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Abstract

The solvability of the two-phase quasilinear elliptic boundary value problems with Venttsel interface condition is established under the assumption that the interface surface meets the exterior boundary of a medium transversally. Bibliography: 12 titles.

The boundary value problems with Venttsel type conditions appear in the description of various physical processes in media containig a thin film of a material having high permeability. The examination of the one-phase Venttsel problems was initiated by N. S. Trudinger in the late 80th. During the last two decades the theory of solvability of the one- and two-phase Venttsel problems has been actively developing by many authors. For historical remarks and bibliography the reader is referred to review [1].

This paper continues a series of the authors' publications (see [2] and [3]) devoted to the two-phase Venttsel problems. Such problems can be considered as models of some processes in a medium separated into two parts by a thin film. The condition on the interface in this case is specified by an equation of the second order with the principal term being an elliptic (parabolic) operator in tangential variables and with the first order term being a "jump" operator across the separating film.

In [3], the existence result in Sobolev and Hölder spaces was established for parabolic and elliptic quasilinear two-phase Venttsel problems in the case, where the interface surface does not intersect the exterior boundary of a medium.

Our objective in this paper is to study the solvabilty of the quasilinear elliptic two-phase Venttsel problems under the assumption that the separating film meets the exterior boundary of a medium transversally. It follows from this assumption that both parts of a medium are domains with smooth closed edges. So, we can consider, first of all, the auxiliary Dirichlet problem on the interface surface, and then reduce the proof of a priori estimates for solutions of our two-phase problem to corresponding estimates for solutions of two auxiliary Dirichlet problems in domains with edges. It is known (see [4] for details) that the natural functional spaces for studing the Dirichlet problems in domains with edges are the weight Sobolev spaces, where the weight is equal to some power of the distance from a point to the edge. By this reason, we have to solve the Dirichlet problem on the separating film also

in the weight spaces where the weight is equal to some power of the distance (in the intrinsic metric) from a point to the boundary of the film. We note that a priori estimates for solutions of the Dirichlet problems in domains with edges were obtained in [5] and [4], while all the necessary estimates for solutions of the Dirichlet problem on films were obtained in [6].
The paper is divided into three sections. In Sec. 1, we state the problem and formulate the main result. In Sec. 2, we consider the corresponding linear problem. First, we derive an auxiliary result about an extension operator acting in the weight spaces. Then we obtain the coercive estimates for solutions and prove the existence and uniqueness theorem in the weight spaces. Finally, in Sec. 3, estimates for the gradients of solutions of the quasilinear problem are established and the main existence result is proved.

Notation.
Throughout the paper, we use the following notation:
$x = (x', x'')$ is a vector in $\mathbb{R}^n$, $x' = (x_1, x_2) \in \mathbb{R}^2$, $x'' = (x_3, \ldots, x_n) \in \mathbb{R}^{n-2}$;
$|x|$, $|x'|$, $|x''|$ are the Euclidean norms in the corresponding spaces;
$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 > 0\}$;
$\mathbb{R}^n_{+,+} = \{x \in \mathbb{R}^n : x_1 > 0,\ x_2 > 0\}$;
$\mathbb{R}^{n-1}_{+,2} = \{(x_2, x'') \in \mathbb{R}^{n-1} : x_2 > 0\}$;
Ω is a bounded domain in $\mathbb{R}^n$ with compact closure $\overline{\Omega}$ and $(n-1)$-dimensional boundary $\partial\Omega$;
$|\Omega|$ denotes the Lebesgue measure of Ω;
Σ is a sufficiently smooth hypersurface separating Ω into two subdomains: Ω_1 and Ω_2;
$\mathbf{n}(x)$ is the unit vector of the outward (with respect to Ω_1) normal to Σ at the point x;
$B^n_\rho(x^0)$ is the open ball in $\mathbb{R}^n$ with center x^0 and radius ρ;
$\Pi_\rho = \{x \in \mathbb{R}^n : |x'| < \rho, |x''| < \rho, x_2 > 0\}$;
$\Gamma(\Pi_\rho)$ is the part of Π_ρ lying on the hyperplane $x_1 = 0$.
We adopt the convention that the indices i, j, and s run from 1 to n, and the indices k and m run from 2 to n. We also adopt the convention regarding summation with respect to repeated indices.
D_i denotes the operator of differentiation with respect to the variable x_i;
$Du = (D_i u)$ is the gradient of u; $\quad D^*u = (D_2 u, D'' u) = (D_2 u, D_3 u, \ldots, D_n u)$;
$D^2 u = D(Du)$ is the Hessian of u.
For $p \in \mathbb{R}^n$ and $x \in \Sigma$ we define $\widetilde{p} = p - (p \cdot \mathbf{n}(x))\, \mathbf{n}(x)$.
$\widetilde{D}_i$ denotes the tangent differential operator on the manifold Σ;
$\widetilde{D}u = (\widetilde{D}_i u)$ is the tangential gradient of u.
We introduce the following spaces:

$C(\overline{\Omega})$ is the space of continuous functions with the norm $\|\cdot\|_{\Omega}$;
$C^2(\overline{\Omega})$ is the space of functions continuous in Ω together with their derivatives up to the second order;
$C^{1+\gamma}(\overline{\Omega})$ is the Hölder space with the norm

$$\|u\|_{C^{1+\gamma}(\overline{\Omega})} = \|u\|_{\Omega} + \|Du\|_{\Omega} + [Du]_{\gamma,\Omega}\,,$$

where $[\cdot]_{\gamma,\Omega}$ is the Hölder constant with exponent γ;
$W^2_p(\Omega)$ $(1 \leqslant p \leqslant \infty)$ is the Sobolev space with the norm

$$\|u\|_{W^2_p(\Omega)} = \|D^2u\|_{p,\Omega} + \|u\|_{p,\Omega},$$

where $\|\cdot\|_{p,\Omega}$ denotes the standard norm in $L_p(\Omega)$;
We set

$$f_+ = \max\ \{f, 0\}\,, \qquad f_- = \max\ \{-f, 0\}\,, \qquad \operatorname*{osc}_{\Omega} f = \sup_{\Omega} f - \inf_{\Omega} f,$$

We assume that $q > n$ and define

$$\widehat{\alpha}(q) = 1 - \frac{n}{q}, \qquad q\,' = \frac{q}{q-1}.$$

We use letters M, N, C (with or without indices) to denote various constants. To indicate that, say, N depends on some parameters, we list them in the parentheses: $N(\dots)$.

§1. Statement of the problem

We assume that Σ meets $\partial\Omega$ transversally. Namely, we suppose that $\mathcal{M} = \Sigma \cap \partial\Omega$ is an $(n-2)$-dimensional submanifold without boundary satisfying the following condition: for every point $x^0 \in \mathcal{M}$ there exists a neighborhood $U(x^0)$ in $\mathbb{R}^n$ and a diffeomorphism $\Psi_{(x^0)}$ such that

(1) $\quad \Psi_{(x^0)}\,(U(x^0) \cap \Omega) = \Pi_{\rho_0}$
($\rho_0 \leqslant 1$ is a constant independent of x^0),

(2) $\quad \Psi_{(x^0)}(x^0) = 0, \qquad (\Psi'_{(x^0)}(x^0))^{\top} \cdot \Psi'_{(x^0)}(x^0) = I_n,$

(3) $\quad$ the norms of the Jakobi matrices $\Psi'_{(x^0)}(x)$ and $\left(\Psi^{-1}_{(x^0)}\right)'\left(\Psi_{(x^0)}(x)\right)$ are uniformly bounded with respect to $x^0 \in \mathcal{M}$ and $x \in U(x^0)$,

(4) $\quad \Psi_{(x^0)}\,(U(x^0) \cap \partial\Omega) = \{x \in \partial\Pi_{\rho_0} : x_2 = 0\},$

(5) $\quad \Psi_{(x^0)}\,(U(x^0) \cap \Sigma) = \{x \in \Pi_{\rho_0} : \operatorname{arcctg}(x_1/x_2) = 2\vartheta(x^0)\},$

(6) $\quad \max\{\sup_{\mathcal{M}} \vartheta(x^0), \sup_{\mathcal{M}}(\pi/2 - \vartheta(x^0)\} = \theta < \pi/2.$

Let $d(x)$ denote the distance between a point x and the submanifold $\mathcal{M}$, and let $\widetilde{d}(x)$ denote the distance with respect to the intrinsic metric in the manifold Σ between a point $x \in \Sigma$ and the submanifold $\mathcal{M}$. We introduce the following spaces:

$L_{p,(\alpha)}(\Omega)$ is the weight space with the norm $\|\cdot\|_{p,(\alpha),\Omega}$, where

$$\|u\|_{p,(\alpha),\Omega} = \||x'|^\alpha u\|_{p,\Omega};$$

$\mathbb{L}_{p,(\alpha)}(\Omega)$ is the weight space with the norm $\|\!|\cdot|\!\|_{p,(\alpha),\Omega}$, where

$$|\!|\!|u|\!|\!|_{p,(\alpha),\Omega} = \|\,(d(x))^\alpha\, u\|_{p,\Omega};$$

$\mathbb{L}_{p,(\alpha)}(\Sigma)$ is the weight space with the norm $|\!|\!|\cdot|\!|\!|_{p,(\alpha),\Sigma}$, where

$$|\!|\!|u|\!|\!|_{p,(\alpha),\Sigma} = \|(\widetilde{d}(x))^\alpha u\|_{p,\Sigma};$$

$\mathbb{V}^2_{p,(\alpha)}(\Omega)$ is the space with the norm

$$\|u\|_{\mathbb{V}^2_{p,(\alpha)}(\Omega)} = |\!|\!|D^2u|\!|\!|_{p,(\alpha),\Omega} + |\!|\!|\,(d(x))^{-1}\,Du|\!|\!|_{p,(\alpha),\Omega} + |\!|\!|\,(d(x))^{-2}\,u|\!|\!|_{p,(\alpha),\Omega};$$

$\widetilde{\mathbb{V}}^2_{p,(\alpha)}(\Omega)$ is the set of functions vanishing on $\partial\Omega$ with the finite semi-norm $|\!|\!|D^2u|\!|\!|_{p,(\alpha),\Omega}$;

$\widetilde{\mathbb{V}}^2_{p,(\alpha)}(\Sigma)$ is the set of functions vanishing on $\mathcal{M}$ with the finite semi-norm $|\!|\!|\widetilde{D}^2u|\!|\!|_{p,(\alpha),\Sigma}$.

