mu(\nabla u) + h(\nabla u)$ with h of growth order q satisfying $0 \leq D^2h(Z)(Y,Y) \leq const(1+|Z|^2)^{\frac{q-2}{2}}|Y|^2$ and

$$\Phi_\mu(Z) = \int_0^{|Z|}\int_0^s (1+t^2)^{-\mu/2}\, dt\, ds\,.$$

Note that on account of $D^2h(Z) \geq 0$ (allowing degeneracy of D^2h) the ellipticity estimate, i.e. the left-hand side of (5), is a consequence of the corresponding inequality valid for the function Φ_μ, see e.g. [BFM].

We have the following regularity results

Theorem 1 *((s,μ,q)-GROWTH CONDITIONS) Let u denote the unique solution of problem ($\mathcal{P}$). Suppose that (4)–(6) hold together with $q < 2 - \mu + \frac{2}{n}s$.*

i) *(scalar case) If $N = 1$, then we have $u \in C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$.*

ii) *Let $N > 1$ and assume $f(Z) = g(|Z|^2)$. Then the statement of i) holds.*

iii) *In the vectorial case $N > 1$ without additional structure we have partial $C^{1,\alpha}$-regularity.*

Remark 1 i) *The results easily extend to locally minimizing maps. Moreover, for the scalar case we can include (double) obstacles.*

ii) *The detailled discussion and the comparison with the known results given in [BFM], [BF1], [BF3] and [Bi1] shows that Theorem 1 provides a unified and extended approach to the regularity theory of convex variational problems having nonstandard growth.*

Under the additional requirement

$$u_0 \in L^\infty(\Omega;\mathbb{R}^N) \tag{7}$$

we can improve Theorem 1 as follows:

Theorem 2 *If u denotes the solution of ($\mathcal{P}$) and if we assume (4), (5), (7) together with $q < 4 - \mu$, then i), ii) of Theorem 1 hold.*

If $N > 1$, then we get from $q < 4 - \mu$: for any $\kappa \in (q, 4-\mu)$ and for any $\Omega' \Subset \Omega$ there is a positive number c such that

$$\int_{\Omega'} |\nabla u|^\kappa\, dx \leq c < \infty\,, \tag{8}$$

provided that the structure condition $f(Z_1,\dots,Z_n) = g(|Z_1|,\dots,|Z_n|)$, $Z \in \mathbb{R}^{nN}$, g increasing in each argument, holds true.

If we additionally replace $q < 4 - \mu$ in case $n \geq 3$ by the requirement $q < \min\{(2-\mu)n/(n-2), 4-\mu\}$, then we have iii) of Theorem 1.

Remark 2 i) *In the scalar case we again can include obstacle problems.*

ii) *Theorem 2 substantially extends the results of [BF3].*

iii) *Note that the condition* $q < 4 - \mu$ *is in complete accordance with the ellipticity constraint*

$$\mu < 3$$

in the case of linear growth problems (see [Bi1], [Bi2], [BF5]). Moreover, we like to remark that the analogous constraint $q < 2 + p$ *first appeared in [ELM], where higher integrability (up to a certain extend) in the anisotropic, superquadratic* (p, q)*-case was proved under some extra boundedness condition.*

iv) *In terms of anisotropic integrands with* (p, q)*-growth (recall (2) of the introduction and let* $\mu = 2 - p$*), the main assumption of Theorem 1 reads as*

$$q < p\frac{n+2}{n}\,, \tag{a}$$

and if (a) holds, then there is no need to impose L^∞*-bounds on the solution. Hence, at first sight one may wonder about the case*

$$2 + p < p\frac{n+2}{n} \tag{b}$$

since then the hypothesis $q < 2+p$ *occurring in Theorem 2 implies (a), thus the results of Theorem 2 holds without an additional boundedness condition. But (b) is equivalent to* $p > n$*, hence, by Sobolev's embedding theorem, boundedness becomes no restriction at all.*

