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Abdellah Elfanni

Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken
Germany
E-Mail: elfanni@math.uni-sb.de

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Edited by
FR 6.1 – Mathematik
Im Stadtwald
D–66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

Abstract

In this note, we consider a nonconvex variational problem for which the admissible deformations are located between two obstacles. It turns out that the value of the minimization problem is equal to zero when the obstacles do not touch each other, otherwise it might be positive. A numerical analysis of such problems is also considered.

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1. Introduction.

Let Ω be a bounded polyhedral domain in $\mathbf{R^n}$ of boundary $\partial\Omega$ and of closure $\overline{\Omega}$. Let $w_i \in \mathbf{R^n},\ i = 1, \dots, p,\ p \geq 2$ and consider a function $\varphi :\ \mathbf{R^n} \to \mathbf{R}$ such that

$$\varphi(w_i) = 0\ \forall i = 1, \dots, p, \tag{1.1}$$

$$\varphi(w) > 0\ \forall w \neq w_i,\ i = 1, \dots, p, \tag{1.2}$$

$$w_i - w_p,\ i = 1, \dots, p-1 \text{ are linearly independent.} \tag{1.3}$$

For instance, in a physical setting, φ could be some stored energy density that vanishes at wells w_i's. These wells stand for natural states with low or no energy (see for example [B.J$_1$] and [B.J$_2$] for the physical background). We refer to [A.] for details and notations on Sobolev spaces. If G, α and β are Lipschitz continuous functions i.e. if

$$\alpha,\ \beta,\ G \in W^{1,\infty}(\Omega)$$

such that

$$\alpha(x) \leq G(x) \leq \beta(x) \text{ on } \Omega,$$

we denote by $\mathcal{K}$ the following set

$$\mathcal{K}(\Omega) := \mathcal{K} = \{v \in W^{1,\infty}(\Omega) :\ \alpha(x) \leq v(x) \leq \beta(x) \text{ in } \Omega \text{ and } v(x) = G(x) \text{ on } \partial\Omega\}.$$

Let Ω_1 and Ω_2 denote the following sets

$$\Omega_1 := \left\{x \in \Omega \mid G(x) > \frac{\alpha(x) + \beta(x)}{2}\right\}$$

$$\Omega_2 := \Omega \backslash \Omega_1$$

and M_1, M_2 the following nonegative constants

$$M_1 := \inf_{x \in \Omega_1} (G(x) - \alpha(x)), \tag{1.4}$$

$$M_2 := \inf_{x\in\Omega_2} (\beta(x) - G(x)). \tag{1.5}$$

Then we consider the following problem

$$I = \inf_{v\in\mathcal{K}} \int_\Omega \varphi(\nabla v(x))dx. \tag{1.6}$$

We assume that

$$\nabla G(x) \in \ \mathrm{Co}(w_i) \text{ a.e. in } \Omega \tag{1.7}$$

($\mathrm{Co}(w_i)$ denotes the convex hull of the w_i's).
For the numerical purpose, let $(\mathcal{T}_h)$ be a family of triangulations of Ω (see [R.T.]), this means

$$\forall\ h > 0 \begin{cases} \forall\ K \in\ \mathcal{T}_h,\ K \text{ is a N-simplex,} \\ \max_{K\in\ \mathcal{T}_h} (h_K) = h, \\ \exists\nu > 0 \text{ such that } \forall K \in\ \mathcal{T}_h\ \frac{h_K}{\rho_K}\ \leq\ \nu. \end{cases}$$

where h_K is the diameter of the N-simplex K and ρ_K its roundness (i.e. the largest diameter of the balls that could fit in K). If $P_1(K)$ is the space of polynomials of degree 1 on K, set

$$V_h = \{v\ :\ \Omega \longrightarrow \mathbf{R} \text{ continuous, } v/_K\ \in\ P_1(K),\ \forall\ K\ \in\ \mathcal{T}_h\},$$

$$\Sigma_h = \{p \in \overline{\Omega}\ /\ p \text{ is a vertex of } K \in \mathcal{T}_h\},$$

$$\Sigma_h^0 = \{p \in \Sigma_h\ /\ p \notin \partial\Omega\},$$

and

$$\mathcal{K}_h^A = \{v \in V_h\ /\ \alpha(p) \leq v(p) \leq \beta(p)\ \forall p \in \Sigma_h^0 \text{ and } v(p) = A(p)\ \forall p \in \Sigma_h \cap \partial\Omega\}$$