It should be noted that the semi-norms $|\!|\!|D^2u|\!|\!|_{p,(\alpha),\Omega}$ and $|\!|\!|\widetilde{D}^2u|\!|\!|_{p,(\alpha),\Sigma}$ become the norms since u vanishes on $\partial\Omega$ and on $\mathcal{M}$, respectively.

We also introduce the function space

$$\mathcal{V}_{p,(\alpha)}(\Omega,\Sigma) = \left\{u \in \widetilde{\mathbb{V}}^2_{p,(\alpha)}(\Omega_1\cup\Omega_2)\cap\widetilde{\mathbb{V}}^2_{p-1,(\alpha p\,')}(\Sigma)\cap C(\overline{\Omega}) : u|_{\partial\Omega} = 0\right\}$$

with the norm

$$\|u\|_{\mathcal{V}_{p,(\alpha)}(\Omega)} = |\!|\!|D^2u|\!|\!|_{p,(\alpha),\Omega_1} + |\!|\!|D^2u|\!|\!|_{p,(\alpha),\Omega_2} + |\!|\!|\widetilde{D}^2u|\!|\!|_{p-1,(\alpha p\,'),\Sigma}.$$

The notation $\partial\Omega_h \in \widetilde{\mathbb{V}}^2_{q,(\alpha)}$ with $\alpha < \widehat{\alpha}$ and $h = 1, 2$ means that $\partial\Omega_h \setminus \mathcal{M} \in W^2_{q,loc}$ and for all points $x^0 \in \mathcal{M}$ the matrices $D^2\Psi_{(x^0)}$ belong to $\mathbb{L}_{q,(\alpha)}\,(U(x^0)\cap\Omega_h)$. Moreover the norms $|\!|\!|D^2\Psi_{(x^0)}|\!|\!|_{q,(\alpha)}$ are bounded uniformly with respect to $x^0 \in \mathcal{M}$.

We consider solutions of the boundary value problem

$$
\begin{aligned}
& -a_{[1]}^{ij}(x,u,Du)D_iD_ju + a_{[1]}(x,u,Du) = 0 \qquad \text{in} \quad \Omega_1, && (1.1)\\
& -a_{[2]}^{ij}(x,u,Du)D_iD_ju + a_{[2]}(x,u,Du) = 0 \qquad \text{in} \quad \Omega_2, && (1.2)\\
& -a_{[0]}^{ij}(x,u,\widetilde{D}u)\widetilde{D}_i\widetilde{D}_ju + a_{[0]}(x,u,\widetilde{D}u) + Ju = 0 \qquad \text{on} \quad \Sigma, && (1.3)\\
& u|_{\partial\Omega} = 0, && (1.4)
\end{aligned}
$$

where J denotes a jump-operator

$$
\begin{aligned}
Ju \equiv & \beta_{[1]}\left(x,u,\widetilde{D}u + \mathbf{n}(x)\lim_{\varepsilon\to-0}\frac{\partial u}{\partial \mathbf{n}}\left(x+\varepsilon\mathbf{n}(x)\right)\right)\\
& - \beta_{[2]}\left(x,u,\widetilde{D}u + \mathbf{n}(x)\lim_{\varepsilon\to+0}\frac{\partial u}{\partial \mathbf{n}}\left(x+\varepsilon\mathbf{n}(x)\right)\right).
\end{aligned}
$$

Remark. The null boundary condition (1.4) is used for simplicity only. Obviously, it can be replaced by $u|_{\partial\Omega} = \phi$, where ϕ belongs to an appropriate function space dictated by the embedding theorem.

We assume that the matrices $\left(a_{[h]}^{ij}\right)$, $h = 0,1,2$, are symmetric, and the functions $a_{[h]}^{ij}$, $h = 0,1,2$, have the first-order derivatives with respect to all the arguments.
We assume that the functions involved in Eqs. (1.1) and (1.2) satisfy the following structure conditions:

$$
\nu_h|\xi|^2 \leqslant a_{[h]}^{ij}(x,z,p)\xi_i\xi_j \leqslant \nu_h^{-1}|\xi|^2 \quad \forall\xi\in\mathbb{R}^n, \tag{A0}
$$

$$
|a_{[h]}(x,z,p)| \leqslant \mu_h|p|^2 + b_{[h]}(x)|p| + \Phi_1^{[h]}(x), \tag{A1}
$$

$$
|p|\left|\frac{\partial a_{[h]}^{ij}(x,z,p)}{\partial p_s}\right| \leqslant \mu_h \quad \text{for} \quad |p|\geqslant 1, \tag{A2}
$$

$$
\left|\frac{\partial a_{[h]}^{ij}(x,z,p)}{\partial z}p_s + D_s\left(a_{[h]}^{ij}(x,z,p)\right)\right| \leqslant \mu_h|p| + \Phi_2^{[h]}(x), \tag{A3}
$$

$$
b_{[h]}, \Phi_1^{[h]}, \Phi_2^{[h]} \in \mathbb{L}_{q,(\alpha)}(\Omega_h) \tag{A4}
$$

for $h=1,2$ and for any $x\in\Omega_h$, $z\in\mathbb{R}^1$, and $p\in\mathbb{R}^n$, where ν_h and μ_h are some positive constants.
We assume that the functions $a_{[0]}^{ij}$ and $a_{[0]}$ in Eq. (1.3) satisfy the following

structure conditions:

$$\nu_0|\widetilde{\xi}|^2 \leqslant a^{ij}_{[0]}(x,z,\widetilde{p})\widetilde{\xi}_i\widetilde{\xi}_j \leqslant \nu_0^{-1}|\widetilde{\xi}|^2 \quad \forall \xi \in \mathbb{R}^n, \tag{B0}$$

$$|a_{[0]}(x,z,\widetilde{p})| \leqslant \mu_0|\widetilde{p}|^2 + b_{[0]}(x)|\widetilde{p}| + \Phi^{[0]}_1(x), \tag{B1}$$

$$|\widetilde{p}|\left|\frac{\partial a^{ij}_{[0]}(x,z,\widetilde{p})}{\partial \widetilde{p}_s}\right| \leqslant \mu_0 \quad \text{for} \quad |\widetilde{p}| \geqslant 1, \tag{B2}$$

$$\left|\frac{\partial a^{ij}_{[0]}(x,z,\widetilde{p})}{\partial z}\widetilde{p}_s + \widetilde{D}_s\left(a^{ij}_{[0]}(x,z,\widetilde{p})\right)\right| \leqslant \mu_0|\widetilde{p}| + \Phi^{[0]}_2(x), \tag{B3}$$

$$b_{[0]}, \Phi^{[0]}_1, \Phi^{[0]}_2 \in \mathbb{L}_{q-1,(\alpha q\,')}(\Sigma) \tag{B4}$$

for any $x \in \Sigma$, $z \in \mathbb{R}^1$, and $p \in \mathbb{R}^n$, where ν_0 and μ_0 are some positive constants.
Finally, we assume that the coefficients of the jump-operator J satisfy the following structure conditions:

$$0 \leqslant \frac{\partial \beta_{[h]}(x,z,p)}{\partial p_i} \cdot \mathbf{n}_i(x) \leqslant b_{[0]}(x), \tag{J0}$$

$$|\beta_{[h]}(x,z,p)| \leqslant b_{[0]}(x)|p| + \Phi^{[0]}_1(x) \tag{J1}$$

for $h = 1,2$ and for any $x \in \Sigma$, $z \in \mathbb{R}^1$, and $p \in \mathbb{R}^n$.

Remark. Without loss of generality we may assume that $\nu_0 = \nu_1 = \nu_2 = \nu$ and $\mu_0 = \mu_1 = \mu_2 = \mu$.

Let $\Theta = \Theta(\theta,\nu)$ be a solution to the equation

$$\mathrm{ctg}(\Theta) = \nu\ \mathrm{ctg}(\theta), \quad \Theta \in]0,\pi/2[, \tag{1.5}$$

where ν is the ellipticity constant in (A0) and θ is the parameter in (5).
We set

$$\widehat{q} = \max\left\{n, \frac{n-2}{\frac{\pi}{2\Theta}-1}\right\}.$$

Theorem 1. *Let the following conditions hold:*
(i) $\widehat{q} < q < \infty, \quad \max\left\{-\frac{1}{q}, 2-\frac{2}{q}-\frac{\pi}{2\Theta}\right\} < \alpha < \widehat{\alpha}(q)$;
(ii) $\partial\Omega_h \in \widetilde{\mathbb{V}}^2_{q,(\alpha)}$ *for* $h = 1,2$;

(iii) *every solution* $\widehat{u}^{[\tau]} \in \mathcal{V}_{q,(\alpha)}(\Omega,\Sigma)$ *to the problem*

$$
\begin{aligned}
&\tau\left(-a^{ij}_{[1]}(x,u,Du)D_iD_ju + a_{[1]}(x,u,Du)\right) - (1-\tau)\Delta u = 0 \ \mathit{in}\ \Omega_1,\\
&\tau\left(-a^{ij}_{[2]}(x,u,Du)D_iD_ju + a_{[2]}(x,u,Du)\right) - (1-\tau)\Delta u = 0 \ \mathit{in}\ \Omega_2,\\
&\tau\left(-a^{ij}_{[0]}(x,u,\widetilde{D}u)\widetilde{D}_i\widetilde{D}_ju + a_{[0]}(x,u,\widetilde{D}u) + Ju\right) - \\
&\qquad\qquad\qquad\qquad - (1-\tau)\widetilde{\Delta}u = 0 \ \mathit{on}\ \Sigma,\\
&u|_{\partial\Omega} = 0,
\end{aligned}
\tag{1.6}
$$

where $\tau \in [0,1]$, *satisfies the estimate*

$$\|\widehat{u}^{[\tau]}\|_{\Omega} \leqslant M_0;$$

(iv) *if* $|z| \leqslant M_0$, *then conditions* (A0), (A1), (A2), (A3), (A4), (B0), (B1), (B2), (B3), (B4), (J0), *and* (J1) *hold*;
(v) *for* $h=1,2$ *the functions* $a_{[h]}(\cdot,z,p)$ *regarded as elements of the spaces* $\mathbb{L}_{q,(\alpha)}(\Omega_h)$ *as well as the functions* $\beta_{[h]}(\cdot,z,p)$ *regarded as elements of the space* $\mathbb{L}_{q-1,(\alpha q')}(\Sigma)$, *are continuous with respect to* (z,p).
(vi) *the function* $a_{[0]}(\cdot,z,\widetilde{p})$ *regarded as an element of the space* $\mathbb{L}_{q-1,(\alpha q')}(\Sigma)$ *is continuous with respect to* $(z,\widetilde{p})$.
Then the problem (1.6) *has at least one solution* $\widehat{u}^{[\tau]} \in \mathcal{V}_{q,(\alpha)}(\Omega,\Sigma)$ *for each* $\tau\in[0,1]$. *In particular,* $\widehat{u}^{[1]}$ *is a solution of the problem* (1.1)-(1.4).