v) *With iv) it is clear that Theorem 2 extends to the case where* $(4 - \mu)$ *is replaced by*

$$\max\left\{4 - \mu, (2 - \mu)\frac{n+2}{n}\right\}.$$

Note that, if $4-\mu < (2-\mu)(n+2)/n$*, then* $2-\mu > n$*, in particular* $\mu < 1$ *and* f *is at least of growth rate* $s = 2-\mu$ *(see the short discussion leading to formula (15) below). Hence, in this case we may apply Theorem 1 to obtain Theorem 2 (without imposing condition (7)).*

vi) *We do not dare to state a conjecture on sharpness. Nevertheless, the linear growth example of [Bi1], [BF5], the previous remarks iii) and iv) and a detailled comparison with the known results (recall the above mentioned references) at least show that our results are reasonable and consistent.*

Example. Let us shortly discuss an example which emphasizes the essential improvements obtained by Theorem 2. Let $Z = (Z_1, Z_2) \in \mathbb{R}^{kN} \times \mathbb{R}^{(n-k)N}$, $1 \leq k < n$. Moreover, suppose that we are given exponents $1 < p < q < 2$ and

$$f(Z) = (1 + |Z_1|^2)^{\frac{p}{2}} + (1 + |Z_2|^2)^{\frac{q}{2}} \, .$$

In this subquadratic case (by elementary calculations) the estimate

$$\lambda(1 + |Z|^2)^{\frac{p-2}{2}} |Y|^2 \leq D^2 f(Z)(Y, Y) \leq \Lambda |Y|^2$$

is seen to be the best possible one. As a consequence, no regularity results are available from Theorem 1 if p is close to 1 – even if $(q - p)$ becomes very small. Hence, with the trivial assumption $2 < p + 2$ (let $q = 2$, $\mu = 2 - p$), Theorem 2 really provides some completely new results.

A final theorem covers the anisotropic vectorial case in two dimensions.

Theorem 3 *Let $n = 2$, and consider $1 < s < q < \infty$ such that (4), (5), (6) hold with $\mu = 2 - s$. Then, if $q < 2s$, the solution u of problem $(\mathcal{P})$ is smooth on Ω.*

Example. In particular we get regularity in case

$$f(\nabla u) = |\nabla u|^2 + (1 + |\partial_1 u|^2)^{\frac{q}{2}}$$

with $q \in (2, 4)$.

Remark 3 *Note that the assumption $q < 2s$ of Theorem 3 formally coincides with the "(s, μ, q)-condition" of Theorem 1.*

Remark 4 *In Theorem 1 – Theorem 3 we concentrated on the case of integrands having superlinear growth so that the existence of minimizers in appropriate Orlicz-Sobolev spaces is easily established. However, it should be noted, that it is also possible to discuss μ-elliptic integrands of linear growth (see [BF2], [Bi2]) or even anisotropic problems of mixed linear/superlinear growth ([Bi4]). Of course then one has to look at suitable generalized minimizers from the space of functions having bounded variation (see [BF6] for three formally different approaches leading to the same set of generalized minimizers). For a short overview of the regularity results in the case of linear growth problems we refer the reader to [BF5].*

3 Some remarks on the proofs

A complete proof of Theorem 1 already appeared in [BFM] and [BF1]. Theorem 3 essentially relies on a lemma due to Frehse and Seregin (see [FrS]) and is published in [BF4]. The first claim of Theorem 2 is contained in [Bi3] (compare also [Bi1]).

In our paper we concentrate on the general vectorial setting and give a rigorous proof of the higher integrabilty assertion (8) of Theorem 2. This is done by refining some of the ideas given by Choe ([Ch]) and by combining this with a more appropriate Caccioppoli-type inequality. Note that we do not impose the restriction $f(Z) = g(|Z|^2)$ as considered in [Ch], where also the constraint $q < 1 + p$ is taken as a further assumption.

