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THE (SUB/SUPER)ADDITIVITY ASSERTION OF CHOQUET

HEINZ KÖNIG

Abstract. The assertion in question comes from the short final section in the Theory of Capacities of Choquet 1953/54, in connection with his prototype of the subsequent Choquet integral. The problem was whether and when this formation is additive. Choquet had the much more abstract idea that all functionals in a certain wide class must be subadditive, and the counterpart with superadditive. His treatment of this point was kind of an outline, and his proof limited to a rather narrow special case. Thus the adequate context and scope of the assertion remained open even up to now. In this paper we present a counterexample which shows that the initial context has to be modified, and then in new context a comprehensive theorem which fulfils all needs turned up so far.

In section 48 of his famous Theory of Capacities [2] Gustave Choquet introduced a certain class of functionals with the flavour of an *integral*, but invented for an important issue in capacities and not at all for the sake of measure and integration. Yet the concept showed basic qualities in that other respect too: It was in the initial spirit of Lebesgue [10] to construct the integral via decomposition into horizontal strips rather than into vertical ones, which had fallen into oblivion in the course of the 20th century, and was simpler and much more comprehensive than the usual constructions. Thus in subsequent decades the concept developed into a universal one in measure and integration, called the *Choquet integral*. One could even wonder why the Choquet integral did not become the foundation for all of integration theory.

But the fact that this did not happen had an immediate reason: The basic hardship with the Choquet integral is that it is a priori obscure whether and when it is *additive*, which one best even subdivides into *subadditive* and *superadditive*. To this issue Choquet contributed in his final section 54 a spectacular, because much more abstract idea: On certain lattice cones *all submodular and positive-homogeneous real-valued functionals must be subadditive*, and the same for *super* in place of *sub*. It is this assertion which forms the theme of the present paper (in the sequel the two cases will be united via an obvious *sub/super* shorthand notation). The treatment of Choquet was kind of an outline, and his proof limited to a rather narrow special case. While the Choquet integral has been explored in subsequent decades, the abstract assertion remained unsettled up to now.

In recent years the present author became motivated because he needed an assertion of this kind for the further development of his extended Daniell-Stone and Riesz representation theorems [7]. In 1998 he obtained an intermediate result which

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was sufficient for that purpose [8]. In this paper we present a counterexample which shows that the initial context for the abstract assertion has to be modified, and then in new context a comprehensive theorem which fulfils all needs turned up so far. For the basic step within the so-called *finite situation* there will be two proofs. One of them is a distributional version of the initial proof due to Choquet, while the other one furnishes, via a remarkable fact on convex functions, an essential fortification in the finite situation.

1. **Introduction and Fundamentals**

The Choquet Integral. The Choquet integral as evolved in the second half of the 20th century exists in different versions. The present version is from the author's textbook [7] section 11. It features *two* classes of admissible functions. The reason is that the two variants are in perfect accord with the two extension theories in measure and integration, the inner and the outer one, developed in [7].

Let X be a nonvoid set and $\mathfrak{S}$ be a lattice of subsets with $\varnothing\in\mathfrak{S}$ in X. We define $\mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ to consist of the functions $f\in[0,\infty]^X$ such that $[f\geqq t]/[f>t]\in\mathfrak{S}$ for all $0<t<\infty$, called *upper/lower measurable* $\mathfrak{S}$. We fix an increasing set function $\varphi:\mathfrak{S}\to[0,\infty]$ with $\varphi(\varnothing)=0$ and define the *Choquet integral*

$$\fint f d\varphi := \int_{0\leftarrow}^{\to\infty} \varphi([f\geqq t])dt\in[0,\infty] \quad \text{for } f\in\mathrm{UM}(\mathfrak{S}),$$

$$\fint f d\varphi := \int_{0\leftarrow}^{\to\infty} \varphi([f> t])dt\in[0,\infty] \quad \text{for } f\in\mathrm{LM}(\mathfrak{S}),$$

both times as an improper Riemann integral of a monotone function with values in $[0,\infty]$. It is well-defined since in case $f\in\mathrm{UM}(\mathfrak{S})\cap\mathrm{LM}(\mathfrak{S})$ the two second members are equal. Thus for $A\in\mathfrak{S}$ we have $\chi_A\in\mathrm{UM}(\mathfrak{S})\cap\mathrm{LM}(\mathfrak{S})$ with $\fint\chi_A d\varphi=\varphi(A)$. If $\mathfrak{S}$ is a σ algebra then $\mathrm{UM}(\mathfrak{S})=\mathrm{LM}(\mathfrak{S})$ consists of the $f\in[0,\infty]^X$ *measurable* $\mathfrak{S}$ in the usual sense, and in case of a measure φ then $\fint f d\varphi$ is the usual integral $\int f d\varphi$.