Our objective is to compute the value of I in (1.6) and try to obtain estimates for the following discrete problem

$$I_h = \inf_{\mathcal{K}_h^A} \int_\Omega \varphi(\nabla v(x))dx \tag{1.8}$$

in terms of the mesh size h. If one assumes that φ is a continuous function then by a compactness argument, the discrete problem (1.8) admits a minimizer. Instead the continuous problem (1.6), as we will see, does not admit in general a minimizer. This kind of problems were studied in [C.], [C.L.] and [C.E.] but where the obstacles are out of consideration.

2. A particular case.

For convenience we consider in this section the particular where

$$M := \inf_{x\in\Omega}(G(x) - \alpha(x)) > 0. \tag{2.1}$$

The result of this section will be used when we will study at the end of this note the case where the two obstacles touch each other. Let $(v_1, \ldots, v_{p-1})$ be the dual basis of $(w_1 - w_p, \ldots, w_{p-1} - w_p)$ i.e. the basis such that

$$(w_i - w_p) \cdot v_j = \delta_{ij} \ \forall i, j = 1, \ldots, p-1.$$

Then we have the following theorem:

Theorem 2.1. *Let us assume that Ω is convex and that φ is a function bounded on bounded subsets of $\mathbf{R}^n$ satisfying (1.1), (1.2). Then if (1.3), (1.7) and (2.1) hold, one has*

$$I = 0.$$

Moreover, there exists a constant C independent of h, $0 < h < \inf\left(\left(\frac{M}{C^*}\right)^2, 1\right)$, such that

$$I_h = \inf_{\mathcal{K}_h^A} \int_\Omega \varphi(\nabla v(x))dx \leq Ch^{\frac{1}{2}}$$

where $C^* = 2(p-1)\max_i \|w_i\| \max_i \|v_i\|$ ($\|\ \|$ denotes the Euclidian norm in $\mathbf{R}^n$).

To prove Theorem2.1 some preliminary lemmas are needed. First we have

Lemma 2.1. *If $G \in W^{1,\infty}(\Omega)$ satisfies (1.7), then there exists a Lipschitz continuous function $\tilde{G}$ defined in $\mathbf{R}^n$ such that*

$$\tilde{G} = G \ in \ \Omega$$

and

$$\tilde{G}(x) - \tilde{G}(y) \geq \bigwedge_{i=1}^{p} w_i \cdot (x-y) \ for\ all\ x, y \in \mathbf{R}^n \tag{2.2}$$

where $\wedge$ denotes the infimum of functions.

Proof. Applying the mean value theorem after regularization one has (see [E.] for details)

$$G(x) - G(y) \geq \bigwedge_{i=1}^{p} w_i \cdot (x-y) \text{ for a.e. } x, y \in \Omega. \tag{2.3}$$

Then let

$$\tilde{G}(x) = \inf_{y\in\Omega}\Big\{G(y) - \bigwedge_{i=1}^{p} w_i \cdot (y-x)\Big\} \; x \in \mathbf{R}^n.$$

It is clear that $\tilde{G}(x) \leq G(x)$ in Ω. Moreover one has using (2.3)

$$\tilde{G}(x) \geq G(x) \text{ in } \Omega$$

so that

$$\tilde{G}(x) = G(x) \text{ in } \Omega.$$

Let us now prove that

$$\tilde{G}(x) - \tilde{G}(y) \geq \bigwedge_{i=1}^{p} w_i \cdot (x-y) \text{ for a.e. } x, y \in \mathbf{R}^n. \tag{2.4}$$