Remark 5 *Having established (8), the partial regularity result will then be an immediate corollary following the ideas of [BF1]. In fact, we observe that the blow-up arguments of [BF1] remain unchanged once a Caccioppoli-type inequality and higher local integrability of the gradient are verified. Some more details are outlined in [BF3], here we just note that the way of regularizing the problem (which will slightly differ from [BF1]) is irrelevant since these ingredients are formulated in terms of the solution u. The restriction*

$$q < (2-\mu)\frac{n}{n-2} \quad \textit{if } n \geq 3 \tag{$*$}$$

is due to the needed properties of the auxiliary functions ψ_m introduced in [BF1] (compare also [FO]). Since our boundedness condition does not improve the Caccioppoli-type inequality, which in turn is the basis of the discussion of ψ_m, we can not expect to get rid of assumption $(*)$.

From now on we assume that the general hypotheses of Theorem 2 are valid. We then consider a ball $B_R(x_0) \Subset \Omega$ and an ε-mollification $(u)^\varepsilon$ of u, where $\varepsilon > 0$ is chosen sufficiently small. Moreover, for any $\delta \in (0,1)$ let

$$f_\delta(Z) := f(Z) + \delta(1+|Z|^2)^{\frac{t}{2}}\,, \quad Z \in \mathbb{R}^{nN}\,,$$

with some exponent $t > \max\{2, q\}$, and denote by $v_\varepsilon = v_{\varepsilon,\delta}$ the unique solution of the Dirichlet problem

$$J_\delta[w] := \int_{B_R(x_0)} f_\delta(\nabla w)\,dx \to \min\,,\ w \in (u)^\varepsilon_{|B_R(x_0)} + \overset{\circ}{W}{}^1_t(B_R(x_0);\mathbb{R}^N)\,. \tag{$\mathcal{P}_\delta$}$$

Then, if $\delta = \delta(\varepsilon)$ is chosen sufficiently small (see [BF3]), the main properties of the regularizing sequence $\{v_\varepsilon\}$ are summarized in

Lemma 1 *With the above notation we have*

i) $v_\varepsilon \in W^1_{\infty,loc} \cap W^2_{2,loc}(B_R(x_0);\mathbb{R}^N)$;

ii) $\|v_\varepsilon\|_{W^1_A(B_R(x_0);\mathbb{R}^N)} \le c < \infty$, *where the constant* c *is independent of* ε;

iii) $v_\varepsilon \rightharpoondown u$ *in* $W^1_1(B_R(x_0);\mathbb{R}^N)$ *and a.e. as* $\varepsilon \to 0$;

iv) $\sup\limits_{B_R(x_0)} |v_\varepsilon| \le \sup\limits_{B_{R+\varepsilon}(x_0)} |u| < \infty$;

v) $\delta(\varepsilon) \displaystyle\int_{B_R(x_0)} (1 + |\nabla v_\varepsilon|^2)^{t/2}\, dx \to 0$ *as* $\varepsilon \to 0$;

vi) $\displaystyle\int_{B_R(x_0)} f(\nabla v_\varepsilon)\, dx \to \int_{B_R(x_0)} f(\nabla u)\, dx$ *as* $\varepsilon \to 0$;

vii) $\displaystyle\int_{B_R(x_0)} f_{\delta(\varepsilon)}(\nabla v_\varepsilon)\, dx \to \int_{B_R(x_0)} f(\nabla u)\, dx$.

Proof. The proof is quite standard and outlined, for instance, in [BFM], [BF1], [BF3] or [Bi1]. Let us just give two comments: first, since the regularization is done w.r.t. the exponent $t > \max\{2, q\}$, the discussion of asymptotically regular integrands (compare [CE] or the generalization given in [GM], Theorem 5.1) immediately yields i). As a second remark, we like to mention that the structure condition $f = g(|Z_1|, \dots |Z_n|)$ with g as above provides the convex hull property (see [BF3] for a detailled proof), hence iv) follows from the boundedness of u_0. Note that once iv) is established, we no longer make use of the L^∞-bounds w.r.t. u_0 and the structure condition imposed on f. ■