§2. Linear two-phase Venttsel problems

Lemma 2.1. *Let* $q>n$, $\alpha<\widehat{\alpha}(q)$, *and let* $\partial\Omega_h \in \widetilde{\mathbb{V}}^2_{q,(\alpha)}$ *for* $h=1,2$. *Then there exists an extension operator*

$$\mathcal{P}_h : \widetilde{\mathbb{V}}^2_{q-1,(\alpha q')}(\Sigma) \to \widetilde{\mathbb{V}}^2_{q,(\alpha)}(\Omega_h)$$

such that

$$\|D^2\left(\mathcal{P}_h u\right)\|_{q,(\alpha),\Omega_h} \leqslant C_h \|\widetilde{D}^2 u\|_{q-1,(\alpha q'),\Sigma},$$

where the constant C_h *depends only on* q, α, *and the characteristics of* $\partial\Omega_h$.

Proof. *1.* We start from construction an extension operator for a model case.
Let $u=u(x_2,x'')$ be a function with compact support, let $(D^*)^2u \in L_{q-1,(\alpha q')}(\mathbb{R}^{n-1}_{+,2})$, and let $u(0,x'')=0$. We extend the function u with respect

to x_2 by the odd reflection, i.e., by setting it as $u(x_2, x'') = -u(-x_2, x'')$ for $x_2 < 0$, and define

$$\widetilde{u}(x) = \zeta(x_1) \int_{\mathbb{R}^{n-1}} u(x_2 - x_1 z_2, x'' - x_1 z'') \eta(z_2, z'') dz_2 dz''.$$

Here ζ stands for a smooth cut-off function which equals 1 near the origin, while η denotes the following averaging kernel:

$$\eta \in C_0^\infty\left(\mathbb{R}^{n-1}\right), \quad \eta \geqslant 0, \quad \eta(-z_2, z'') = \eta(z_2, z''),$$

$$\eta(z_2, z'') = 0 \quad \text{for} \quad |z_2| + |z''| \geqslant 1, \qquad \int_{\mathbb{R}^{n-1}} \eta(z_2, z'') dz_2 dz'' = 1.$$

Obviously, $\widetilde{u}$ has a compact support, $\widetilde{u}(x_1, 0, x'') = 0$, and $\widetilde{u}(0, x_2, x'') = u$. It remains to show that

$$\|D^2 \widetilde{u}\|_{q,(\alpha)} \leqslant C \|(D^*)^2 u\|_{q-1,(\alpha q')}. \tag{2.1}$$

Differentiating $\widetilde{u}$ we obtain

$$D_i D_j \widetilde{u}(x) = I_1 + I_2,$$

where

$$I_1 = \zeta(x_1) \int_{\mathbb{R}^{n-1}} D_k D_m u(x_2 - x_1 z_2, x'' - x_1 z'') \eta(z_2, z'') \tau_{ijkm}(z_2, z'') dz_2 dz'',$$

$$\tau_{ijkm}(z_2, z'') = \delta^{ik}\delta^{jm} - \delta^{j1} z_m \delta^{ik} - \delta^{i1} z_m \delta^{jk} + \delta^{i1}\delta^{j1} z_k z_m,$$

and I_2 stands for the terms containing derivatives of ζ.
Note that the terms in I_2 contain only the first derivatives of u and the function u itself. Since I_2 vanishes near the points $x \in \mathbb{R}^n$ with $x' = 0$, we can estimate the norm of I_2 in $\mathbb{L}_{q,(\alpha)}$ by $\|D^* u\|$. Moreover, from the assumptions on q and α it follows that $\|D^* u\|$ majorizes by $\|(D^*)^2 u\|_{q-1,(\alpha q')}$. Thus, to prove (2.1) it suffices to estimate the norm of I_1 in the weight space $\mathbb{L}_{q,(\alpha)}$.
By the Hölder inequality we have

$$\int_{\mathbb{R}^n_+} |x'|^{q\alpha} |I_1|^q dx \leqslant$$

$$\int_{\mathbb{R}^1_+} \zeta^q(x_1) \left[\int_{\mathbb{R}^{n-1}} |x'|^{q\alpha} \left(\mathcal{I}_1(x)\right)^{\frac{q}{q-1}} \left(\mathcal{I}_2(x)\right)^{\frac{q(q-2)}{q-1}} dx_2 dx'' \right] dx_1, \tag{2.2}$$

where

$$\mathcal{I}_1(x) = \int_{B_1^{n-1}} |x_2 - x_1 z_2|^{\alpha q} |(D^*)^2 u(x_2 - x_1 z_2, x'' - x_1 z'')|^{q-1} dz_2 dz'',$$

$$\mathcal{I}_2(x) = \int_{B_1^{n-1}} \left(\eta(z_2, z'') |\tau_{ijkm}(z)|\right)^{\frac{q-1}{q-2}} dz'' \frac{dz_2}{|x_2 - x_1 z_2|^{\frac{\alpha q}{q-2}}}.$$

It is easily seen that

$$|\mathcal{I}_2(x)| \leqslant N_1 \int_{-1}^{1} \frac{dz_2}{|x_2 - x_1 z_2|^{\frac{\alpha q}{q-2}}}, \tag{2.3}$$

where N_1 is an absolute constant. Since $\frac{q-n}{q-2} \leqslant 1$ the integral in the right-hand side of (2.3) converges, and the homogeneity reasons give

$$|\mathcal{I}_2(x)| \leqslant N_2(q,\alpha)\, |x'|^{-\frac{\alpha q}{q-2}}. \tag{2.4}$$

Substituting (2.4) into (2.2) and changing the variables we get

$$\int_{\mathbb{R}^n_+} |x'|^{q\alpha} |I_1|^q dx \leqslant N_3 \int_{\mathbb{R}^1_+} \left[\int_{\mathbb{R}^{n-1}} \left(\widetilde{\mathcal{I}}_1\right)^{\frac{q}{q-1}} dx_2 dx'' \right] \frac{\zeta^q(x_1) dx_1}{(x_1)^{\frac{q(n-1+\alpha)}{q-1}}}, \tag{2.5}$$

where N_3 depends on q and α, and

$$\widetilde{\mathcal{I}}_1 = \widetilde{\mathcal{I}}_1(x_2, x'') = \int_{B^{n-1}_{x_1}} |x_2 - y_2|^{\alpha q} |(D^*)^2 u(x_2 - y_2, x'' - y'')|^{q-1} dy_2 dy''.$$

By Minkowskii's inequality we can estimate the integral in the square braces by $N_4(q,\alpha)\|D^2 u\|^q_{q-1,(\alpha q')} \left|B^{n-1}_{x_1}\right|$. Hence (2.5) takes the form

$$\begin{aligned} \int_{\mathbb{R}^n_+} |x'|^{q\alpha} |I_1|^q dx &\leqslant N_5(q,\alpha) \|(D^*)^2 u\|^q_{q-1,(\alpha q')} \int_{\mathbb{R}^1_+} \frac{\zeta^q(x_1) dx_1}{(x_1)^{\frac{n-1+\alpha q}{q-1}}} \\ &\leqslant C \|(D^*)^2 u\|^q_{q-1,(\alpha q')}. \end{aligned} \tag{2.6}$$

We note that $\frac{n-1+\alpha q}{q-1} < 1$, since $\alpha < \widehat{\alpha}(q)$. Consequently, the second integral in (2.6) converges and guarantees the second inequality in (2.6).
Now, combined with the above remark on majorant of the norm I_2, the estimate (2.6) yields (2.1).
One can see from the above argument that the constant C in (2.1) depends only on q, α, and the diameters of $supp(u)$ and $supp(\widetilde{u})$.
We note also that for $\alpha = 0$ the operator $\mathcal{P}$ acts in the standard (non-weight) Sobolev spaces. In this case the existence of $\mathcal{P}$ can be deduced from the embedding theorems and the extension theorems (see Ch. 17-18, [7]):

$$W^2_{q-1}(\mathbb{R}^{n-1}) \to \mathcal{B}^2_{q-1,q-1}(\mathbb{R}^{n-1}) \to \mathcal{B}^{2+\frac{1}{q-1}}_{q-1,q-1}(\mathbb{R}^n_+) \to W^2_q(\mathbb{R}^n_+),$$

(here the notation of the Besov space $\mathcal{B}$ corresponds to the monograph [7]).