For every $x, y \in \mathbf{R}^n$ one has

$$\tilde{G}(x) = \inf_{z\in\Omega}\Big\{G(z) - \bigwedge_{i=1}^{p} w_i \cdot (z-y) + \bigwedge_{i=1}^{p} w_i \cdot (z-y) - \bigwedge_{i=1}^{p} w_i \cdot (z-x)\Big\}.$$

Since

$$\bigwedge_{i=1}^{p} w_i \cdot (z-y) - \bigwedge_{i=1}^{p} w_i \cdot (z-x) \geq \bigwedge_{i=1}^{p} w_i \cdot (x-y)$$

one gets

$$\tilde{G}(x) \geq \inf_{z\in\Omega}\Big\{G(z) - \bigwedge_{i=1}^{p} w_i \cdot (z-y)\Big\} + \bigwedge_{i=1}^{p} w_i \cdot (x-y)$$

which implies (2.4). This completes the proof of the lemma.

■

Denote by x_z the points of the lattice of size $h^{\frac{1}{2}}$ spanned by the v_i's i.e. for any $z = (z_1, \ldots, z_{p-1}) \in \mathbf{Z}^{p-1}$ set

$$x_z = \sum_{i=1}^{p-1} z_i h^{\frac{1}{2}} v_i. \tag{2.5}$$

Then let us define the function Λ_h by

$$\Lambda_h(x) = \bigvee_{z\in\mathbf{Z}^{p-1}} \Big(\bigwedge_{i=1}^{p} w_i \cdot (x - x_z) + \tilde{G}(x_z)\Big) \tag{2.6}$$

where $\tilde{G}$ is the extension of G obtained in Lemma 2.1 and $\bigvee$ denotes the supremum of functions. By a unit cell of the lattice spanned by the $h^{\frac{1}{2}}v_i$ we mean a set of the type

$$C_z = x_z + \{\sum_{i=1}^{p-1} \beta_i h^{\frac{1}{2}} v_i \;/ \beta_i \in [0,1]\}$$

where x_z is defined by (2.5). Then one has

Lemma 2.2. *Let us assume that Ω is convex. Under the above assumptions, denote by C_{z_0} a unit cell spanned by $h^{\frac{1}{2}}v_i$'s and by E the set*

$$E = \{z \in \mathbf{Z}^{p-1} \;/\; z_i = 0 \text{ or } 1 \;\forall i = 1, \dots, p-1\},$$

then one has

$$\Lambda_h(x) = \bigvee_{z' \in z_0 + E} \Big(\bigwedge_{i=1}^{p} w_i \cdot (x - x_{z'}) + \tilde{G}(x_{z'})\Big).$$

Proof. We give here an astute proof for th case of two wells i.e. the case when $p = 2$ and we refer to [C.E.] for the general case. Let $z_0, z \in \mathbf{Z}$ and $x \in C_{z_0} := [x_{z_0}, x_{z_0+1}]$. One has either $x_{z_0} \in [x_z, x]$ or $x_{z_0+1} \in [x_z, x]$. Let us assume that $x_{z_0} \in [x_z, x]$ the other case can be handled similarly. There exists $\lambda \in [0,1]$ such that

$$x_{z_0} = \lambda x_z + (1-\lambda)x.$$

Therefore

$$x - x_{z_0} = \lambda(x - x_z) \text{ and } x_{z_0} - x_z = (1-\lambda)(x - x_z).$$