The prototype of the Choquet integral defined in [2] was for the lattice $\mathfrak{S}=\mathrm{Comp}(X)$ of the compact subsets in a locally compact Hausdorff topological space X and under the assumption $\varphi<\infty$, but restricted to the function class

$$\mathrm{CK}(X,[0,\infty[)\subset\mathrm{USCK}(X,[0,\infty[)\subset\mathrm{UM}(\mathrm{Comp}(X))\cap[0,\infty[^X,$$

with these classes defined to consist of the continuous and of the upper semicontinuous functions $X\to[0,\infty[$ with compact support. Therefore the set functions φ had sometimes to be restricted to the downward τ continuous ones, that is to the *capacities* in the sense of [2].

We return to the full Choquet integral. For the basic properties noted below we refer to [7] section 11 and [8] section 2.

1.1 Remark. *For a function $f\in[0,\infty[^X$ with finitely many values the following are equivalent.*

i) $f\in\mathrm{UM}(\mathfrak{S})$. ii) $f\in\mathrm{LM}(\mathfrak{S})$.

iii) $f=\sum\limits_{l=1}^{r}t_l\chi_{A(l)}$ *for some real* $t_1,\cdots,t_r>0$ *and* $A(1),\cdots,A(r)\in\mathfrak{S}$.

iv) *The same as* iii) *with* $A(1) \supset \cdots \supset A(r)$.

We define $\mathrm{S}(\mathfrak{S})$ to consist of these functions; note from iii) that $\mathrm{S}(\mathfrak{S})$ is stable under addition. *For the representations* iv) *of* $f \in \mathrm{S}(\mathfrak{S})$ *then* $⨍ f d\varphi = \sum\limits_{l=1}^{r} t_l \varphi(A(l))$.

1.2 Properties. 1) $\mathrm{UM}(\mathfrak{S})$ *and* $\mathrm{LM}(\mathfrak{S})$ *are positive-homogeneous* (under multiplication with real numbers $0 < t < \infty$) *with* 0 *and stable under pointwise maximum* $\vee$ *and pointwise minimum* $\wedge$.

2) *If* $\mathfrak{S}$ *is stable under countable intersections then* $\mathrm{UM}(\mathfrak{S})$ *is stable under addition and* $\mathrm{UM}(\mathfrak{S}) \supset \mathrm{LM}(\mathfrak{S})$. *If* $\mathfrak{S}$ *is stable under countable unions then* $\mathrm{LM}(\mathfrak{S})$ *is stable under addition and* $\mathrm{LM}(\mathfrak{S}) \supset \mathrm{UM}(\mathfrak{S})$.

3) *For an increasing* $\varphi : \mathfrak{S} \to [0, \infty]$ *with* $\varphi(\varnothing) = 0$ *the Choquet integral* $I : I(f) = ⨍ f d\varphi$ *on* $\mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ *is positive-homogeneous and increasing under the pointwise order* $\leqq$.

The next point is important enough to be included, although this time it will not be needed except for a historical remark at the end of the present subsection: The Beppo Levi theorem carries over to the Choquet integral, and in fact in most comprehensive sequential and nonsequential forms, provided that one adopts the integral in the version of [7]. We quote from [7] 11.17/18 in the obvious old notations.

1.3 Theorem. *Let* $\varphi : \mathfrak{S} \to [0, \infty]$ *be increasing with* $\varphi(\varnothing) = 0$ *and* $I : I(f) = ⨍ f d\varphi$ *on* $\mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$. *For* $\bullet = \sigma\tau$ *then*

Inn) φ *is almost downward* $\bullet$ *continuous* $\Longleftrightarrow$ I *is almost downward* $\bullet$ *continuous on* $\mathrm{UM}(\mathfrak{S})$.

Out) φ *is upward* $\bullet$ *continuous* $\Longleftrightarrow$ I *is upward* $\bullet$ *continuous on* $\mathrm{LM}(\mathfrak{S})$.

