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Abstract

Let $\Omega \subset \mathbb{R}^2$ denote a bounded Lipschitz domain and consider some portion Γ_0 of $\partial\Omega$ representing the austenite-twinned martensite interface which is not assumed to be a straight segment. We prove

$$\inf_{u \in \mathcal{W}(\Omega)} \int_\Omega \varphi(\nabla u(x,y))dxdy = 0$$

for an elastic energy density $\varphi : \mathbb{R}^2 \to [0,\infty)$ such that $\varphi(0,\pm 1) = 0$. Here $\mathcal{W}(\Omega)$ consists of all functions u from the Sobolev class $W^{1,\infty}(\Omega)$ such that $|u_y| = 1$ a.e. on Ω together with $u = 0$ on Γ_0. Moreover some minimizing sequences vanishing on the whole boundary $\partial\Omega$ are constructed, that is, one can even take $\Gamma_0 = \partial\Omega$. We also show that the existence or nonexistence of minimizers depends on the shape of the austenite-twinned martensite interface Γ_0.

AMS classification: 49, 74

Keywords: microstructure, martensitic phase transformation, elastic energy, minimizing sequences, Young measures.

1 Introduction.

In solid-solid phase transformations one often observes certain characteristical microstructural features involving fine mixtures of the phases. If we consider martensitic phase transformations, then one usually has a plane interface which separates one homogeneous phase called austenite from a very fine mixture of twins of the other phase termed martensite. We now consider a two-dimensional section and assume that for some physical reasons the interface which seperates the two phases is not a segment but a curve not necessarily being smooth.

For instance, it is known that some applied small loads easily change the austenite-martensite interface. For further details concerning the physical background of martensitic phase transformation and also the mathematical modelling we refer the reader to the papers [B.J.$_1$] and [B.J.$_2$] and the references quoted therein. To give a more precise formulation of the problem we like to investigate, let us consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ representing the martensitic configuration, and let Γ_0 denote a part of $\partial\Omega$ with positive measure having the meaning of the austenite-twinned martensite interface. Let $\varphi : \mathbb{R}^2 \to [0,\infty)$ denote a Borel function such that

$$\varphi(0,1) = \varphi(0,-1) = 0. \tag{1.1}$$

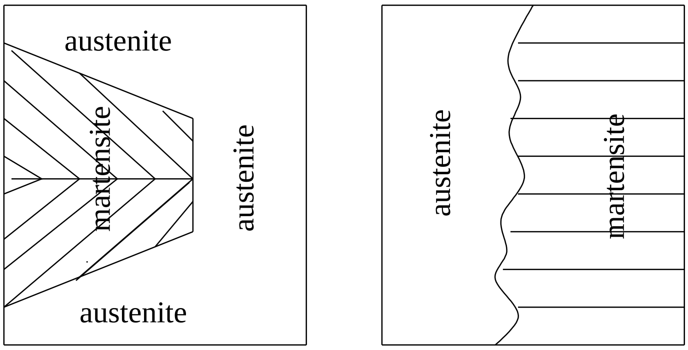

Figure 1. The austenite−twinned martensite interface

For example, φ could be the elastic energy density of the martensite with wells in $(0, \pm 1)$ corresponding to the stress-free states of two possible variants of the martensite. We then would like to consider the problem

$$I^\infty := \inf_{u \in \mathcal{W}(\Omega)} \int_\Omega \varphi(\nabla u(x,y))dxdy \tag{1.2}$$

in the class of admissible comparison functions

$$\mathcal{W} := \mathcal{W}(\Omega) := \{u \in W^{1,\infty}(\Omega) \ : \ |u_y| = 1 \text{ a.e. in } \Omega \text{ and } u = 0 \text{ on } \Gamma_0\}.$$