One has

$$\bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \tilde{G}(x_{z_0}) = \bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \tilde{G}(x_{z_0}) - \tilde{G}(x_z) + \tilde{G}(x_z)$$

so that

$$\bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \tilde{G}(x_{z_0}) \geq \bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \bigwedge_{i=1}^{p} w_i \cdot (x_{z_0} - x_z) + \tilde{G}(x_z)$$

but

$$\bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \bigwedge_{i=1}^{p} w_i \cdot (x_{z_0} - x_z) = \lambda \bigwedge_{i=1}^{p} w_i \cdot (x - x_z) + (1-\lambda) \bigwedge_{i=1}^{p} w_i \cdot (x - x_z)$$

so that

$$\bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \bigwedge_{i=1}^{p} w_i \cdot (x_{z_0} - x_z) = \bigwedge_{i=1}^{p} w_i \cdot (x - x_z).$$

Hence

$$\bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0}) + \tilde{G}(x_{z_0}) \geq \bigwedge_{i=1}^{p} w_i \cdot (x - x_z) + \tilde{G}(x_z).$$

This completes the proof of the lemma.

■

Remark 2.1. Notice that the above proof can be applied to functions φ having more than two wells but spanning a one dimensional space. Hence the condition (1.3) is not necessary in this case. Nevertheless this condition is crucial when the space spanned by the wells has a dimension greater than one. As we mentioned before we refer to [C.E.] for details.

We need also the following lemma:

Lemma 2.3. *Let us assume that Ω is convex, under the preceding assumptions one has*

$$G(x) - C^* h^{\frac{1}{2}} \leq \Lambda_h(x) \leq G(x) \;\forall x \in \Omega, \tag{2.7}$$

where $C^* = 2(p-1) \max_i \|w_i\| \max_i \|v_i\|$.

Proof. Let $z \in \mathbf{Z}^{p-1}$ and $x \in \Omega$. Using Lemma 2.1. one has

$$\bigwedge_{i=1}^{p} w_i \cdot (x - x_z) + \tilde{G}(x_z) \leq G(x) \;\forall x_z$$

so that

$$\Lambda_h(x) \leq G(x).$$

Now let us denote by x' the component of x on $P_W(\Omega)$ the orthogonal projection of Ω onto W the space spanned by the w_i's. There exists z_0 such that $x' \in C_{z_0}$, then x' can be written as follows

$$x' = x_{z_0} + \sum_{i=1}^{p-1} \beta_i h^{\frac{1}{2}} v_i, \;\beta_i \in [0,1].$$

Hence

$$\|x' - x_{z_0}\| \leq (p-1) \max_i \|v_i\| h^{\frac{1}{2}}$$

and

$$|\bigwedge_{i=1}^{p} w_i \cdot (x - x_{z_0})| \leq (p-1) \max_i \|w_i\| \max_i \|v_i\| h^{\frac{1}{2}}.$$

Since

$$\Lambda_h(x) \geq \bigwedge_{i=1}^{p} w_i \cdot (x' - x_{z_0}) + \tilde{G}(x_{z_0})$$

one gets

$$\Lambda_h(x) \geq -(p-1) \max_i \|w_i\| \max_i \|v_i\| h^{\frac{1}{2}} + \tilde{G}(x_{z_0}).$$

Since

$$\tilde{G}(x_{z_0}) = \tilde{G}(x_{z_0}) - G(x) + G(x) \geq \bigwedge_{i=1}^{p} w_i \cdot (x_{z_0} - x) + G(x)$$

one obtains

$$\Lambda_h(x) \geq -2(p-1) \max_i \|w_i\| \max_i \|v_i\| h^{\frac{1}{2}} + G(x).$$

This completes the proof of the lemma.

■

Remark 2.2. We have seen that the condition (1.7) implies (2.3). The two conditions are actually equivalent. Indeed, if (2.3) is verified then due to (2.7) the sequence (Λ_h) converges uniformly to G. Since $\nabla\Lambda_h(x) = w_i$ a.e. in Ω one has at least for a subsequence that $\nabla\Lambda_h \rightharpoonup \nabla G$ in $L^\infty(\Omega)$ weak $*$. Let B any ball included in Ω. Since

$$\frac{1}{|B|} \int_B \nabla\Lambda_h(x) dx \in \mathrm{Co}(w_i)$$

and $\mathrm{Co}(w_i)$ is a closed set one has

$$\frac{1}{|B|} \int_B \nabla G(x) dx \in \mathrm{Co}(w_i).$$

Using the Lebesgue differentiation theorem one obtains (1.7).