2. The condition $\partial\Omega_h \in \widetilde{\mathbb{V}}^2_{q,(\alpha)}$ implies that Σ can be covered by a finite sum $\bigcup U_l$ of open sets $U_l \subset \mathbb{R}^n$ such that

(a) the sets $\Omega_h \cap U_l$ are diffeomorphic to $B_1^n \cap \mathbb{R}^n_{+,+}$ if $U_l \cap \mathcal{M} \neq \emptyset$; otherwise the diffeomorphic images of $\Omega_h \cap U_l$ coincide with $B_1^n \cap \mathbb{R}^n_+$;
(b) all the diffeomorphisms have generalized derivatives of the second order, and the norms of these derivatives are bounded in $\mathbb{L}_{q,(\alpha)}$.
Therefore, the "transplantation" operators generated by above-mentioned diffeomorphisms are bounded operators from $\widetilde{\mathbb{V}}^2_{q-1,(\alpha q')}(\Sigma \cap U_l)$ to $\widetilde{\mathbb{V}}^2_{q-1,(\alpha q')}(B_1^{n-1} \cap \mathbb{R}^{n-1}_+)$ if $U_l \cap \mathcal{M} \neq \emptyset$, and from $W^2_{q-1}(\Sigma \cap U_l)$ to $W^2_{q-1}(B_1^{n-1})$ if otherwise.
Using an appropriate partition of unity, now we can glue the desired operator $\mathcal{P}_h$ from local operators constructed in the first part of the proof. $\square$

Throughout this section we suppose that $\mathcal{L}_h$, $h = 1, 2$, are the uniformly elliptic linear operators

$$\mathcal{L}_h u \equiv -a^{ij}_{[h]}(x) D_i D_j u + b^i_{[h]}(x) D_i u, \qquad x \in \Omega_h,$$
$$a^{ij}_{[h]} = a^{ji}_{[h]},$$
$$\nu|\xi|^2 \leqslant a^{ij}_{[h]}\xi_i\xi_j \leqslant \nu^{-1}|\xi|^2 \quad \text{for} \quad \xi \in \mathbb{R}^n, \quad \nu = \text{const} > 0.$$

Suppose also that $\mathcal{L}_{[0]}$ is a uniformly elliptic linear interface operator

$$\mathcal{L}_0 u \equiv -a^{ij}_{[0]}(x) \widetilde{D}_i \widetilde{D}_j u + b^i_{[0]}(x) \widetilde{D}_i u, \qquad x \in \Sigma,$$
$$a^{ij}_{[0]} = a^{ji}_{[0]},$$
$$\nu|\widetilde{\xi}|^2 \leqslant a^{ij}_{[0]}\widetilde{\xi}_i\widetilde{\xi}_j \leqslant \nu^{-1}|\widetilde{\xi}|^2 \quad \text{for} \quad \xi \in \mathbb{R}^n.$$

Suppose, finally, that $\mathcal{J}$ is a linear jump-operator

$$\mathcal{J} u \equiv \beta_{[1]}(x) \lim_{\varepsilon \to -0} \frac{\partial u}{\partial \mathbf{n}}(x + \varepsilon \mathbf{n}(x)) - \beta_{[2]}(x) \lim_{\varepsilon \to +0} \frac{\partial u}{\partial \mathbf{n}}(x + \varepsilon \mathbf{n}(x)),$$

where $\beta_{[1]}(x) \geqslant 0$ and $\beta_{[2]}(x) \geqslant 0$ for $x \in \Sigma$.
We introduce the notation $\mathbf{b}_{[h]}(x) = \left(b^i_{[h]}(x)\right)$, $h = 0, 1, 2$.

Theorem 2.2. *Let $\widehat{q}$ and Θ be the same quantities as in Sec.*1. *Let $\widehat{q} < q < \infty$, and let* $\max\left\{-\frac{1}{q}, 2 - \frac{2}{q} - \frac{\pi}{2\Theta}\right\} < \alpha < \widehat{\alpha}(q)$. *Suppose that $\partial\Omega_1$ and $\partial\Omega_2$ belong to $\widetilde{\mathbb{V}}_{q,(\alpha)}$.*
Suppose also that $u \in \mathcal{V}_{q,(\alpha)}(\Omega, \Sigma)$ is a function such that

$$\mathcal{L}_{[1]} u = f_{[1]}(x) \quad \textit{in} \quad \Omega_1, \tag{2.7}$$
$$\mathcal{L}_{[2]} u = f_{[2]}(x) \quad \textit{in} \quad \Omega_2, \tag{2.8}$$
$$\mathcal{L}_{[0]} u + \mathcal{J} u = f_{[0]}(x) \quad \textit{on} \quad \Sigma, \tag{2.9}$$
$$u|_{\partial\Omega} = 0. \tag{2.10}$$

If, in addition,

$$a^{ij}_{[h]} \in C(\overline{\Omega}_h), \quad f_{[h]}, |\mathbf{b}_{[h]}| \in \mathbb{L}_{q,(\alpha)}(\Omega_h), \quad h = 1, 2,$$
$$a^{ij}_{[0]} \in C(\overline{\Sigma}), \quad f_{[0]}, |\mathbf{b}_{[0]}|, \beta_{[1]}, \beta_{[2]} \in \mathbb{L}_{q-1,(\alpha q')}(\Sigma),$$

then

$$\|u\|_{\mathcal{V}_{q,(\alpha)}(\Omega,\Sigma)} \leqslant C_3\{\|f_{[1]}\|_{q,(\alpha),\Omega_1} + \|f_{[2]}\|_{q,(\alpha),\Omega_2} + \|f_{[0]}\|_{q-1,(\alpha q'),\Sigma} + \|u\|_{q,(\alpha),\Omega} + \|u\|_{q-1,(\alpha q'),\Sigma}\}, \tag{2.11}$$

where C_3 depends only on n, ν, q, α, $\operatorname{diam}\Omega$, the characteristics of $\partial\Omega_h$, the numbers $\|b^i_{[h]}\|_{q,(\alpha),\Omega_h}$, $\|\beta_{[h]}\|_{q-1,(\alpha q'),\Sigma}$ for $h = 1, 2$ and $\|b^i_{[0]}\|_{q-1,(\alpha q'),\Sigma}$, and on the moduli of continuity of the coefficients $a^{ij}_{[h]}(x)$ for $h = 0, 1, 2$.

Proof. Taking into account the condition $u|_{\partial\Sigma} = 0$ which follows from (2.10) and regarding (2.9) as a self-governing equation on the hypersurface Σ, we can consider the following Dirichlet problem:

$$\begin{aligned} &\mathcal{L}_{[0]}u = f_{[0]} - \mathcal{J}u \quad \text{on} \quad \Sigma, \\ &u|_{\partial\Sigma} = 0. \end{aligned} \tag{2.12}$$

By the choice of α, we have $\alpha q' \in]-\frac{1}{q-1}, 2 - \frac{1}{q-1}[$. Therefore, the inequality (35) from [6] (with the natural changes $n \to n-1$, $p \to q-1$, and $\alpha \to \alpha q'$) is applicable to a solution of (2.12). It should be noted that in [6] we studied equation in a domain. However, all the arguments from [6], which prove (35), remain valid, if we replace a domain by the manifold Σ. As a result we have the estimate

$$\|\widetilde{D}^2 u\|_{q-1,(\alpha q'),\Sigma} \leqslant N_6 \left(\|f_{[0]} - \mathcal{J}u\|_{q-1,(\alpha q'),\Sigma} + \|u\|_{q-1,(\alpha q'),\Sigma}\right), \tag{2.13}$$

where N_6 is determined by n, ν, q, α, $\|b^i_{[0]}\|_{q-1,(\alpha q'),\Sigma}$, as well as by the moduli of continuity of the coefficients $a^{ij}_{[0]}(x)$ and the properties of Σ.
According to our definition of the linear jump-operator $\mathcal{J}$, the inequality (2.13) takes the form

$$\|\widetilde{D}^2 u\|_{q-1,(\alpha q'),\Sigma} \leqslant N_6(\|f_{[0]}\|_{q-1,(\alpha q'),\Sigma} + \|u\|_{q-1,(\alpha q'),\Sigma} + \|\beta_{[1]}\|_{q-1,(\alpha q'),\Sigma}\|Du\|_{\Omega_1} + \|\beta_{[2]}\|_{q-1,(\alpha q'),\Sigma}\|Du\|_{\Omega_2}). \tag{2.14}$$

Using Lemma 2.1 we extend the function $u|_\Sigma$ to Ω_1 and Ω_2, respectively, so as to satisfy

$$\|D^2\overline{u}_h\|_{q,(\alpha),\Omega_h} \leqslant C_h\|\widetilde{D}^2 u\|_{q-1,(\alpha q'),\Sigma}, \tag{2.15}$$

where $h = 1, 2$ and $\overline{u}_h$ denote the corresponding extended functions, whereas C_h are the constants from Lemma 2.1.
From (2.7), (2.8), and (2.10) it follows that for $h = 1, 2$ the functions $v_h = u - \overline{u}_h$ are the solutions of the boundary value problems

$$\begin{aligned} &\mathcal{L}_{[h]} v_h = f_{[h]} - \mathcal{L}_{[h]} \overline{u}_h \quad \text{in} \quad \Omega_h, \\ &v_h\big|_{\partial\Omega_h} = 0. \end{aligned}$$

We note that the dihedral angles lying in Ω_h, $h = 1, 2$ and touching $\partial\Omega_h$ at the point $x^0 \in \mathcal{M}$ have the openings $2\vartheta(x^0)$ and $\pi - 2\vartheta(x^0)$, respectively. The linear transformations reducing the operators $\mathcal{L}_{[h]}$ to the canonical form at the point x^0 transform these dihedral angles. However, the conditions (6), (A0) together with the relation (1.5) guarantee that both the openings of the "new" dihedral angles do not exceed Θ. In addition, by the choice of α, we have $\alpha \in]2 - \frac{2}{q} - \frac{\pi}{2\Theta}, 2 - \frac{2}{q} + \frac{\pi}{2\Theta}[$.
Hence we can apply the results from the linear elliptic theory in nonsmooth domains (see [8]) which give us for $h = 1, 2$ the inequalities

$$\begin{aligned} |\!|\!| D^2 v_h |\!|\!|_{q,(\alpha),\Omega_h} \leqslant \|v_h\|_{\mathbb{V}^2_{q,(\alpha),\Omega_h}} &\leqslant N_{7,[h]} \big(|\!|\!| f_{[h]} |\!|\!|_{q,(\alpha),\Omega_h} \\ &+ |\!|\!| D^2 \overline{u}_h |\!|\!|_{q,(\alpha),\Omega_h} + |\!|\!| \mathbf{b}_{[h]} |\!|\!|_{q,(\alpha),\Omega_h} \|D\overline{u}_h\|_{\Omega_h} \big), \end{aligned} \tag{2.16}$$

with $N_{7,[h]}$ depending on n, ν, q, α, $|\!|\!| b^i_{[h]} |\!|\!|_{q,(\alpha),\Omega_h}$, the characteristics of $\partial\Omega_h$, and on the moduli of continuity of $a^{ij}_{[h]}(x)$.