Here $W^{1,\infty}(\Omega)$ is the Sobolev space of all weakly differentiable functions $u : \Omega \to \mathbb{R}$ such that u, $|\nabla u| \in L^\infty(\Omega)$. Since Ω is a bounded Lipschitz domain, Sobolev's embedding theorem implies $W^{1,\infty}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, and the requirement $u = 0$ on Γ_0 has to be understood in the pointwise sense. If $u = 0$ on the whole of $\partial\Omega$, we just say that u is of class $W_0^{1,\infty}(\Omega)$. For a further discussion of Sobolev spaces we refer the reader to [A.].
We remark that the boundary condition occurring in $\mathcal{W}$ refers to elastic compatibility with the austenitic phase in the extreme case of complete rigidity of the austenite (see [B.J.$_1$], [B.J.$_2$] and [Ko.]). Problems of the type (1.2) have been investigated by Chipot and Collins (compare [C.] and [C.C.]) but without the constraint $|u_y| = 1$. This constraint was introduced by Kohn and Müller (see [K.M.$_1$] and [K.M.$_2$]): they considered a functional consisting of an elastic energy plus a surface energy term for the case that the martensitic configuration is a rectangle like $(0, L) \times (0, 1)$ and the austenite-martensite interface is the segment $\{0\} \times (0, 1)$.
Problem (1.2) was studied in [E.F.] for the case when no loads are applied, i.e. the austenite-martensite interface is given by a segment Γ_0. We proved

that the value of I^∞ is zero by constructing suitable minimizing sequences from the class $\mathcal{W}(\Omega)$ which represent, according to the Ball-James theory, the microstructure. The minimizing sequences discussed in [E.F.] differ for the case when the segment Γ_0 is vertical and for the case when Γ_0 is oblique. In particular, for non-vertical segments we could even replace the set $\mathcal{W}(\Omega)$ by a smaller class by adding the additional constraint

$$|u_{yy}| \text{ is a Radon measure of finite mass}$$

which is not true in the vertical case (see [W.]).
In the present note we want to extend the result of [E.F.] to the general case of curved boundary portions, precisely we have:

THEOREM 1.1 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ and consider a non empty portion Γ_0 of $\partial\Omega$ having positive measure . If φ satisfies (1.1), then we have*

$$I^\infty := \inf_{u \in \mathcal{W}(\Omega)} \int_\Omega \varphi(\nabla u(x,y))dxdy = 0.$$

Moreover, we can find a minimizing sequence $(u_n)_n \subset \mathcal{W}(\Omega)$ such that $u_n = 0$ on the whole boundary $\partial\Omega$.

For the proof we first discuss in Section 2 the case when the Lipschitz domain Ω is replaced by some elementary domain, e.g. the domain enclosed by a triangle or a square. Then, in Section 3, we consider the general situation by covering every bounded open set with a countable number of such elementary domains.

2 The case of some elementary domains.

Here we prove Theorem 1.1 for some special cases. First we let Δ denote the interior of the triangle with vertices in $(-1,0)$, $(1,0)$ and $(0,1)$.

THEOREM 2.1 *Assume that φ satisfies (1.1). Then there exists a sequence $v_n \in W_0^{1,\infty}(\Delta)$ satisfying $|\partial_y v_n| = 1$ a.e. for each n and such that*

$$\lim_{n\to\infty} \int_\Delta \varphi(\nabla v_n(x,y))dxdy = 0.$$

Proof. Given $N \in \mathbb{N}$ we will define $u \in W_0^{1,\infty}(\Delta)$, $|u_y| = 1$, such that

$$\int_\Delta \varphi(\nabla u(x,y))dxdy$$

is of order $\frac{1}{N}$. Let $\delta := \frac{1}{N}$ and consider the δ-periodic extension to the whole line of

$$h(t) := \begin{cases} t & \text{if } 0 \le t \le \frac{\delta}{2}, \\ \delta - t & \text{if } \frac{\delta}{2} \le t \le \delta. \end{cases}$$

We then let

$$u(x,y) := \begin{cases} (x+1-y) \wedge h(y) & \text{if } (x,y) \in \Delta, -1 \le x \le 0, \\ (1-x-y) \wedge h(y) & \text{if } (x,y) \in \Delta, 0 \le x \le 1. \end{cases}$$

Here we write $\alpha \wedge \beta$ for the minimum of two numbers $\alpha,\ \beta \in \mathbb{R}$. Figure 2 below shows the situation for $N = 3$.