Proof of Thoerem 2.1. Let us consider the following function

$$u_h(x) = \Lambda_h(x) \vee (G(x) - \mathrm{dist}(x, \partial\Omega)).$$

where we have denoted by $\mathrm{dist}(x, \partial\Omega)$ the distance from x to the boundary $\partial\Omega$. According to Lemma 2.3. one has

$$u_h(x) \leq G(x) \leq \beta(x) \ \forall x \in \Omega.$$

On the other hand one has

$$u_h(x) \geq (G(x) - C^* h^{\frac{1}{2}}) \vee (G(x) - \text{ dist}(x, \partial\Omega)) \geq G(x) - C^* h^{\frac{1}{2}}$$

or again

$$u_h(x) \geq G(x) - \alpha(x) - C^* h^{\frac{1}{2}} + \alpha(x)$$

so that

$$u_h(x) \geq M - C^* h^{\frac{1}{2}} + \alpha(x)$$

where M is defined by (2.1). Since $0 < h < (\frac{M}{C^*})^2$ one gets

$$u_h(x) \geq \alpha(x).$$

Then the function u_h belongs to $\mathcal{K}$. Now we can prove that

$$I = 0. \tag{2.8}$$

Indeed, due to (2.7) one has $\nabla u_h = w_i$ except in a neighbourhood of the boundary of measure less than $Ch^{\frac{1}{2}}$. Therefore $0 \leq I \leq Ch^{\frac{1}{2}}$ for every $0 < h < \inf((\frac{M}{C^*})^2, 1)$ which obviously implies (2.9). Now the rest of the proof follows as in [C.E.]. Due to (2.7) one has

$$|G(x) - \Lambda_h(x)| \leq C^* h^{\frac{1}{2}}$$

so that

$$|G(x) - u_h(x)| \leq Ch^{\frac{1}{2}}.$$

where C is a constant independent of h. Let $\hat{u}_h$ denote the interpolate of u_h. Clearly $\hat{u}_h \in \mathcal{K}_h^A$ and

$$|u_h - \hat{u}_h| \leq Ch.$$

Therefore (Recall that $h < 1$)

$$|\hat{u}_h - G(x)| \leq Ch^{\frac{1}{2}}.$$

Notice that

$$\nabla \hat{u}_h = w_i$$

except maybe on the set S composed of simplices where interpolation occured. Since on this set $\nabla \hat{u}_h$ remains bounded (see [B.C.]) one has

$$\int_\Omega \varphi(\nabla \hat{u}_h(x)) dx = \int_S \varphi(\nabla \hat{u}_h(x)) dx \leq C|S| \tag{2.9}$$

where $|S|$ is the Lebesgue measure of S and C is a constant which only depends on the wells w_i. When

$$\mathrm{dist}(x, \partial\Omega) \geq C^* h^{\frac{1}{2}}$$

one has

$$u_h(x) = \Lambda_h(x).$$

So, when $\mathrm{dist}(x, \partial\Omega) \geq C^* h^{\frac{1}{2}} + h$ one has

$$\hat{u}_h = \text{the interpolate of } \Lambda_h.$$

Let us denote by S_1 the set

$$S_1 = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) \geq C^* h^{\frac{1}{2}} + h\}.$$