By the choice of α, we have for $\gamma = \widehat{\alpha}(q) - \alpha_+$ the embeddings of $\widetilde{\mathbb{V}}^2_{q,(\alpha)}(\Omega_h)$ into $C^{1+\gamma}(\overline{\Omega}_h)$. Therefore, for $h = 1, 2$ we get

$$|\!|\!| \mathbf{b}_{[h]} |\!|\!|_{q,(\alpha),\Omega_h} \|D\overline{u}_h\|_{\Omega_h} \leqslant N_{8,[h]} |\!|\!| D^2 \overline{u}_h |\!|\!|_{q,(\alpha),\Omega_h}, \tag{2.17}$$

where $N_{8,[h]}$ depends on n, q, α, $\operatorname{diam}\Omega$, the numbers $|\!|\!| b^i_{[h]} |\!|\!|_{q,(\alpha),\Omega_h}$, and on the characteristics of $\partial\Omega_h$. Moreover, for an arbitrary $\varepsilon > 0$ and $h = 1, 2$ we make use the well-known interpolation inequalities

$$\|Du\|_{\Omega_h} \leqslant \varepsilon |\!|\!| D^2 u |\!|\!|_{q,(\alpha),\Omega_h} + N_{9,[h]}(\varepsilon, n, q, \alpha, \operatorname{diam}\Omega, \partial\Omega_h) |\!|\!| u |\!|\!|_{q,(\alpha),\Omega_h}. \tag{2.18}$$

Combining (2.14), (2.15), (2.16), and (2.17) we get

$$\begin{aligned} \|u\|_{\mathcal{V}_{q,(\alpha)}(\Omega,\Sigma)} \leqslant\; & N_{10} \big(|\!|\!| \beta_{[1]} |\!|\!|_{q-1,(\alpha q'),\Sigma} \|Du\|_{\Omega_1} + |\!|\!| \beta_{[2]} |\!|\!|_{q-1,(\alpha q'),\Sigma} \|Du\|_{\Omega_2} \big) \\ & + N_{11} \big\{ |\!|\!| f_{[1]} |\!|\!|_{q,(\alpha),\Omega_1} + |\!|\!| f_{[2]} |\!|\!|_{q,(\alpha),\Omega_2} \\ & + |\!|\!| f_{[0]} |\!|\!|_{q-1,(\alpha q'),\Sigma} + |\!|\!| u |\!|\!|_{q-1,(\alpha q'),\Sigma} \big\}, \end{aligned} \tag{2.19}$$

with $N_{10} = N_6 \left(1 + C_1 + C_2 + \sum_{h=1}^{2} \left[C_h N_{7,[h]}(1 + N_{8,[h]})\right]\right)$ and $N_{11} = N_{7,[1]} + N_{7,[2]} + N_{10}$.
Let us take $\varepsilon = \{2N_{10} \left(\|\!|\beta_{[1]}|\!\|_{q-1,(\alpha q'),\Sigma} + \|\!|\beta_{[2]}|\!\|_{q-1,(\alpha q'),\Sigma} \right)\}^{-1}$. Then substituting (2.18) into (2.19) we arrive at (2.11). $\square$

Theorem 2.3. *Let all the assumptions of Theorem* 2.2 *be valid, and let the boundary value problem*

$$\begin{aligned} &\mathcal{L}_{[1]}u = 0 \quad in \quad \Omega_1, \\ &\mathcal{L}_{[2]}u = 0 \quad in \quad \Omega_2, \\ &\mathcal{L}_{[0]}u + \mathcal{J}u = 0 \quad on \quad \Sigma, \\ &u|_{\partial\Omega} = 0 \end{aligned} \tag{2.20}$$

have only the trivial solution. Then the boundary value problem (2.7)-(2.10) *has a unique solution* $u \in \mathcal{V}_{q,(\alpha)}(\Omega, \Sigma)$.

Proof. Consider the problem

$$\begin{aligned} &\widehat{\mathcal{L}}_{[0]}u \equiv \widetilde{D}_i \left(\widehat{a}^{ij}_{[0]} \widetilde{D}_j u \right) = f_{[0]} \quad \text{on} \quad \Sigma, \\ &u|_{\partial\Sigma} = 0, \end{aligned} \tag{2.21}$$

where the coefficients $\widehat{a}^{ij}_{[0]}$ satisfy the following conditions:

$$\widetilde{D}_i(\widehat{a}^{ij}_{[0]}) \in \mathbb{L}_{q-1,(\alpha q')}(\Sigma), \quad a^{ij}_{[0]} = a^{ji}_{[0]}, \quad \nu|\widetilde{\xi}|^2 \leqslant a^{ij}_{[0]}\widetilde{\xi}_i\widetilde{\xi}_j \leqslant \nu^{-1}|\widetilde{\xi}|^2 \quad \forall \xi \in \mathbb{R}^n,$$

and $\widehat{a}^{ij}_{[0]}$ take the form δ^{ij} after the straightening the boundary $\partial\Sigma$ in a neighborhood of an arbitrary point $x^0 \in \mathcal{M}$.
For any C_0^∞-function $f_{[0]}$ the problem (2.21) has a generalized solution. Moreover, if we straighten $\partial\Sigma$ in a neighborhood of an arbitrary point $x^0 \in \mathcal{M}$ then this solution is a smooth function of "new" local variables. Therefore, it is an easy matter to see that this solution belongs to $\widetilde{\mathbb{V}}^2_{q-1,(\alpha q')}(\Sigma)$.
So, the inverse operator $\widehat{\mathcal{L}}^{-1}_{[0]}$ is defined on everywhere dense set in the weight space $\mathbb{L}_{q-1,(\alpha q')}(\Sigma)$. We note that the estimate (2.13) (with $Ju \equiv 0$) holds, if we replace $\mathcal{L}_{[0]}$ by $\widehat{\mathcal{L}}_{[0]}$, and guarantees that $\widehat{\mathcal{L}}^{-1}_{[0]}$ can be continuously extended on the whole space $\mathbb{L}_{q-1,(\alpha q')}(\Sigma)$.
We denote by $\widehat{u}$ a solution of the problem (2.21). Using Lemma 2.1 we extend the function $\widehat{u}$ to Ω_1 and Ω_2, respectively, and denote by $\overline{u}_h$, $h = 1, 2$ the corresponding extended function.

From the linear elliptic theory in nonsmooth domains (see [8]) it follows that for $h = 1, 2$ each of the boundary value problems

$$\begin{aligned} &\mathcal{L}_{[h]}v = f_{[h]} - \mathcal{L}_{[h]}\overline{u}_h \quad \text{in} \quad \Omega_h, \\ &v\big|_{\partial\Omega_h} = 0 \end{aligned}$$

has a unique solution $v_h \in \widetilde{\mathbb{V}}^2_{q,(\alpha)}(\Omega_h)$.
Therefore, the function

$$u(x) = \begin{cases} v_1 + \overline{u}_1, & x \in \Omega_1, \\ v_2 + \overline{u}_2, & x \in \Omega_2 \end{cases}$$

belongs to $\mathcal{V}_{q,(\alpha)}(\Omega, \Sigma)$, and is a unique solution of the problem

$$\begin{aligned} &\mathcal{L}_{[1]}u = f_{[1]} \quad \text{in} \quad \Omega_1, \\ &\mathcal{L}_{[2]}u = f_{[2]} \quad \text{in} \quad \Omega_2, \\ &\widehat{\mathcal{L}}_{[0]}u = f_{[0]} \quad \text{on} \quad \Sigma, \\ &u\big|_{\partial\Omega} = 0. \end{aligned} \tag{2.22}$$

Finally, using the estimate (2.11) and the method of extending by continuity with respect to the parameter we deduce the unique solvability of the problem (2.7)-(2.10) from the unique solvability of the problem (2.22). This finishes the proof. □

Theorem 2.4. *Let* q, α, $\partial\Omega_1$ and $\partial\Omega_2$ *satisfy the assumptions of Theorem* 2.2. *If, in addition,*

$$\begin{aligned} a^{ij}_{[0]} \in C(\overline{\Sigma}), \quad a^{ij}_h \in C(\overline{\Omega}_h), \\ b_{[0]}, \beta_{[h]} \in L_{\infty,\mathrm{loc}}(\Sigma), \quad b_{[h]} \in L_{\infty,\mathrm{loc}}(\Omega_h) \quad h = 1, 2, \end{aligned} \tag{2.23}$$

then the boundary value problem (2.20) *has only the trivial solution.*

Proof. Let u be a nontrivial solution of the boundary value problem (2.20). In view of (2.11), we have $u \in \mathcal{V}_{q,(\alpha)}(\Omega, \Sigma)$ and for $h = 1, 2$, in view of the embedding theorem, we have $Du \in C(\overline{\Omega}_h)$.
Let u attain its maximum at some point x^0. Then $x^0 \in \Sigma$. From the application of the Hopf lemma (see Lemma 3.1 [9]) to Ω_1 and Ω_2, it follows that

$$\lim_{\varepsilon\to-0}\frac{\partial u}{\partial \mathbf{n}}(x^0 + \varepsilon\mathbf{n}(x^0)) > 0, \qquad \lim_{\varepsilon\to+0}\frac{\partial u}{\partial \mathbf{n}}(x^0 + \varepsilon\mathbf{n}(x^0)) < 0.$$

Since $Du \in C(\overline{\Sigma})$ the inequality $\mathcal{J}u \geqslant 0$ is true in a neighborhood of the point x^0. Hence $\mathcal{L}_{[0]}u \leqslant 0$ in the neighborhoods of those points of Σ, where

u attains its maximum. But the last inequality contradicts the Hopf lemma on Σ. This finishes the proof. $\square$

Remark 2.1. It should be noted that Lemma 3.1 in [9] was actually proved for $u \in C^2(\overline{\Omega})$. The case $u \in W^2_{n,\mathrm{loc}}(\Omega)$ can be treated similarly; the only difference is that we must use the Aleksandrov maximum principle instead of the Hopf maximum principle.