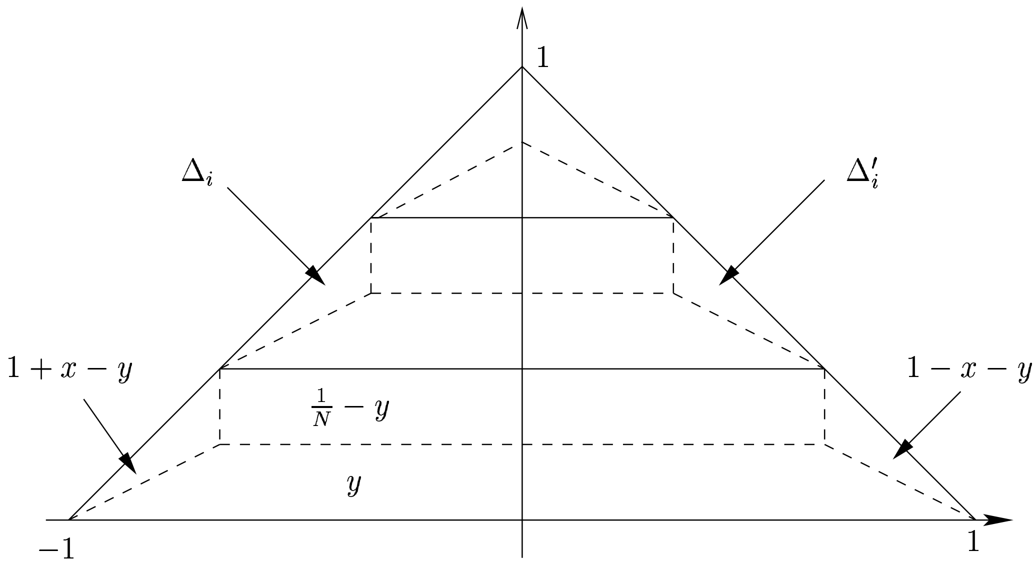

Figure 2: the function u for $N = 3$

Clearly $u \in W_0^{1,\infty}(\Delta)$ and

$$\nabla u(x,y) = (0, \pm 1)$$

for points (x,y) not belonging to the $2N$ triangles Δ_i and Δ_i', $i = 1, \ldots, N$. It is easy to check that

$$\nabla u(x,y) = (1,-1) \text{ on } \Delta_i$$

whereas

$$\nabla u(x,y) = (-1,-1) \text{ on } \Delta_i'.$$

Therefore $|u_y| = 1$ a.e. on Δ and (1.1) implies

$$\int_\Delta \varphi(\nabla u(x,y))dxdy = \sum_{i=1}^{N}\Big[\int_{\Delta_i} \varphi(\nabla u(x,y))dxdy + \int_{\Delta_i'} \varphi(\nabla u(x,y))dxdy\Big]$$

$$= \sum_{i=1}^{N}\Big[\mathcal{L}^2(\Delta_i)\varphi(1,-1) + \mathcal{L}^2(\Delta_i')\varphi(-1,-1)\Big]$$

$$= N\frac{\delta^2}{4}[\varphi(1,-1) + \varphi(-1,-1)],$$

thus

$$0 \le I^\infty \le \int_\Delta \varphi(\nabla u(x,y))dxdy = \frac{1}{4N}[\varphi(1,-1) + \varphi(-1,-1)],$$

and Theorem 2.1 is established.

■

Let S now denote the set of points (x,y) such that $(x,y) \in \overline{\Delta}$ or $(x,-y) \in \overline{\Delta}$, i.e. S is the closed square with vertices in $(\pm 1, 0)$ and $(0, \pm 1)$. Then we have the following

Corollary 2.1 *Assume that φ satisfies (1.1). Then there exists a sequence $v_n \in W_0^{1,\infty}(\overset{\circ}{S})$ satisfying $|\partial_y v_n| = 1$ a.e. for each n and such that*

$$\lim_{n\to\infty} \int_S \varphi(\nabla v_n(x,y))dxdy = 0.$$

Proof. Let us define on S the following function

$$v(x,y) := \begin{cases} u(x,y) & \text{if } (x,y) \in \Delta, \\ \\ u(x,-y) & \text{if } (x,y) \in S\backslash\Delta \end{cases}$$

where the function $u : \Delta \to \mathbb{R}$ is defined in the proof of Theorem 2.1. One can easily check that

$$\int_S \varphi(\nabla v(x,y))dxdy = \int_\Delta \varphi(\nabla u(x,y))dxdy + \int_\Delta \tilde{\varphi}(\nabla u(x,y))dxdy$$

where

$$\tilde{\varphi}(x,y) = \varphi(x,-y).$$

Thus

$$\int_S \varphi(\nabla v(x,y))dxdy = \frac{1}{4N}[\varphi(1,-1)+\varphi(-1,-1)+\tilde{\varphi}(1,-1)+\tilde{\varphi}(-1,-1)]$$

$$= \frac{1}{4N}[\varphi(1,-1)+\varphi(-1,-1)+\varphi(1,1)+\varphi(-1,1)],$$

and Corollary 2.1 is proved.