In the present context the basic question is when the Choquet integral $I : I(f) = ⨍ f d\varphi$ on $\mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ is (sub/super)additive, and also when it is (sub/super) modular. It is adequate to define a functional $I : S \to [0, \infty]$ on a nonvoid function system $S \subset [0, \infty]^X$ to be *(sub/super)additive* iff

$$I(u+v) \leqq/\geqq I(u) + I(v) \quad \text{for all } u, v \in S \text{ such that } u + v \in S,$$

so that S need not be stable under addition; and to define I, under the condition that S be stable under $\vee\wedge$, to be *(sub/super)modular* iff

$$I(u \vee v) + I(u \wedge v) \leqq/\geqq I(u) + I(v) \quad \text{for all } u, v \in S.$$

In case of the Choquet integral I these properties will of course be related to the respective behaviour of the set function φ. For set functions on *lattices* the adequate notion is *(sub/super)modular*, defined to be

$$\varphi(A \cup B) + \varphi(A \cap B) \leqq/\geqq \varphi(A) + \varphi(B) \quad \text{for all } A, B \in \mathfrak{S}.$$

We note two simple observations.

1.4 Remark. *The Choquet integral* $I : I(f) = ⨍ f d\varphi$ *on* $\mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ *fulfils*

(A) $\quad \varphi$ *(sub/super)modular* $\Longleftarrow$ I *(sub/super)additive,*

(M) $\quad \varphi$ *(sub/super)modular* $\Longleftrightarrow$ I *(sub/super)modular.*

Proof. One obtains the two implications $\Longleftarrow$ for $A, B \in \mathfrak{S}$ from the equations

$$I(\chi_A + \chi_B) = I(\chi_{A\cup B} + \chi_{A\cap B}) = \varphi(A \cup B) + \varphi(A \cap B),$$
$$I(\chi_A \vee \chi_B) + I(\chi_A \wedge \chi_B) = I(\chi_{A\cup B}) + I(\chi_{A\cap B}) = \varphi(A \cup B) + \varphi(A \cap B),$$

the first of which follows from 1.1, both times combined with $I(\chi_A)+I(\chi_B)=\varphi(A)+\varphi(B)$. To obtain $\Longrightarrow$ in (M) one notes for $u,v\in\mathrm{UM}(\mathfrak{S})$ that

$$\begin{aligned} I(u\vee v)+I(u\wedge v) &= \int_{0\leftarrow}^{\rightarrow\infty}\big(\varphi([u\vee v\geqq t])+\varphi([u\wedge v\geqq t])\big)dt \\ &= \int_{0\leftarrow}^{\rightarrow\infty}\big(\varphi([u\geqq t]\cup[v\geqq t])+\varphi([u\geqq t])\cap[u\geqq t])\big)dt \\ &\leqq/\geqq \int_{0\leftarrow}^{\rightarrow\infty}\big((\varphi([u\geqq t])+\varphi([v\geqq t])\big)dt=I(u)+I(v), \end{aligned}$$

and the same on $\mathrm{LM}(\mathfrak{S})$. □

We note that for the prototype in [2] with its restricted domain the two implications $\Longleftarrow$ require that φ be a *capacity*, and under this assumption follow from 1.3.Inn). The decisive question is whether $\Longrightarrow$ holds true in (A). It will be dealt with in the sequel.

The Work of Choquet 1953/54. Choquet in [2] noted that for his prototype the implication $\Longrightarrow$ in (A) holds true, and hence that the three properties involved in (A)(M) are equivalent for *capacities* φ. But what is more and deserves to be called *spectacular*, he had the idea that the implication

$$I \text{ (sub/super)modular } \Longrightarrow I \text{ (sub/super)additive}$$

must be valid for a much wider class of functionals (he also knew that this cannot be true for the converse implication). His precise formulation 54.1 was as follows.

1.5 Choquet's Vision. *Let E be an ordered vector space with order $\sqsubseteq$ and positive cone E^+, and assume that E (or at least E^+) is a lattice under $\sqsubseteq$ with lattice operations $\sqcup\sqcap$. Let $I:E^+\to\mathbb{R}$ be positive-homogeneous. If I is* (sub/super)modular *under $\sqcup\sqcap$ then it must be* (sub/super)additive*.*

¿From this vision 1.5 applied to $E=\mathrm{CK}(X,\mathbb{R})$ on the locally compact Hausdorff X with pointwise order $\leqq$ and lattice operations $\vee\wedge$, and to the restricted Choquet integral $I:I(f)=⨍ fd\varphi$ on $E^+=\mathrm{CK}(X,[0,\infty[)$ with an arbitrary φ, and combined with $\Longrightarrow$ in (M), it follows indeed that Choquet's prototype fulfils the desired implication $\Longrightarrow$ in (A).