Remark 2.2. If $\Omega_2 = \Omega \cap \mathbb{R}^n_+$ then we can weaken the assumptions of Theorem 2.4. Namely, instead of the condition (2.23) we can assume

$$b_{[0]}, \beta_{[h]} \in L_{n,\,\mathrm{loc}}(\Sigma), \quad b_{[h]} \in L_{n,\,\mathrm{loc}}(\Omega_h) \quad h = 1, 2. \tag{2.24}$$

This fact follows from the Aleksandrov type maximum principle established in [2].
Obviously, if there exists a diffemorphism $\widetilde{\Psi} : \Omega \to \widetilde{\Omega}$ such that $\widetilde{\Psi}(\Omega_2) = \widetilde{\Omega} \cap \mathbb{R}^n_+$ then Theorem 2.4 is valid. By this reason, it looks like that the condition (2.24) is sufficient in the general case as well. However, we do not know the proof of the Hopf lemma for operators with unbounded coefficients of lower-order derivatives. So, in the general case the question on the possibility of replacing (2.23) by (2.24) is still open.

§3. The solvability of the quasilinear problems

Theorem 3.1. *Let $n < q < \infty$, and let $\delta = \widehat{\alpha}(q) - \alpha_+ > 0$. Suppose that $\partial\Omega_1$ and $\partial\Omega_2$ belong to the space $\widetilde{\mathbb{V}}_{q,(\alpha)}$.*
Suppose also that a function $u \in \mathcal{V}_{q,(\alpha)}(\Omega, \Sigma)$ is a solution to the problem (1.1)-(1.4), $\|u\|_\Omega \leqslant M_0$, *and the conditions* (A0), (A1), (A2), (A3), (A4), (B0), (B1), (B2), (B3), (B4), (J0), *and* (J1) *hold for* $|z| \leqslant M_0$.
Then u satisfies the estimate

$$\sup_{x\in\Sigma} \left| \lim_{\varepsilon\to\pm 0} \frac{\partial u}{\partial \mathbf{n}}(x + \varepsilon \mathbf{n}(x)) \right| \leqslant C_4, \tag{3.1}$$

where the constant C_4 depends on n, ν, μ, q, δ, M_0, the characteristics of $\partial\Omega_h$, the numbers $\|b_{[h]}\|_{q,(\alpha),\Omega_h}$, $\|\Phi_1^{[h]}\|_{q,(\alpha),\Omega_h}$, $\|\Phi_2^{[h]}\|_{q,(\alpha),\Omega_h}$ for $h = 1, 2$, as well as on $\|b_{[0]}\|_{q-1,(\alpha q'),\Sigma}$, $\|\Phi_1^{[0]}\|_{q-1,(\alpha q'),\Sigma}$, and $\|\Phi_2^{[0]}\|_{q-1,(\alpha q'),\Sigma}$.

Remark 3.1. Let us note that if we straighten the hypersurface Σ in a neighborhood of an arbitrary point $x \in \Sigma$ then all the structure conditions listed in the statement of the theorem remain valid. The "new" constants appearing in these conditions are determined only by the "old" constants and the characteristics of $\partial\Omega_h$. Therefore we will keep the notation after the straightening Σ.

Remark 3.2. Suppose that we apply the diffeomorphism $\Psi_{(x^0)}$, defined in (1)-(6), to a neighborhood $U(x^0)$ of a point $x^0 \in \mathcal{M}$. By the choice of ρ_0, we may assume without loss of generality that the norms of the Jakobi matrices $\Psi'_{(x^0)}(x)$ and $\left(\Psi^{-1}_{(x^0)}\right)' \left(\Psi_{(x^0)}(x)\right)$ are not greater than 2.
Next, we apply to the set $\Psi_{(x^0)}\left(U(x^0) \cap \Omega\right)$ a linear mapping Υ which keeps the coordinates x_2 and x'' unchanged and transforms the hyperplane $\{\operatorname{arcctg}(x_1/x_2) = 2\vartheta(x^0)\}$ into the hyperplane $\{x_1 = 0\}$. It is evident that the norm of the mapping Υ is bounded by a constant depending only on θ. Moreover, there exists a positive constant $R_1 \leqslant 1$ depending only on ν, δ, the number $\|b_{[0]}\|_{q-1,(\alpha q'),\Sigma}$, and on the characteristics of $\mathcal{M}$, such that after the transform $\Upsilon \circ \Psi_{(x^0)}$ the inequality

$$\|b_{[0]}\|_{q-1,(\alpha q'),\Gamma(\Pi_{\rho_0 R_1})} \leqslant \sigma_1 \tag{3.2}$$

hold independently of the particular point x^0. Here ρ_0 is the radius introduced in (1), while $\sigma_1 = \sigma_1(n-1,\nu,1/2,\delta q')$ is the constant from Lemma 6 [6].

Proof. Let $\rho < \min\{1,\rho_0\}$ be an arbitrary positive number, and let $R \leqslant R_1$ be a positive constant completely determined by the known parameters listed in the statement of the theorem. The precise value of R will be specified later. For $x \in \Sigma$ we define

$$M_1^{\pm}(x) = \left|\lim_{\varepsilon\to\pm 0}\frac{\partial u}{\partial \mathbf{n}}(x+\varepsilon\mathbf{n}(x))\right|, \qquad M_1(x) = \max\left\{M_1^+(x), M_1^-(x)\right\}.$$

Next, we choose $x^* \in \Sigma$ such that $M_1(x)$ attains its supremum over Σ at this point. For the definiteness we assume that

$$M_1 := \sup_{x\in\Sigma} M_1(x) = M_1(x^*) = M_1^-(x^*). \tag{3.3}$$

Now the proof is divided into two cases.

<u>CASE 1. $\quad \widetilde{d}(x^*) \leqslant \frac{\rho R}{288}$.</u>

In this case we consider a point $y^* \in \mathcal{M}$ which is the nearest point to x^* in the intrinsic metric in Σ. We apply the mapping $\Upsilon \circ \Psi_{(y^*)}$ defined in

Remark 3.2 to a neighborhood of y^*. It is obvious that

$$M_1 \leqslant N_{12} \left| \lim_{\varepsilon \to -0} D_1 u(\varepsilon, \varkappa\rho, 0, \dots, 0) \right|, \tag{3.4}$$

where

$$(0, \varkappa\rho, 0, \dots, 0) = \Upsilon \circ \Psi_{(y^*)}(x^*), \qquad \varkappa \in \left]0, \frac{R}{144}\right[,$$

and N_{12} depends only on θ.

Further, our arguments are based on the reasoning of Theorem 1 [10]. We consider the cylinder $\Pi_{\rho R}$. By change of the scale we transform $\Pi_{\rho R}$ into Π_R. In the "dilatated" coordinates the problem (1.1)-(1.3) takes the form

$$-\widetilde{a}^{ij}_{[1]}(x, u_\rho, Du_\rho) D_i D_j u_\rho + \widetilde{a}_{[1]}(x, u_\rho, Du_\rho) = 0 \quad \text{in } \Pi_R \cap \{x_1 < 0\}, \tag{3.5}$$

$$-\widetilde{a}^{ij}_{[2]}(x, u_\rho, Du_\rho) D_i D_j u_\rho + \widetilde{a}_{[2]}(x, u_\rho, Du_\rho) = 0 \quad \text{in } \Pi_R \cap \{x_1 > 0\}, \tag{3.6}$$

$$-\widetilde{a}^{km}_{[0]}(x, u_\rho, D^* u_\rho) D_k D_m u_\rho + \widetilde{a}_{[0]}(x, u_\rho, D^* u_\rho) + \widetilde{J} u_\rho = 0 \text{ on } \Gamma(\Pi_R), \tag{3.7}$$

where

$$u_\rho(x) = u(\rho x),$$
$$\widetilde{a}^{ij}_{[h]}(x, z, p) = a^{ij}_{[h]}(\rho x, z, p/\rho),\ \widetilde{a}_{[h]}(x, z, p) = \rho^2 a_{[h]}(\rho x, z, p/\rho) \text{ for } h = 0, 1, 2,$$
$$\widetilde{J} u_\rho = \widetilde{\beta}_{[1]}(x, u_\rho, D^* u_\rho + \lim_{x_1 \to -0} D_1 u_\rho) - \widetilde{\beta}_{[2]}(x, u_\rho, D^* u_\rho + \lim_{x_1 \to +0} D_1 u_\rho),$$
$$\widetilde{\beta}_{[h]}(x, z, p) = \rho^2 \beta_{[h]}(\rho x, z, p/\rho) \quad \text{for } h = 1, 2.$$

It is easy to see that the constants ν and μ, appearing in the structure conditions, remain without changes under dilatation, whereas the functions appearing on the right-hand sides of (A1), (A3), (B1), (B3), (J0) and (J1) become as follows:

$$\begin{gathered} \widetilde{b}_{[h]}(x) = \rho b_{[h]}(\rho x), \quad \widetilde{\Phi}^{[h]}_1(x) = \rho^2 \Phi^{[h]}_1(\rho x), \\ \widetilde{\Phi}^{[h]}_2(x) = \rho \Phi^{[h]}_2(\rho x), \quad h = 0, 1, 2. \end{gathered} \tag{3.8}$$

Moreover, it follows that

$$\|\widetilde{b}_{[0]}\|_{q-1,(\alpha q'),\Gamma(\Pi_R)} \leqslant \rho^{\delta q'} \|b_{[0]}\|_{q-1,(\alpha q'),\Gamma(\Pi_{\rho_0 R_1})}. \tag{3.9}$$

Similarly, for $h = 1, 2$ and $l = 1, 2$ we have

$$\begin{gathered} \|\widetilde{b}_{[h]}\|_{q,(\alpha),\Pi_R \cap \{x_1 < 0 (>0)\}} \leqslant \|b_{[h]}\|_{q,(\alpha),\Pi_{\rho_0 R_1} \cap \{x_1 < 0 (>0)\}}, \\ \|\widetilde{\Phi}^{[h]}_l\|_{q,(\alpha),\Pi_R \cap \{x_1 < 0 (>0)\}} \leqslant \|\Phi^{[h]}_l\|_{q,(\alpha),\Pi_{\rho_0 R_1} \cap \{x_1 < 0 (>0)\}}, \\ \|\widetilde{\Phi}^{[0]}_l\|_{q-1,(\alpha q'),\Gamma(\Pi_R)} \leqslant \|\Phi^{[0]}_l\|_{q-1,(\alpha q'),\Gamma(\Pi_{\rho_0 R_1})}. \end{gathered} \tag{3.10}$$