■

REMARK 2.1 *Notice that for the elementary domains we considered above one can add the constraint*

$$|u_{yy}| \textit{ is a Radon measure of finite mass.}$$

One can also consider other elementary domains like squares with sides parallel to the x and y axis or discs and construct minimizing sequences using the principle of branching. But for these domains it is not possible to incorporate the above constraint.

3 The construction of minimizing sequences for general domains.

Here we are going to prove Theorem 1.1. To this purpose we need the following lemmas

LEMMA 3.1 *Let Ω denote a bounded open subset of $\mathbb{R}^2$. Then there exist points $(x_n, y_n) \in \Omega$ and positive numbers r_n such that*

$$S_n := r_n S + (x_n, y_n) \subset \Omega \textit{ and } \mathring{S}_l \cap \mathring{S}_k = \emptyset \textit{ for } l \neq k,$$

where S is the square with vertices in $(\pm 1, 0)$ and $(0, \pm 1)$. Moreover, we have

$$\Omega = \bigcup_{n=0}^{+\infty} S_n.$$

Proof. A multi-dimensional proof can be found in [S.]. Nevertheless for our two-dimensional case we give an alternative proof showing the evolution of the microstructure when it approaches the boundary. We put $\Omega_0 = \Omega$ and cover it with a scaled copy (with diameter δ) of the square S. We divide this square into four squares by joining the midpoints of its sides and denote by $\mathcal{S}_0$ the union of all squares which are inside Ω_0. We then let

$$\Omega_1 = \Omega_0 \backslash \mathcal{S}_0$$

and divide the squares which intersect Ω_1 as above and put

$$\Omega_2 = \Omega_1 \backslash \mathcal{S}_1$$

where $\mathcal{S}_1$ is the union of all squares inside Ω_1. Repeating the above procedure, we inductively obtain two sequences $(\Omega_n)_n$ and $(\mathcal{S}_n)_n$ such that

$$\begin{cases} \Omega_0 = \Omega, \\ \\ \Omega_{n+1} = \Omega_n \backslash \mathcal{S}_n \end{cases}$$

where $\mathcal{S}_n$ is the union of all squares inside Ω_n obtained at the (n+1)$^{\text{th}}$ step. Notice that the squares composing $\mathcal{S}_n$ are of diameter $\frac{\delta}{4^{n+1}}$. It is clear that

$$\Omega = \Omega_0 = \mathcal{S}_0 \cup \Omega_1 = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \Omega_2 = \ldots = \Big(\bigcup_{i=0}^{n} \mathcal{S}_i\Big) \cup \Omega_{n+1} \text{ for all } n \in \mathbb{N}.$$

We claim that

$$\Omega = \bigcup_{n=0}^{+\infty} \mathcal{S}_n.$$

We proceed by contradiction, assuming that there exists $x \in \Omega$ such that

$$x \notin S_n \text{ for every } n \in \mathbb{N}.$$

But if $x \notin S_n$, then x would belong to a square of diameter $\frac{\delta}{4^{n+1}}$ encountering the boundary of Ω. Thus

$$\operatorname{dist}(x, \partial\Omega) \leq \frac{\delta}{4^{n+1}} \text{ for every } n \in \mathbb{N}$$

where $\operatorname{dist}(x, \partial\Omega)$ denotes the distance from x to the boundary $\partial\Omega$. Hence

$$\operatorname{dist}(x, \partial\Omega) = 0, \text{ i.e. } x \in \partial\Omega,$$

which is not possible. This completes the proof of the lemma.