Hence

$$|S| \leq |S \cap S_1| + |\Omega \backslash S_1|.$$

First we have

$$|\Omega \backslash S_1| \leq C h^{\frac{1}{2}}. \tag{2.10}$$

where C is a constant.
To estimate $|S \cap S_1|$ one can see that the interpolation occurs on a $h-$neighbourhood of the set where Λ_h has a discontinuity in its gradient. Clearly Λ_h has a jump in its gradient on a unit cell of the lattice spanned by $h^{\frac{1}{2}} v_i$ when one of the functions

$$w_i \cdot (x - x_z) + \tilde{G}(x_z)$$

is equal to an other. These two functions are equal on a set of $(p-2)-$dimensional measure bounded by $C h^{\frac{p-2}{2}}$ where C is a constant. Since in (2.6) the supremum is taken on a finite number of functions it is clear that

$$|S \cap S_1| \leq C h^{\frac{p-2}{2}} . h . N(h)$$

where $N(h)$ is the number of cells of size $h^{\frac{1}{2}}$ included in $P_W(\Omega)$. Clearly

$$N(h) h^{\frac{p-1}{2}} \leq C$$

where C is a constant. Therefore

$$|S \cap S_1| \leq C h^{\frac{1}{2}}. \tag{2.11}$$

Combining (2.9), (2.10) and (2.11) one obtains

$$\int_\Omega \varphi(\nabla \hat{u}_h) dx \leq C h^{\frac{1}{2}}.$$

where C is a constant. This completes the proof of the theorem.

$\blacksquare$

Remark 2.3. Notice that a polyhedral domain can be divided into a finite number of disjoint convex domains. Using the same construction as above in every such subdomain one can see that the estimate we obtained in theorem 2.1 is obviously still valid.

Remark 2.4. The continuous problem (1.6) does not admit in general a minimizer. Indeed, let us assume that $p < n+1$ and

$$\nabla G \neq w_i \text{ in a set of positive measure.} \tag{2.12}$$

There exists $\nu \in \mathbf{R}^n$ such that

$$w_i \cdot \nu = 0 \ \forall i = 1, 2, \ldots, p. \tag{2.13}$$

If the problem (1.6) admits a minimizer u, by (2.8), (1.1) and (1.2) one has

$$\nabla u = w_i \text{ a.e. in } \Omega. \tag{2.14}$$

Using a variant of Poincaré's inequality one gets

$$\int_\Omega |u(x) - G(x)| dx \leq C \int_\Omega |(\nabla u(x) - \nabla G(x)) \cdot \nu| dx.$$

Using (1.7), (2.13) and (2.14) one deduces that

$$u = G.$$

But the assertions (2.12) and (2.14) are incompatible. Therefore the problem (1.6) cannot admit a minimizer.

3. The general case.

In this section we assume that M_1 and M_2 defined in (1.4) and (1.5) are positive. Before stating our estimate theorem we begin with some lemmas. First we have

Lemma 3.1. *Let $\tilde{G}$ be the extension of G defined in section 2. Then one has*

$$\bigwedge_{i=1}^{p} w_i \cdot (x-y) \leq \tilde{G}(x) - \tilde{G}(y) \leq \bigvee_{i=1}^{p} w_i \cdot (x-y) \quad \forall x, y \in \mathbf{R}^n \tag{3.1}$$

where $\bigwedge$ and $\bigvee$ denote respectively the infimum and supremum of functions.

Proof. It is easy to check that

$$\bigvee_{i=1}^{p} w_i \cdot (y-x) = -\bigwedge_{i=1}^{p} w_i \cdot (x-y) \quad \forall x,y \in \mathbf{R}^n.$$

Using (2.2) one obtains

$$\bigvee_{i=1}^{p} w_i \cdot (y-x) \geq \tilde{G}(y) - \tilde{G}(x) \quad \forall x,y \in \mathbf{R}^n.$$

This completes the proof of the lemma.

■

Let V_h be the function defined by

$$V_h(x) = \bigwedge_{z\in\mathbf{Z}^{p-1}} (\bigvee_{i=1}^{p} w_i \cdot (x-x_z) + \tilde{G}(x_z))$$

where the x_z's are defined by (2.5). Then one has the following lemma

Lemma 3.2 *Under the assumptions and notations of lemma 2.3 one has*

$$\tilde{G}(x) \leq V_h(x) \leq C^* h^{\frac{1}{2}} + \tilde{G}(x)$$

Proof. The proof follows like the one of lemma 2.3.