Given Lemma 1 it is obvious that (8) follows from our main

Lemma 2 *With the above stated hypotheses, for any* $\kappa \in (q, 4 - \mu)$ *and for any ball* $B_r(x_0)$, $r < R$, *there is a constant* c *just depending on the data, on* $sup_{B_R(x_0)}|(u)^\varepsilon|$ *and on* r *and* κ, *such that*

$$\int_{B_r(x_0)} |\nabla v_\varepsilon|^\kappa\, dx \le c < \infty\,.$$

Proof of Lemma 2. In the following we abbreviate $f_\varepsilon = f_{\delta(\varepsilon)}$, moreover we always take the sum w.r.t. repeated Greek indices $\gamma = 1, \dots, n$ and w.r.t. repeated Latin indices $i = 1, \dots, N$.

By definition, v_ε is a solution of the Euler equation

$$\int_{B_R(x_0)} \nabla f_\varepsilon(\nabla v_\varepsilon) : \nabla\varphi \, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(B_R(x_0); \mathbb{R}^N)\,,$$

and on account of Lemma 1, i), we may differentiate this equation (taking $\partial_\gamma \varphi$ as test function and performing a partial integration) with the result

$$\int_{B_R(x_0)} D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \nabla\varphi)\, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(B_R(x_0); \mathbb{R}^N)\,. \tag{9}$$

Given a smooth cut-off function η, standard approximation arguments prove $\eta^2 \partial_\gamma v_\varepsilon$ to be admissible in (9) and, as a result, we obtain the Caccioppoli-type inequality

Lemma 3 *There is a real number $c > 0$, independent of ε, such that for any $\eta \in C_0^\infty(B_R(x_0))$, $0 \le \eta \le 1$,*

$$\begin{aligned}&\int_{B_R(x_0)} D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \partial_\gamma \nabla v_\varepsilon)\eta^2\, dx \\ &\quad\le\; c \int_{B_R(x_0)} \big|D^2 f_\varepsilon(\nabla v_\varepsilon)\big| |\nabla v_\varepsilon|^2 |\nabla\eta|^2\, dx\,.\end{aligned}$$

Proceeding with the proof of Lemma 2 we fix κ as given there, hence it is possible to define

$$q + \mu - 4 < \alpha := \kappa + \mu - 4 < 0\,, \tag{10}$$

where the negative sign of α ensures that

$$0 < \sigma := 2 + \alpha - \frac{\mu}{2} < 2 + \frac{\alpha - \mu}{2} =: \sigma'\,. \tag{11}$$

Note that we may suppose w.l.o.g. that $|\alpha|$ is sufficiently small in order to obtain the positive sign of σ. Alternatively, we observe that in the case of a negative sign the second integral on the right-hand side of inequality (17) below is trivially bounded.

By (11) we may choose in addition $k \in \mathbb{N}$ sufficiently large satisfying

$$2k\frac{\sigma}{\sigma'} < 2k - 2\,.$$

Now, given $\eta \in C_0^\infty(B_R(x_0))$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(x_0)$, $|\nabla\eta| \le c/(R - r)$, we introduce the function $\Gamma_\varepsilon = 1 + |\nabla v_\varepsilon|^2$ and recall the starting integrability

i) of Lemma 1, thus v_ε is smooth enough to perform the following partial integration

$$\begin{aligned}\int_{B_R(x_0)} |\nabla v_\varepsilon|^2 \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx &= -\int_{B_R(x_0)} v_\varepsilon^i \cdot \nabla\Big[\nabla v_\varepsilon^i \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} \eta^{2k}\Big]\, dx \\ &\le c\int_{B_R(x_0)} |\nabla^2 v_\varepsilon| \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx \\ &\quad + c\int_{B_R(x_0)} \Gamma_\varepsilon^{\frac{3+\alpha-\mu}{2}} \eta^{2k-1} |\nabla \eta|\, dx\, .\end{aligned}$$