■

We now return to our plane domain Ω. Applying the construction of Lemma 3.1 we find $r_n > 0$, $(x_n, y_n) \in \Omega$ such that the sets $S_n = r_n S + (x_n, y_n) \subset \Omega$ have the stated properties. Given a function $u_0 \in W_0^{1,\infty}(\overset{\circ}{S})$, we let

$$\begin{cases} u_n \ : \ S_n \ \to \mathbb{R}, \ u_n(x,y) := r_n u_0(\frac{1}{r_n}(x - x_n, y - y_n)), \\ \\ u \ : \ \Omega \ \to \mathbb{R}, \ u(x,y) := \sum_{n=1}^{\infty} (\chi_{\overset{\circ}{S}_n} u_n)(x,y) \end{cases} \tag{3.1}$$

where $\chi_{\overset{\circ}{S}_n}$ denotes the characteristic function of the set $\overset{\circ}{S}_n$. Then we claim:

LEMMA 3.2 *The function u defined in (3.1) is in the space $W_0^{1,\infty}(\Omega)$, and we have the following formula*

$$\nabla u(x,y) = \sum_{n=1}^{\infty} (\chi_{\overset{\circ}{S}_n} \nabla u_n)(x,y) = \sum_{n=1}^{\infty} \chi_{\overset{\circ}{S}_n} \nabla u_0(\frac{1}{r_n}(x - x_n, y - y_n)) \ \textit{a.e. on } \Omega.$$

REMARK 3.1 *If we know $|\partial_y u_0| = 1$ a.e. on $\overset{\circ}{S}$, then we deduce from the disjointness of the family $\{\overset{\circ}{S}_n\}$ that also $|u_y| = 1$ is true a.e. on Ω.*

Proof of Lemma 3.2: On account of $(x_n, y_n) \in \Omega$, $S_n \subset \Omega$, the sequence $(r_n)_n$ stays bounded, thus

$$||u||_{L^\infty(\Omega)} \leq \sup_{n \in \mathbb{N}} r_n \ ||u_0||_{L^\infty(S)} < \infty.$$

In order to prove weak differentiability of the function u, we fix $\psi \in C_0^\infty(\Omega)$ and get from Lebesgue's theorem on dominated convergence

$$\int_\Omega u(x,y) \nabla \psi(x,y) dxdy = \sum_{n=1}^{\infty} \int_{\overset{\circ}{S}_n} u_n(x,y) \nabla \psi(x,y) dxdy.$$

Observing that $u_n = 0$ on ∂S_n, we can write

$$\int_{\overset{\circ}{S}_n} u_n(x,y) \nabla \psi(x,y) dxdy = - \int_{\overset{\circ}{S}_n} \nabla u_n(x,y) \psi(x,y) dxdy$$

and by the same reasoning as above (note: $||\nabla u_n||_{L^\infty(S_n)} = ||\nabla u_0||_{L^\infty(S)}$ and therefore $||\sum_{n=1}^{M} \chi_{\overset{\circ}{S}_n} \nabla u_n||_{L^\infty(\Omega)} = ||\nabla u_0||_{L^\infty(S)}$ for all $M \geq 1$)

$$- \sum_{n=1}^{\infty} \int_{\overset{\circ}{S}_n} \nabla u_n(x,y) \psi(x,y) dxdy = - \int_\Omega (\sum_{n=1}^{\infty} \chi_{\overset{\circ}{S}_n} \nabla u_n(x,y)) \psi(x,y) dxdy,$$

which proves that

$$\sum_{n=1}^{\infty} \chi_{\overset{\circ}{S}_n} \nabla u_n \in L^{\infty}(\Omega, \mathbb{R}^2)$$

is the weak derivative of u. Again by dominated convergence it is obvious that

$$\sum_{n=1}^{M} \chi_{\overset{\circ}{S}_n} u_n \to u, \ \sum_{n=1}^{M} \chi_{\overset{\circ}{S}_n} \nabla u_n \to \nabla u$$

as M goes to infinity in $L^p(\Omega)$ for any finite p. Since the compact sets S_n are included in Ω, we have

$$\sum_{n=1}^{M} \chi_{\overset{\circ}{S}_n} u_n \in W_0^{1,p}(\Omega),$$

thus $u \in W_0^{1,p}(\Omega),\ p < \infty$. Lipschitz boundary of Ω guarantees that

$$W_0^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega) \ : \ B(v) = 0\},$$

where $B : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is the trace operator. Recalling that for functions $v \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$, $B(v)$ is the pointwise trace, we finally deduce $u \in W_0^{1,\infty}(\Omega)$.