However, Choquet did not prove his vision 1.5 in its full extent. His proof was restricted to the case $E=\mathbb{R}^n$ with pointwise order $\leqq$ and lattice operations $\vee\wedge$, and to the positive-homogeneous functions $I:E^+=[0,\infty[^n\to\mathbb{R}$ which are continuous on $[0,\infty[^n$ and C^2 on $]0,\infty[^n$. The explanation is that the entire context was at the end and outside the mainstream of the memoir [2]. Nevertheless Choquet's proof was so well-founded that after half a century it is capable, as we shall see, to furnish a proof of the basic step for the present new main theorem.

The Need for Reconsideration. We have seen that Choquet's vision 1.5 furnishes the decisive implication $\Longrightarrow$ in (A) for his prototype. But we have to note that it does *not* furnish this implication for the full Choquet integral. An obvious reason is that the functions $f\in\mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ and the functional $I:I(f)=⨍ fd\varphi$

can attain the value ∞, and another one that the domains $\mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ need not be stable under addition. But there is a deeper reason: Let for example $X = [0,1]$ and $\mathfrak{S} = \mathrm{Comp}(X)$. Then $\mathrm{USC}(X, [0,\infty[) = \mathrm{UM}(\mathfrak{S}) \cap [0,\infty[^X$ is a convex cone in the vector space $E = \mathrm{B}(X,\mathbb{R})$ of bounded functions, *pointed* and *salient* in the usual sense and hence the positive cone $E^+ \subset E$ in the so-called *intrinsic order* $\sqsubseteq$ on E which it produces. We shall prove that this E^+ is *not a lattice* under $\sqsubseteq$. Thus 1.5 cannot be applied.

1.6 Remark. *On* $X = [0,1]$ *the convex cone* $S = \mathrm{USC}(X,[0,\infty[) \subset \mathrm{B}(X,\mathbb{R})$ *is not a lattice in its intrinsic order* $\sqsubseteq$.

Proof. We fix $0 < a < b < 1$ and form $u = \chi_{[0,b]}$ and $v = \chi_{[a,1]}$ in S. We claim that there is no $w \in S$ such that

i) $u, v \sqsubseteq w$, and

ii) each $h \in S$ with $u, v \sqsubseteq h$ fulfils $w \sqsubseteq h$.

To see this we form $f = 1 + \chi_{\{a,b\}}$ and $g = 1 + \chi_{[a,b]}$ in S. We check three little facts.

0) $h \in S$ with $u, v \sqsubseteq h \Rightarrow f \leqq h$. In fact, we have $u, v \leqq h$ and hence have to show that $h(a), h(b) \geqq 2$. Now for instance $h - u \in S$ and $]b,1] \subset [h - u \geqq 1]$ and hence $[b,1] \subset [h - u \geqq 1]$, so that $h(b) \geqq 2$.

1) $u, v \sqsubseteq f$. In fact, for instance $f - u = \chi_{]b,1]} + \chi_{\{a,b\}} = \chi_{[b,1]} + \chi_{\{a\}} \in S$.

2) $u, v \sqsubseteq g$. In fact, we have $g = u + v$ and thus for instance $g - u = v \in S$.

Now assume that $w \in S$ fulfils i)ii). Then it follows

from i) and 0) applied to $h = w$ that $f \leqq w$,
from 1) and ii) applied to $h = f$ that $w \sqsubseteq f$,
from 2) and ii) applied to $h = g$ that $w \sqsubseteq g$,

which combine to $f = w \sqsubseteq g$. But this is false since $g - f = \chi_{]a,b[} \notin S$. $\square$

To be sure, the implication $\Longrightarrow$ in (A) holds true for the full Choquet integral, even though this does not follow from Choquet's vision 1.5. In the second half of the 20th century the implication has been proved a number of times for the different versions of the Choquet integral. As far as the author is aware, the first proof is due to Topsøe [12] in 1974. For other proofs see [1] [6] [11] [3]. The implication for the present version is in [7] 11.11.