Here we already made use of the fact that v_ε is uniformly bounded. If a positive constant M is fixed, then the left-hand side is immediately estimated by

$$\begin{aligned}\int_{B_R(x_0)} |\nabla v_\varepsilon|^2 \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx &\ge c\int_{B_R(x_0)\cap[|\nabla v_\varepsilon|\ge M]} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx \\ &\ge c\int_{B_R(x_0)} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx - c(M)\, ,\end{aligned}$$

therefore the starting inequality reads as

$$\begin{aligned}\int_{B_R(x_0)} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx &\le c\Bigg\{1 + \int_{B_R(x_0)} |\nabla^2 v_\varepsilon| \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx \\ &\quad + \int_{B_R(x_0)} \Gamma_\varepsilon^{\frac{3+\alpha-\mu}{2}} \eta^{2k-1} |\nabla\eta|\, dx\Bigg\} \\ &=: c\big\{1 + I + II\big\}\, .\end{aligned} \tag{12}$$

At this point we like to emphasize that the choice (10) of α gives

$$2 + \frac{\alpha-\mu}{2} = \frac{\kappa}{2} > \frac{q}{2}\, . \tag{13}$$

Next, for $\gamma > 0$ sufficiently small, Young's inequality yields a bound for II

$$\begin{aligned} II &\le \gamma \int_{B_R(x_0)} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx \\ &\quad + \gamma^{-1} \int_{B_R(x_0)} \Gamma_\varepsilon^{-2-\frac{\alpha-\mu}{2}} \Gamma_\varepsilon^{3+\alpha-\mu} \eta^{2k-2} |\nabla\eta|^2\, dx \\ &\le \gamma \int_{B_R(x_0)} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx + \frac{c\gamma^{-1}}{(R-r)^2} \int_{B_R(x_0)} \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} \eta^{2k-2}\, dx\, .\end{aligned} \tag{14}$$

Note that the first integral on the right-hand side of (14) may be absorbed on the left-hand side of (12), whereas the second one remains uniformly bounded: in fact, if $\mu \geq 1$, then $(2-\mu)/2 < 1/2$ and the claim is trivial (recall the negative sign of α). In case $\mu < 1$ we assume w.l.o.g. that $\nabla f(0) = 0$ (replace f by $\tilde{f}(Z) = f(Z) - \nabla f(0) : Z$) and that $f(0) = 0$. As a consequence, assumption (5) yields

$$\nabla f(Z) : Z = \int_0^1 D^2 f(Z)(\theta Z)(Z, Z)\, d\theta \geq a|Z|^{2-\mu} - b$$

for some real numbers $a > 0$, b. The equality on the left-hand side in particular shows that $\nabla f(Z) : Z \geq 0$, and we may proceed by estimating

$$f(Z) \geq \int_{1/2}^1 \nabla f(\theta Z) : \theta Z\, d\theta \geq c_1 |Z|^{2-\mu} - c_2 \tag{15}$$

for some other constants $c_1 > 0$, c_2. Thus f is at least of growth rate $2-\mu$, whenever $\mu < 1$, and Lemma 1, vi), proves our claim in this case as well.

Hence, Lemma 2 is established once an appropriate estimate for I is found. To this purpose we observe (again $\gamma > 0$ is sufficiently small and Young's inequality is applied)

$$\begin{aligned} I &\leq \gamma \int_{B_R(x_0)} \Gamma_\varepsilon^{-\frac{\mu}{2}} |\nabla^2 v_\varepsilon|^2 \eta^{2k+2}\, dx + \gamma^{-1} \int_{B_R(x_0)} \Gamma_\varepsilon^{2+\alpha-\frac{\mu}{2}} \eta^{2k-2}\, dx \\ &=: \gamma I_1 + \gamma^{-1} I_2\,. \end{aligned} \tag{16}$$