■

We have also the following lemma

Lemma 3.3. *Under the assumptions and notations of Lemma 2.2 one has*

$$V_h(x) = \bigwedge_{z\in z_0+E} (\bigvee_{i=1}^{p} w_i \cdot (x-x_z) + \tilde{G}(x_z)) \tag{3.2}$$

Proof. Reversing the order of the inequalities in the proof of Lemma 2.2 in this note and in [C.E.] for the general case one easily obtains (3.2).

■

Now we can prove the following theorem

Theorem 3.1. *Under the assumptions and notations of theorem 2.1, there exists a constant* C *independent of* h, $0 < h < \inf((\frac{M_1}{C^*})^2, (\frac{M_2}{C^*})^2, 1)$ *such that*

$$I_h = \inf_{K_h^A} \int_\Omega \varphi(\nabla v(x))dx \leq Ch^{\frac{1}{2}}.$$

Proof. Let us consider the following Lipschitz functions

$$u_1^h(x) = \Lambda_h \vee (G(x) - \mathrm{dist}(x, \partial\Omega_1)) \text{ if } x \in \Omega_1,$$

$$u_2^h(x) = V_h(x) \wedge (G(x) + \mathrm{dist}(x, \partial\Omega_2)) \text{ if } x \in \Omega_2.$$

Using respectively lemma 2.3 and lemma 3.2 it is clear that the u_j's coincide with G at the boundaries of Ω_j's. Moreover one can easily check that

$$\alpha(x) \leq u_1^h(x) \leq G(x) \text{ on } \Omega_1,$$

and

$$G(x) \leq u_2^h(x) \leq \beta(x) \text{ on } \Omega_2.$$

Since $\partial\Omega \subset \partial\Omega_1 \cup \partial\Omega_2$ the following function

$$u_h(x) = \begin{cases} u_1^h(x) \text{ if } x \in \Omega_1, \\ \\ u_2^h(x) \text{ if } x \in \Omega_2. \end{cases}$$

coincides with G at the boundary of Ω. Moreover u_h is a Lipschitz function. Indeed, let $x \in \Omega_1$ and $y \in \Omega_2$. There exists $z \in \partial\Omega_1 \cap \partial\Omega_2$ such that

$$z \in [x, y].$$

Thus

$$|u_h(x) - u_h(y)| = |u_1^h(x) - u_2^h(y)| \leq |u_1^h(x) - u_1^h(z)| + |u_2^h(z) - u_2^h(y)|$$

since $u_1^h(z) = u_2^h(z) = G(z)$. Hence

$$|u_h(x) - u_h(y)| \leq C\{|x - z| + |z - y|\} = C|x - y|$$

where C is a constant independent of x, y and h.
Let $\hat{u}_h$ be the interpolate of u_h. Interpolation occurs now in some neighbourhoods N_j of the boundaries of $\Omega_j,\ j = 1, 2$ of measure less than $Ch^{\frac{1}{2}}$ and in the subset of $\Omega \backslash \{N_1 \cup N_2\}$ where the function

$$w_i \cdot (x - x_z) + \tilde{G}(x_z)$$

is equal to another. Arguing as in the proof of theorem 2.1 one obtains

$$\inf_{v \in K_h^A} \int_\Omega \varphi(\nabla u(x))dx \leq \int_\Omega \varphi(\nabla \hat{u}_h(x))dx \leq Ch^{\frac{1}{2}}$$