We start with considering the interface equation (3.7). It should be noted that Eq. (3.7) rewritten as

$$-\widetilde{a}_{[0]}^{km}(x,u_\rho,D^*u_\rho)D_kD_mu_\rho+\widehat{a}_{[0]}(x,u_\rho,D^*u_\rho)=0 \tag{3.11}$$

with

$$\widehat{a}_{[0]}(x,u_\rho,D^*u_\rho)=\widetilde{a}_{[0]}(x,u_\rho,D^*u_\rho)+\widetilde{J}u_\rho \tag{3.12}$$

can be regarded as a self-govering equation on $\Gamma(\Pi_R)$. In accordance with (B1), (J1), (3.3), (3.4), (3.8) and (3.12) we have the estimate

$$|\widehat{a}_{[0]}(x,u_\rho,D^*u_\rho)|\leqslant\mu|D^*u_\rho|^2+3\widetilde{b}_{[0]}(x)|D^*u_\rho|+\widehat{\Phi}_1^{[0]}(x), \tag{3.13}$$

where

$$\widehat{\Phi}_1^{[0]}\equiv3\widetilde{\Phi}_1^{[0]}(x)+2\widetilde{b}_{[0]}(x)N_{12}\left|\lim_{\varepsilon\to-0}D_1u_\rho(\varepsilon,\varkappa,0,\dots,0)\right|. \tag{3.14}$$

Next, we consider a positive constant R_2 such that

$$R_2\leqslant\min\left\{1,\left(\frac{\beta(1+\beta)\nu}{14(\sigma_1+\mu N_{14})}\right)^{1/\beta}\right\},\qquad\beta=\lambda\gamma_1,$$

where σ_1 is the same constant as in (3.2), $\gamma_1=\gamma_1(n-1,\nu)$ and $\lambda=\lambda(n-1,\nu,q-1,\delta q',\mu)$ are the constants from Theorem 7 [6] and Lemma 8 [6], respectively, whereas N_{14} is the constant from the proof of Theorem 11 [6] completely determined by the values of n, q, δ, M_0, ν, μ, $\|\widetilde{b}_{[0]}\|_{q-1,(\alpha q'),\Gamma(\Pi_R)}$, $\|\widehat{\Phi}_1^{[0]}\|_{q-1,(\alpha q'),\Gamma(\Pi_R)}$ and by the characteristics of $\mathcal{M}$. It should be noted that N_{14} is the increasing function (and, consequently, R_2 is the decreasing one) with respect to the argument $\|\widehat{\Phi}_1^{[0]}\|_{q-1,(\alpha q'),\Gamma(\Pi_R)}$.
Define $R=\min\{R_1,R_2\}$. Taking into account (3.2), (3.9), the last inequalities in (3.10), and the condition $u|_{\{x'=0\}}=0$ which follows from (1.4), we can apply successively Theorem 12 [6] and Theorem 13 [6] (with the natural changes $n\to n-1$, $q\to q-1$, and $\alpha\to\alpha q'$) to (3.11). As a result we get the estimates

$$\|D^*u_\rho\|_{\Gamma(\Pi_{2R/9})}\leqslant N_{13},\qquad[D^*u_\rho]_{\gamma,\Gamma(\Pi_{R/36})}\leqslant N_{15},$$

where the constants $\gamma\in]0,\delta[$, N_{13}, and N_{15} depend on n, q, δ, ν, μ, M_0, $\|\widetilde{b}_{[0]}\|_{q-1,(\alpha q'),\Gamma(\Pi_R)}$, $\|\widehat{\Phi}_1^{[0]}\|_{q-1,(\alpha q'),\Gamma(\Pi_R)}$, and on $\|\widetilde{\Phi}_2^{[0]}\|_{q-1,(\alpha q'),\Gamma(\Pi_R)}$.
Now we can consider Eq. (3.11) as a linear equation with continuous coefficients at the second-order derivatives. Using a local variant of the inequality (35) from [6] we arrive at the estimate

$$\|\left(D^*\right)^2u_\rho\|_{q-1,(\alpha q'),\Gamma(\Pi_{R/36})}\leqslant N_{16}, \tag{3.15}$$

where N_{16} depends on the same arguments as N_{13}. It should be noted also that N_{16} is an increasing function with respect to the argument $\|\widehat{\Phi}_1^{[0]}\|_{q-1,(\alpha q'),\Gamma(\Pi_R)}$.
Using Lemma 2.1 we extend the function $u_\rho|_{\Gamma(\Pi_{R/36})}$ to the cylinders $Q^{(1)} =]-\frac{R}{72},0[\times\Gamma(\Pi_{R/72})$ and $Q^{(2)} =]0,\frac{R}{72}[\times\Gamma(\Pi_{R/72})$ so as to satisfy

$$\|D^2\overline{u}_{\rho,[h]}\|_{q,(\alpha),Q^{(h)}} \leqslant C_h \| \left(D^*\right)^2 u_\rho\|_{q-1,(\alpha q'),\Gamma(\Pi_{R/36})}, \tag{3.16}$$

where $h=1,2$ and $\overline{u}_{\rho,[h]}$ denotes the corresponding extended function, whereas C_h are the constants from Lemma 2.1.
By the choice of α, we have the embeddings of $\widetilde{\mathbb{V}}^2_{q,(\alpha)}(Q^{(h)})$ into $C^{1+\delta}(\overline{Q^{(h)}})$. Therefore, for $h=1,2$ we get

$$\|D\overline{u}_{\rho,[h]}\|_{Q^{(h)}} \leqslant N_{17,[h]}\|D^2\overline{u}_{\rho,[h]}\|_{q,(\alpha),Q^{(h)}}, \tag{3.17}$$

where $N_{17,[h]}$ depends on n, q, δ, and on the characteristics of $\partial\Omega_h$.
Set $v_{[h]}(x) = u_\rho(x) - \overline{u}_{\rho,[h]}(x)$ for $h=1,2$. Taking into account the condition $u_\rho|_{\{x_2=0\}} = 0$ which follows from (1.4), we get that both of the functions $v_{[h]}$ vanishes on $\Gamma(\Pi_{R/72}) \cup \left\{x \in \partial Q^{(h)} : x_2 = 0\right\}$.
Then, using Eq. (3.5), we can write

$$-\widehat{a}^{ij}_{[1]}(x, v_{[1]}, Dv_{[1]})D_iD_jv_{[1]} + \widehat{a}_{[1]}(x, v_{[1]}, Dv_{[1]}) = 0,$$

where

$$\widehat{a}^{ij}_{[1]}(x,z,p) = \widetilde{a}^{ij}_{[1]}(x, z+\overline{u}_{\rho,[1]}, p + D\overline{u}_{\rho,[1]}),$$
$$\widehat{a}_{[1]}(x,z,p) = \widetilde{a}_{[1]}(x, z+\overline{u}_{\rho,[1]}, p + D\overline{u}_{\rho,[1]}) + \widehat{a}^{ij}_{[1]}(x,z,p)D_iD_j\overline{u}_{\rho,[1]}.$$

Moreover, for any $x \in Q^{(1)}$, $z \in \mathbb{R}^1$, $p \in \mathbb{R}^n$ the following structure conditions are satisfied:

$$\nu|\xi|^2 \leqslant \widehat{a}^{ij}_{[1]}(x,z,p)\xi_i\xi_j \leqslant \nu^{-1}|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \tag{A0'}$$

$$|\widehat{a}_{[1]}(x,z,p)| \leqslant 2\mu|p|^2 + \widetilde{b}_{[1]}(x)|p| + \widehat{\Phi}^{[1]}_1(x), \tag{A1'}$$

$$|p|\left|\frac{\partial \widehat{a}^{ij}_{[1]}(x,z,p)}{\partial p_s}\right| \leqslant \mu \quad \text{for} \quad |p| \geqslant 1, \tag{A2'}$$

$$\left|\frac{\partial \widehat{a}^{ij}_{[1]}(x,z,p)}{\partial z}p_s + D_s\left(\widehat{a}^{ij}_{[1]}(x,z,p)\right)\right| \leqslant \mu|p| + \widehat{\Phi}^{[1]}_2(x), \tag{A3'}$$

where

$$\widehat{\Phi}^{[1]}_1(x) = \widetilde{\Phi}^{[1]}_1(x) + 2\mu|D\overline{u}_{\rho,[1]}|^2 + \widetilde{b}_{[1]}(x)|D\overline{u}_{\rho,[1]}| + n\nu^{-1}|D^2\overline{u}_{\rho,[1]}|,$$
$$\widehat{\Phi}^{[1]}_2(x) = \widetilde{\Phi}^{[1]}_2(x) + \mu|D\overline{u}_{\rho,[1]}|.$$

Further, applying successively Theorem 4.3 [4] and Theorem 3.2 [4] to the function $v_{[1]}$, we find that

$$\|Dv_{[1]}\|_{\Gamma(\Pi_{R/144})} \leqslant N_{18}, \tag{3.18}$$

where N_{18} depends on the same arguments as N_{13} and, in addition, on $\|\widetilde{b}_{[1]}\|_{q,(\alpha),\Pi_R}$, and $\|\widetilde{\Phi}_l^{[1]}\|_{q,(\alpha),\Pi_R}$ for $l = 1, 2$. It should be noted that N_{18} is an increasing function with respect to the argument $\|\widehat{\Phi}_1^{[0]}\|_{q-1,(\alpha q'),\Gamma(\Pi_R)}$. In particular, from (3.18) it follows that

$$|Dv_{[1]}(0, \varkappa, 0, \dots, 0)| \leqslant N_{18}.$$

In view (3.17) and (3.16) with $h = 1$ and due to (3.15), we have

$$\begin{aligned}
\varphi(\rho) &= \left| \lim_{\varepsilon \to -0} D_1 u_\rho(\varepsilon, \varkappa, 0, \dots, 0) \right| \\
&\leqslant |Dv_{[1]}(0, \varkappa, 0, \dots, 0)| + |D\overline{u}_{\rho,[1]}(0, \varkappa, 0, \dots, 0)| \\
&\leqslant N_{18} + N_{17,[1]} C_1 N_{16}.
\end{aligned} \tag{3.19}$$

Taking into account the relations (3.19), (3.8), (3.14), and the dependence of the constants N_{14} and $N_{15} - N_{18}$ on their arguments, we conclude that

$$\varphi(\rho) \leqslant \chi_1(\rho, \rho^{\delta q'} \varphi(\rho)), \tag{3.20}$$

where χ_1 is an increasing function of its arguments which is determined by the quantities from the assumptions of the theorem.
Observe that for the case $M_1(x^*) = M_1^+(x^*)$, it sufficies to consider Eq. (3.6) instead of Eq. (3.5) and repeat the above arguments for the function $v_{[2]}$.