Using Lemma 3 together with Lemma 1, v), one obtains

$$\begin{aligned} I_1 &\leq \int_{B_R(x_0)} D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \partial_\gamma \nabla v_\varepsilon)\big(\eta^{k+1}\big)^2\, dx \\ &\leq c \int_{B_R(x_0)} \big|D^2 f_\varepsilon(\nabla v_\varepsilon)\big| |\nabla v_\varepsilon|^2 \eta^{2k} |\nabla \eta|^2\, dx \\ &\leq \frac{c}{(R-r)^2} \Bigg\{ \int_{B_R(x_0)} \Gamma_\varepsilon^{\frac{q}{2}} \eta^{2k}\, dx + \delta(\varepsilon) \int_{B_R(x_0)} \Gamma_\varepsilon^{\frac{t}{2}} \eta^{2k}\, dx \Bigg\} \\ &\leq \frac{c}{(R-r)^2} \Bigg\{ 1 + \int_{B_R(x_0)} \Gamma_\varepsilon^{\frac{q}{2}} \eta^{2k}\, dx \Bigg\}\,. \end{aligned}$$

As a result, (16) yields (recalling (13))

$$I \leq \frac{c\gamma}{(R-r)^2} \Bigg\{ 1 + \int_{B_R(x_0)} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx \Bigg\} + \gamma^{-1} \int_{B_R(x_0)} \Gamma_\varepsilon^{2+\alpha-\frac{\mu}{2}} \eta^{2k-2}\, dx\,. \tag{17}$$

Choosing $\gamma = \hat{\gamma}(R-r)^2$ with $\hat{\gamma} > 0$ sufficiently small, the first integral on the right-hand side of (17) may also be absorbed on the left-hand side of (12), hence it remains to find a bound for the second one. Here, the negative sign of α and, as a consequence, (11) and our choice of k come into play. For $\tilde{\gamma} > 0$ sufficiently small we get with a final application of Young's inequality

$$\begin{aligned}
&\hat{\gamma}^{-1}(R-r)^{-2}\int_{B_R(x_0)} \Gamma_\varepsilon^{2+\alpha-\frac{\mu}{2}}\eta^{2k-2}\,dx \\
\leq\ & c\hat{\gamma}^{-1}(R-r)^{-2}\left\{\tilde{\gamma}\int_{B_R(x_0)} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}}\eta^{2k}\,dx + \tilde{\gamma}^{-\frac{\sigma}{\sigma'-\sigma}}|B_R(x_0)|\right\}.
\end{aligned}$$

Absorbing terms by letting $\tilde{\gamma} = \gamma'\hat{\gamma}(R-r)^2$, $1 \gg \gamma' > 0$, Lemma 2 is proved. ■

References

[Ad] Adams, R. A., Sobolev spaces. Academic Press, New York-San Francisco-London 1975.

[AF1] Acerbi, E., Fusco, N., A regularity theorem for minimizers of quasiconvex integrals. Arch. Rat. Mech. Anal. 99 (1987), 261–281.

[AF2] Acerbi, E., Fusco, N., Local regularity for minimizers of non convex integrals. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 16, no. 4 (1989), 603–636.

[AF3] Acerbi, E., Fusco, N., Partial regularity under anisotropic (p,q) growth conditions. J. Diff. Equ. 107, no. 1 (1994), 46–67.

[Al] Almgren, F. J.,Jr., Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure. Annals of Math. 87 (1968), 321–391.

[AG] Anzellotti, G., Giaquinta, M., Convex functionals and partial regularity. Arch. Rat. Mech. Anal. 102 (1988), 243–272.

[Bi1] Bildhauer, M., Convex Variational Problems with Linear, Nearly Linear and/or Anisotropic Growth Conditions. Habilitationsschrift (submitted 2001), Saarland University, Saarbrücken.

[Bi2] Bildhauer, M., A priori gradient estimates for bounded generalized solutions of a class of variational problems with linear growth. To appear in J. Convex Anal.

[Bi3] Bildhauer, M., Convex variational problems with a wide range of anisotropy. Part I: regularity results. To appear.