■

The proof of Theorem 1.1 can now be carried out as follows. Given $N \in \mathbb{N}$, we constructed in the proof of Corollary 2.1 a function $u_0 \in W_0^{1,\infty}(\overset{\circ}{S})$ such that $|\partial_y u_0| = 1$ on S and

$$\int_S \varphi(\nabla u_0(x,y))dxdy = \frac{1}{4N}[\varphi(1,-1) + \varphi(-1,-1) + \varphi(1,1) + \varphi(-1,1)].$$

Let us consider the function u defined in (3.1) for this particular choice of u_0. Lemma 3.2 implies $u \in W_0^{1,\infty}(\Omega)$, and from the remark after Lemma 3.2 we deduce $|u_y| = 1$ a.e. on Ω, thus $u \in \mathcal{W}(\Omega)$. We further have:

$$\int_{\Omega} \varphi(\nabla u(x,y))dxdy = \sum_{n=1}^{\infty} \int_{\overset{\circ}{S}_n} \varphi(\nabla u_0(\frac{1}{r_n}(x - x_n, y - y_n)))dxdy$$

$$= \sum_{n=1}^{\infty} r_n^2 \int_{\overset{\circ}{S}} \varphi(\nabla u_0(x,y))dxdy$$

so that

$$\int_{\Omega} \varphi(\nabla u(x,y))dxdy = \frac{1}{4N}[\varphi(1,-1) + \varphi(-1,-1) + \varphi(1,1) + \varphi(-1,1)] \sum_{n=1}^{\infty} r_n^2.$$

Finally we observe

$$\mathcal{L}^2(\Omega) = \sum_{n=1}^{\infty} \mathcal{L}^2(r_n S + (x_n, y_n)) = 2\sum_{n=1}^{\infty} r_n^2,$$

hence

$$\int_\Omega \varphi(\nabla u(x,y))dxdy = \frac{1}{2N}\mathcal{L}^2(\Omega)[\varphi(1,-1)+\varphi(-1,-1)+\varphi(1,1)+\varphi(-1,1)],$$

and since N was arbitrary, we have shown that $I^\infty = 0$. Moreover, it should be obvious how to obtain from the above construction a minimizing sequence in the class $\mathcal{W}(\Omega) \cap W_0^{1,\infty}(\Omega)$. This finishes the proof of Theorem 1.1.

■

4 Remarks.

In addition to (1.1) let us assume that the integrand φ satisfies

$$\varphi(p, \pm 1) = 0 \Longrightarrow p = 0. \tag{4.1}$$

Under this condition we like to investigate if the infimum $I^\infty = 0$ is attained by some function $u \in \mathcal{W}(\Omega)$. This heavily depends on the shape of the boundary portion. For example, if $\Gamma_0 \subset \mathbb{R} \times \{b\}$ for some number $b \in \mathbb{R}$, then clearly $u(x,y) = y - b$ vanishes on Γ_0, $\partial_y u \equiv 1$ and $\nabla u(x,y) = (0,1)$, hence $\varphi(\nabla u(x,y)) = 0$ by (1.1). In order to exclude such a behaviour we let Σ denote the union of all rays starting from points $(x_0, y_0) \in \Gamma_0$ into Ω with direction $(1,0)$, and require

$$\Omega_0 := \Omega \cap \Sigma \text{ is open and nonempty.} \tag{4.2}$$

Of course, (4.2) does not hold in case $\Gamma_0 \subset \mathbb{R} \times \{b\}$.

THEOREM 4.1 *Let (1.1), (4.1) and (4.2) hold. Then we have*

$$\int_\Omega \varphi(\nabla u(x,y))dxdy > 0$$

for any $u \in \mathcal{W}(\Omega)$.