Case 2. $\widetilde{d}(x^*) > \frac{\rho R}{288}$.

In this case we straighten the hypersurface Σ in a neighborhood of x^*. We recall that all the properties of such a straightening are discussed in Remark 3.1. It is obvious that

$$M_1 \leqslant N_{19} \left| \lim_{\varepsilon \to -0} D_1 u(\varepsilon, x^*) \right|, \tag{3.4$'$}$$

where N_{19} depends only on properties of Σ.
Consider $\varepsilon_1 = \frac{q(n-1)\delta}{n-1+\alpha q}$. By the Hölder inequality, for $\rho^* \leqslant \frac{\rho R}{576}$, and for any functions $f_h \in L_{q,(\alpha)}(B^n_{\rho^*}(x^*) \cap \Omega_h)$, $h = 1, 2$, and $f \in L_{q-1,(\alpha q')}(B^{n-1}_{\rho^*}(x^*))$

we have

$$
\begin{aligned}
\|f_h\|_{n+\varepsilon_1, B^n_{\rho^*}(x^*)\cap\Omega_h} &\leqslant \|f_h\|_{q,(\alpha),B^n_{\rho^*}(x^*)\cap\Omega_h} \left(\rho^*\right)^{-\alpha} \left|B^n_{\rho^*}(x^*)\right|^{\frac{q-n-\varepsilon_1}{q(n+\varepsilon_1)}} \\
&\leqslant \left(\rho^*\right)^{\delta_1} N_{20}(n) \|f_h\|_{q,(\alpha),B^n_{\rho^*}(x^*)\cap\Omega_h}, \\
\|f\|_{n-1+\varepsilon_1, B^{n-1}_{\rho^*}(x^*)} &\leqslant \|f\|_{q-1,(\alpha q'),B^{n-1}_{\rho^*}(x^*)} \left(\rho^*\right)^{-\alpha q'} \left|B^{n-1}_{\rho^*}(x^*)\right|^{\frac{q-n-\varepsilon_1}{(q-1)(n-1+\varepsilon_1)}} \\
&\leqslant N_{20}(n) \|f\|_{q-1,(\alpha q'),B^{n-1}_{\rho^*}(x^*)},
\end{aligned}
$$

where $\delta_1 = \delta_1(n, q, \alpha) > 0$.
The latter inequalities mean that we can apply the same arguments as in the proof of Theorem 3.1 [10]. This immediately gives us

$$
\varphi(\rho) \leqslant \chi_2(\rho, \rho^{\delta q'}\varphi(\rho)), \tag{3.20$'$}
$$

where χ_2 is an increasing function of its arguments which is determined by the quantities from the assumptions of the theorem.

From (3.20) and (3.20') it follows that, **in either case**, we have the inequality

$$
\varphi(\rho) \leqslant \chi(\rho, \rho^{\delta q'}\varphi(\rho)), \tag{3.21}
$$

where $\chi(\rho, \rho^{\delta q'}\varphi(\rho)) = \max\left\{\chi_1(\rho, \rho^{\delta q'}\varphi(\rho)), \chi_2(\rho, \rho^{\delta q'}\varphi(\rho))\right\}$ is an increasing function of its arguments.
By Lemma 2.3 [11], the inequality (3.21) implies

$$
\varphi(\rho) \leqslant \widehat{\rho}^{\,-\delta q'}, \qquad \forall\ \rho \leqslant \widehat{\rho}, \tag{3.22}
$$

where $\widehat{\rho}$ is determined by the known parameters listed in the statement of the theorem.
The rest of the proof is standard. Combining the inequalities (3.4), (3.4') and (3.22), we get the estimate

$$
M_1 \leqslant N_{21} \frac{\varphi(\widehat{\rho})}{\widehat{\rho}} \leqslant N_{21}\, \widehat{\rho}^{\,-1-\delta q'},
$$

where $N_{21} = \max\{N_{12}, N_{19}\}$. This completes the proof. □

Corollary 3.2. *Under the hypothesis of Theorem 3.1 the following estimates hold:*

$$
\|Du\|_{\Omega_h} \leqslant C_5, \qquad [Du]_{\widetilde{\gamma},\Omega_h} \leqslant C_6, \tag{3.23}
$$

where $h = 1, 2$, while the constants C_5, C_6 and $\widetilde{\gamma} \in]0, \delta[$ depend on the same quantities as C_4.

Proof. Once the estimate (3.1) is established, Eq. (3.8) becomes completely self-govering and the rest of the proof is rather standard. Applying Theorem 12 [6] we obtain the estimate $\|\widetilde{D}u\|_{\Sigma}$, and after that from Theorem 13 [6] we obtain the estimate $[\widetilde{D}u]_{\widetilde{\gamma},\Sigma}$. Then we consider the interface equation (1.3) as a linear equation on Σ and deduce the estimate $\|(\widetilde{D})^2 u\|_{q-1,(\alpha q'),\Sigma}$ from Theorem 14 [6]. Using Lemma 2.1 we extend the function $u|_{\Sigma}$ to Ω_1 and Ω_2. For $h=1,2$ we set $v_{[h]}(x)=u(x)-\overline{u}_{[h]}(x)$, where $\overline{u}_{[h]}$ denotes the corresponding extended function. Then, Theorem 4.3 [4] and Theorem 3.2 [4], applied succesively to the both functions $v_{[h]}$, gives us the Hölder estimate for the full gradient on Σ as well as the interior estimates for the gradient. Finally, combining all the estimates mentioned above, we arrive at (3.23). $\square$

Proof of Theorem 1. We fix an arbitrary function

$$v \in \mathcal{X} = C^{1+\widetilde{\gamma}}(\overline{\Omega}_1) \cap C^{1+\widetilde{\gamma}}(\overline{\Omega}_2) \cap C(\overline{\Omega}),$$

where $\widetilde{\gamma}$ is the constant from Corollary 3.2. We consider the following family of linear problems:

$$\begin{aligned}
&\tau\left(-a^{ij}_{[1]}(x,v,Dv)D_iD_ju + a_{[1]}(x,v,Dv)\right) - (1-\tau)\Delta u = 0 \ \mathit{in}\ \Omega_1,\\
&\tau\left(-a^{ij}_{[2]}(x,v,Dv)D_iD_ju + a_{[2]}(x,v,Dv)\right) - (1-\tau)\Delta u = 0 \ \mathit{in}\ \Omega_2,\\
&\tau\left(-a^{ij}_{[0]}(x,v,\widetilde{D}v)\widetilde{D}_i\widetilde{D}_ju + a_{[0]}(x,v,\widetilde{D}v) + Jv\right) -\\
&\qquad\qquad\qquad\qquad\qquad - (1-\tau)\widetilde{\Delta}u = 0 \ \mathit{on}\ \Sigma,\\
&u|_{\partial\Omega} = 0.
\end{aligned} \tag{3.24}$$

In view of the choice of α, it follows that the structural conditions (A2), (A3) and (A4) guarantee for $h=1,2$ the continuity of the functions $a^{ij}_{[h]}$ with respect to all their arguments. Similarly, we may deduce from the structure conditions (B2), (B3) and (B4) the continuity of the functions $a^{ij}_{[0]}$ with respect to all their arguments. Thus, $a^{ij}_{[h]}(\,\cdot\,,v,Dv) \in C(\overline{\Omega}_h)$ for $h=1,2$, and $a^{ij}_{[0]}(\,\cdot\,,v,\widetilde{D}v) \in C(\overline{\Sigma})$. By (A1) and (A4), we have $a_{[h]}(\,\cdot\,,v,Dv) \in L_{q,(\alpha)}(\Omega_h)$ for $h=1,2$. Finally, from (B1), (B4), (J0) and (J1) we get that the functions $a_{[0]}(\,\cdot\,,v,\widetilde{D}v)$ and Jv belong to the space $L_{q-1,(\alpha q')}(\Sigma)$.
Therefore, Theorems 2.3 and 2.4 are applicable to the problem (3.24) for every $\tau \in [0,1]$. It guarantees that for every $\tau \in [0,1]$ there is a unique solution $u^{[\tau]}$ of the problem (3.24) belonging to the space $\mathcal{V}_{q,(\alpha)}(\Omega,\Sigma)$.
We define the family of nonlinear operators $\mathcal{F}_\tau$, $\tau \in [0,1]$, from $\mathcal{X}$ in $\mathcal{V}_{q,(\alpha)}(\Omega,\Sigma)$. With every function v the operator $\mathcal{F}_\tau$ associates the

solution $u^{[\tau]}$ of the problem (3.24). By the choice of q and α, the embedding of $\mathcal{V}_{q,(\alpha)}(\Omega, \Sigma)$ into $\mathcal{X}$ is compact. Therefore, the operators $\mathcal{F}_\tau$ are compact in $\mathcal{X}$.
It is obvious that $\mathcal{F}_0(v) \equiv 0$. By condition (iii) of the theorem, the set $\mathcal{N}$ of fixed points of the operators $\mathcal{F}_\tau$, $\tau \in [0,1]$, is bounded in $C(\overline{\Omega})$. Corollary 3.2 shows that the set $\mathcal{N}$ is bounded in $\mathcal{X}$. Applying the Leray-Schauder principle (see Theorem 10.1 [12]), we complete the proof. $\square$

Remark. For the case $n = 2$ the condition $q > \widehat{q}$ in Theorem 1 becomes the condition $q > n$ which is standard for domains with smooth boundaries. Moreover, if q is not too large ($\frac{2}{q} > 2 - \frac{\pi}{2\Theta}$), then we can take $\alpha = 0$ and solve the problem in an ordinary (not weight) Sobolev space.

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