[Bi4] Bildhauer, M., Convex variational problems with a wide range of anisotropy. Part II: anisotropic linear/superlinear growth. To appear.

[BF1] Bildhauer, M., Fuchs, M., Partial regularity for variational integrals with (s, μ, q)-growth. Calc. Var. 13 (2001), 537–560.

[BF2] Bildhauer, M., Fuchs, M., On a class of variational integrals with linear growth satisfying the condition of μ-ellipticity. To appear.

[BF3] Bildhauer, M., Fuchs, M., Partial regularity for a class of anisotropic variational integrals with convex hull property. To appear.

[BF4] Bildhauer, M., Fuchs, M., Twodimensional anisotropic variational problems. To appear in Calc. Var.

[BF5] Bildhauer, M., Fuchs, M., Convex Variational Problems with Linear Growth. To appear.

[BF6] Bildhauer, M., Fuchs, M., Relaxation of convex variational problems with linear growth defined on classes of vector-valued functions. To appear in Algebra and Analiz 14, no. 1 (2002).

[BFM] Bildhauer, M., Fuchs, M., Mingione, G., A priori gradient bounds and local $C^{1,\alpha}$-estimates for (double) obstacle problems under nonstandard growth conditions. Z. Anal. Anw. 20, no.4 (2001), 959–985.

[CFM] Carozza, M., Fusco, N., Mingione, G., Partial regularity of minimizers of quasiconvex integrals with subquadratic growth. Ann. Mat. Pura e Appl. 4 (1998), 141–164.

[CE] Chipot, M., Evans, L.C., Linearization at infinity and Lipschitz estimates for certain problems in the calculus of variations. Proc. Roy. Soc. Edinburgh 102 A (1986), 291–303.

[Ch] Choe, H. J., Interior behaviour of minimizers for certain functionals with nonstandard growth. Nonlinear Analysis, Theory, Methods & Appl. 19.10 (1992), 933-945.

[DG1] De Giorgi, E., Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. Mem. Accad. Sci. Torino 3 (1957), 25–43.

[DG2] De Giorgi, E., Frontiere orientate di misura minima. Quaderni Sc. Norm. Sup. Pisa (1960/61).

[DG3] De Giorgi, E., Un esempio di estremali discontinue per un problema variazionale di tipo ellittico. Boll. UMI 4 (1968), 135–137.

[ELM] Esposito, L., Leonetti, F., Mingione, G., Regularity for minimizers of functionals with p-q growth. Nonlinear Diff. Equ. Appl. 6 (1999), 133-148.

[EM] Esposito, L., Mingione, G., Partial regularity for minimizers of convex integrals with $L\log L$-growth. Nonlinear Diff. Equ. Appl. 7 (2000), 107–125.

[EG] Evans, L. C., Gariepy, R., Blowup, compactness and partial regularity in the calculus of variations. Indiana Univ. Math. J. 36 (1987), 361–371.

[Ev] Evans, L. C., Quasiconvexity and partial regularity in the calculus of variations. Arch. Rat. Mech. Anal. 95 (1986), 227–252.

[FrS] Frehse, J. Seregin, G., Regularity of solutions to variational problems of the deformation theory of plasticity with logarithmic hardening. Proc. St. Petersburg Math. Soc. 5 (1998), 184-222. English translation: Amer. Math. Soc. Transl. II, 193 (1999), 127–152.

[FS1] Fuchs, M., Seregin, G., A regularity theory for variational integrals with $L \log L$-growth. Calc. Var. 6 (1998), 171–187.

[FS2] Fuchs, M., Seregin, G., Variational methods for problems from plasticity theory and for generalized Newtonian fluids. Lecture Notes in Mathematics 1749, Springer, Berlin-Heidelberg 2000.

[FH] Fusco, N., Hutchinson, J. E., $C^{1,\alpha}$ partial regularity of functions minimizing quasiconvex integrals. Manus. Math. 54 (1985), 121–143.