∎

We end up this note by considering the case where the two obstacles touch each other. Let us denote by Ω' the following open set

$$\Omega' := \{x \in \Omega \mid \alpha(x) < \beta(x)\}.$$

One has for every $u \in \mathcal{K}$

$$u(x) = G(x) = \alpha(x) = \beta(x) \text{ in } \Omega \backslash \Omega'$$

and

$$\int_\Omega \varphi(\nabla u(x))dx = \int_{\Omega'} \varphi(\nabla u(x))dx + \int_{\Omega\backslash\Omega'} \varphi(\nabla G(x))dx. \tag{3.3}$$

Since

$$u = G \text{ on } \partial\Omega'$$

one has

$$\inf_{u\in\mathcal{K}(\Omega)} \int_\Omega \varphi(\nabla u(x))dx = \inf_{u\in\mathcal{K}(\Omega')} \int_{\Omega'} \varphi(\nabla u(x))dx + \int_{\Omega\backslash\Omega'} \varphi(\nabla G(x))dx. \tag{3.4}$$

Then one has

Theorem 3.2. *Assume that φ is bounded on bounded subsets of $\mathbf{R}^n$ and (1.1), (1.2), (1.3), (1.7) hold. Then*

$$\inf_{u\in\mathcal{K}(\Omega)} \int_\Omega \varphi(\nabla u(x))dx = \int_{\Omega\backslash\Omega'} \varphi(\nabla G(x))dx.$$

Proof. Due to (3.4) it suffices to prove that

$$\inf_{u\in\mathcal{K}(\Omega')} \int_{\Omega'} \varphi(\nabla u(x))dx = 0. \tag{3.6}$$

Let us denote by Ω'_ε the set

$$\Omega'_\varepsilon = \{x \in \Omega' \mid \operatorname{dist}(x, \partial\Omega') > \varepsilon\}.$$

Let $u \in \mathcal{K}(\Omega'_\varepsilon)$, we extend u to Ω' by setting

$$u = G \text{ in } \Omega'\backslash\Omega'_\varepsilon.$$

The extension of u to Ω' belongs to $\mathcal{K}(\Omega')$ and

$$\int_{\Omega'} \varphi(\nabla u(x))dx = \int_{\Omega'_\varepsilon} \varphi(\nabla u(x))dx + \int_{\Omega'\backslash\Omega'_\varepsilon} \varphi(\nabla G(x))dx.$$

Thus

$$\int_{\Omega'_\varepsilon} \varphi(\nabla u(x))dx + \int_{\Omega'\setminus\Omega'_\varepsilon} \varphi(\nabla G(x))dx \geq \inf_{v\in\mathcal{K}(\Omega')} \int_{\Omega'} \varphi(\nabla v(x))dx$$

which implies that

$$\inf_{v\in\mathcal{K}(\Omega')} \int_{\Omega'} \varphi(\nabla v(x))dx \leq \inf_{v\in\mathcal{K}(\Omega'_\varepsilon)} \int_{\Omega'_\varepsilon} \varphi(\nabla v(x))dx + \int_{\Omega'\setminus\Omega'_\varepsilon} \varphi(\nabla G(x))dx.$$

According to section 2 we know that

$$\inf_{v\in\mathcal{K}(\Omega'_\varepsilon)} \int_{\Omega'_\varepsilon} \varphi(\nabla v(x))dx = 0$$

since

$$M := \inf_{x\in\Omega'_\epsilon} [G(x) - \alpha(x)] > 0.$$

Thus

$$0 \leq \inf_{v\in\mathcal{K}(\Omega')} \int_{\Omega'} \varphi(\nabla v(x))dx \leq C'|\Omega'\setminus\Omega'_\varepsilon| \leq C''\varepsilon$$

where C' and C'' are constants independent of ε. Hence

$$\inf_{v\in\mathcal{K}(\Omega')} \int_{\Omega'} \varphi(\nabla v(x))dx = 0$$

and this completes the proof of the theorem.

∎

Remark 3.1. Using (3.3) and arguing as in Remark 2.4 one can prove that the continuous problem I does not admit, in genaral, a minimizer.

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