Proof. If we assume that

$$\int_\Omega \varphi(\nabla u(x,y))dxdy = 0$$

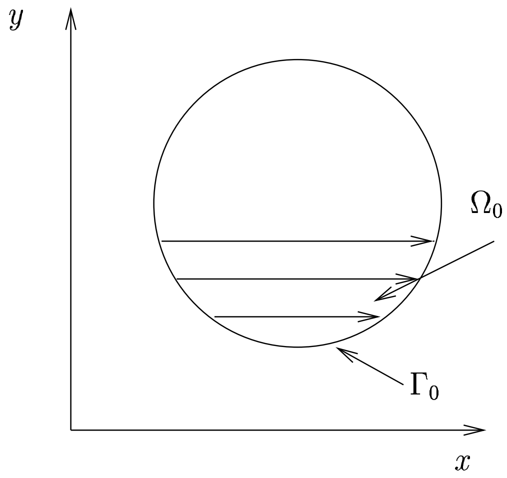

Figure3: Ω = a disc

for some $u \in \mathcal{W}(\Omega)$, then we get from (4.1)

$$u_x = 0 \text{ on } \Omega.$$

This implies the vanishing of u on any ray of the type defined before, hence, by (4.2), $u = 0$ on Ω_0 contradicting $u_y = \pm 1$ a.e.

$\blacksquare$

Next we like to describe minimizing sequences in terms of Young measures (see [P.] for details about the notion Young measure)

THEOREM 4.2 *Let Ω denote a bounded Lipschitz domain in $\mathbb{R}^2$ and assume that the boundary portion Γ_0 is chosen in such a way that $\Omega_0 = \Omega$ (see (4.2)). Suppose that the integrand $\varphi : \mathbb{R}^2 \to [0, \infty)$ is a continuous function such that*

$$\varphi(p, q) = 0 \textit{ if and only if } (p, q) = (0, \pm 1).$$

Let $(u_n)_n$ denote a minimizing sequence of problem (1.2) *such that*

$$||u_n||_{L^\infty(\Omega)},\ ||\nabla u_n||_{L^\infty(\Omega)} \le C$$

for a finite constant C independent of n. Then

$$u_n \to 0 \textit{ uniformly on } \Omega.$$

Moreover, the sequence of gradients $(\nabla u_n)_n$ defines a unique homogeneous Young measure given by

$$\nu_{(x,y)} = \frac{1}{2}\delta_{(0,-1)} + \frac{1}{2}\delta_{(0,1)} \textit{ for a.a. } (x, y) \in \Omega,$$

where $\delta_{(0,\pm 1)}$ are the Dirac measures at $(0, \pm 1)$.

Proof. One proceeds as in [E.F.], we refer also to [C.] for a proof related to multiple-wells problems.

Corollary 4.1 *Let Ω denote a bounded Lipschitz domain in $\mathbb{R}^2$. Suppose that the integrand $\varphi : \mathbb{R}^2 \to [0, \infty)$ is a continuous function such that*

$$\varphi(p,q) = 0 \text{ if and only if } (p,q) = (0, \pm 1).$$

Let $(u_n)_n$ denote a minimizing sequence of problem (1.2) such that

$$||u_n||_{L^\infty(\Omega)},\ ||\nabla u_n||_{L^\infty(\Omega)} \leq C.$$

Suppose further that (4.2) holds. Then

$$u_n \to 0 \text{ uniformly on } \Omega_0.$$

Moreover, the sequence of gradients $(\nabla u_n)_n$ defines a Young measure given by

$$\nu_{(x,y)} = \alpha(x)\delta_{(0,-1)} + (1-\alpha(x))\delta_{(0,1)} \text{ for a.a. } (x,y) \in \Omega,$$

where $\alpha : \Omega \to [0,1]$ is a measurable function such that

$$\alpha(x) = \frac{1}{2} \text{ for a.e. in } \Omega_0$$

Proof. The restriction of (u_n) to Ω_0 is a minimizing sequence of

$$I^\infty(\Omega_0) := \inf_{u \in \mathcal{W}(\Omega_0)} \int_{\Omega_0} \varphi(\nabla u(x,y))dxdy = 0.$$

where $\mathcal{W}(\Omega_0)$ is defined with respect to the boundary portion $\Gamma_0 \cap \partial\Omega_0$. Since $(\Omega_0)_0 = \Omega_0$ with an obvious definition of $(\Omega_0)_0$, one can apply Theorem 4.2 to get Corollary 4.1.

REMARK 4.1 *Note that $\Omega_0 = \Omega$ holds for the particular case $\Gamma_0 = \partial\Omega$. Now if $\Omega_0 \neq \Omega$ then the considered minimizing sequences do not necessarily converge to zero uniformly on the whole domain Ω and the related Young measure is in general not unique (see [E.F.] Remark 6 for an example).*

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