[FM] Fuchs, M., Mingione, G., Full $C^{1,\alpha}$-regularity for free and constrained local minimizers of elliptic variational integrals with nearly linear growth. Manus. Math. 102 (2000), 227–250.

[FO] Fuchs, M., Osmolovskii, V., Variational integrals on Orlicz-Sobolev spaces. Z. Anal. Anw. 17 (1998), 393–415.

[Gia1] Giaquinta, M., Multiple integrals in the calculus of variations and nonlinear elliptic systems. Ann. Math. Studies 105, Princeton University Press, Princeton 1983.

[Gia2] Giaquinta, M., Growth conditions and regularity, a counterexample. Manus. Math. 59 (1987), 245–248.

[GM] Giaquinta, M., Modica, G., Remarks on the regularity of the minimizers of certain degenerate functionals. Manus. Math. 57 (1986), 55–99.

[GiM1] Giusti, E., Miranda, M., Sulla regolarità delle soluzioni deboli di una classe di sistemi ellittici quasi-lineari. Arch. Rat. Mech. Anal. 31 (1968), 173–184.

[GiM2] Giusti, E., Miranda, M., Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni. Boll. UMI 2 (1968), 1–8.

[Giu] Giusti, E., Regolarità parziale delle soluzioni di sistemi ellittici quasi lineari di ordine arbitrario. Ann. Sc. Norm. Sup. Pisa 23 (1969), 115–141.

[LU] Ladyzhenskaya, O.A., Ural'tseva, N.N., Linear and quasilinear elliptic equations. Nauka, Moskow, 1964. English translation: Academic Press, New York 1968.

[Ma1] Marcellini, P., Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. Arch. Rat. Mech. Anal. 105 (1989), 267–284.

[Ma2] Marcellini, P., Regularity and existence of solutions of elliptic equations with (p, q)–growth conditions. J. Diff. Equ. 90 (1991), 1–30.

[Ma3] Marcellini, P., Regularity for elliptic equations with general growth conditions. J. Diff. Equ. 105 (1993), 296–333.

[Ma4] Marcellini, P., Everywhere regularity for a class of elliptic systems without growth conditions. Ann. Scuola Norm. Sup. Pisa 23 (1996), 1–25.

[Ma5] Marcellini, P., Regularity for some scalar variational problems under general growth conditions. J. Optim. Theory Appl. 90 (1996), 161–181.

[Ma6] Marcellini, P., General growth conditions and regularity. In: Variational methods for discontinuous structures (Como 1994), 111–118, Birkhäuser, Basel 1996.

[Mor1] Morrey, C. B., Multiple integrals in the calculus of variations. Grundlehren der math. Wiss. in Einzeldarstellungen 130, Springer, Berlin-Heidelberg-New York 1966.

[Mor2] Morrey, C. B., Partial regularity results for nonlinear elliptic systems. J. Math. Mech. 17 (1968), 649–670.

[Mos] Moser, J., A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. Comm. Pure Appl. Math. 13 (1960), 457–468.

[MS] Mingione, G., Siepe, F., Full $C^{1,\alpha}$ regularity for minimizers of integral functionals with $L \log L$ growth. Z. Anal. Anw. 18 (1999), 1083–1100.

[Na] Nash, J., Continuity of solutions of parabolic and elliptic equations. Amer. J. Math. 80 (1958), 931–954.

[Ne] Nečas, J., Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions of regularity. In: Theory of Non Linear Operators, Abhandlungen Akad. der Wissen. der DDR, Proc. of a summer school held in Berlin (1975), 197–206, Berlin 1977.

[PS] Passarelli Di Napoli, A., Siepe, F., A regularity result for a class of anisotropic systems. Rend. Ist. Mat. Univ. Trieste 28, No.1-2 (1996), 13–31.

[SY] Šverák, V., Yan, X., A singular minimizer of a smooth strongly convex functional in three dimensions. Calc. Var. 10 (2000), 213–221.

[Uh] Uhlenbeck, K., Regularity for a class of nonlinear elliptic systems. Acta Math. 138 (1977), 219–240.