On the other hand Choquet's vision 1.5 remained open in [2], and in fact remained open all the time so far. Now in March 2002 the present author observed that the statement is not true as it stands.

1.7 Example. Let $E = \mathbb{R}^2$ be equipped with the lexicographical order $\sqsubseteq$, that is $u = (u_1, u_2)$ and $v = (v_1, v_2)$ fulfil $u \sqsubseteq v$ iff either $u_1 < v_1$ or $u_1 = v_1$ and $u_2 \leqq v_2$. The order $\sqsubseteq$ is compatible with the standard vector space structure, and E^+ consists of the halfspace $\{x : x_1 > 0\}$ and the halfline $\{x : x_1 = 0 \text{ and } x_2 \geqq 0\}$. Moreover $\sqsubseteq$ is *total* and hence a lattice order with trivial lattice operations $\sqcup\sqcap$. Thus in particular all positive-homogeneous functions $I : E^+ \to \mathbb{R}$ are modular $\sqcup\sqcap$. But of course most of them are *not additive*. A simple example is $I : I(x) = x_2^+$, since $u = (u_1, u_2)$ and $v = (v_1, v_2)$ with $u_1, v_1 > 0$ and $u_2 < 0 < v_2$ are in E^+ and fulfil $I(u + v) = (u_2 + v_2)^+ < v_2 = v_2^+ = I(v) = I(u) + I(v)$. $\square$

The same idea works for any real vector space E of dimension > 1, in that one defines a compatible and total order $\sqsubseteq$ on E via the choice of a basis B of E and of a well-order of B.

Thus there is quite some reason for reconsideration. This does not mean to impose questions upon the wonderful overall implication

$$I \text{ (sub/super)modular } \Longrightarrow I \text{ (sub/super)additive},$$

which will henceforth be called the *fundamental implication.* In the sequel the author wants to develop what he observed since 1998. Section 2 will be devoted to the special case $E = \mathbb{R}^n$ with pointwise order $\leqq$ and lattice operations $\vee\wedge$, henceforth called the *finite* situation, and section 3 to the *full* situation, as we shall see with pointwise order and lattice operations as well. It will be concluded with the application to the Daniell-Stone and Riesz representation theorems mentioned in the introduction.

2. **The Finite Situation**

The Basic Step. The basic step is the result which follows. It is due to Choquet [2] under the assumption that the function I is C^2. We start to sketch his proof.

2.1 Proposition. *Assume that $I :]0,\infty[^n \rightarrow \mathbb{R}$ is positive-homogeneous and continuous. Then I fulfils the fundamental implication.*

Sketch of proof in case that I is C^2. Let $X_1, \cdots, X_n :]0,\infty[^n \rightarrow]0,\infty[$ denote the coordinate functions and $D_1, \cdots, D_n$ the partial derivations. One verifies three facts.

0) For $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$ one has the identity

$$\sum_{k,l=1}^{n} z_k z_l (D_k D_l I) = \sum_{k=1}^{n} \frac{z_k^2}{X_k} \Big(\sum_{l=1}^{n} X_l (D_k D_l I) \Big) - \frac{1}{2} \sum_{k,l=1}^{n} X_k X_l \Big(\frac{z_k}{X_k} - \frac{z_l}{X_l} \Big)^2 (D_k D_l I).$$

1) If I is positive-homogeneous then $\sum_{l=1}^{n} X_l (D_k D_l I) = 0$ for $1 \leqq k \leqq n$.

2) If I is (sub/super)modular $\vee\wedge$ then $D_k D_l I \leqq / \geqq 0$ for $1 \leqq k \neq l \leqq n$.

For I positive-homogeneous and (sub/super)modular $\vee\wedge$ these facts combine with Taylor's formula to furnish that I is convex/concave, hence (sub/super)additive. □

In the sequel we present two proofs of 2.1. The first proof extends Choquet's partial result via distribution theory. This has not been done before, perhaps because one had tried to extend the *result* via regularization, which does not work. By contrast we shall follow Choquet's *proof*, in that we take the above steps 0)1)2) in the distributional sense.

First proof of 2.1. We put $U :=]0,\infty[^n \subset \mathbb{R}^n$. Let as above $X_1, \cdots, X_n : U \rightarrow]0,\infty[$ denote the coordinate functions $x = (x_1, \cdots, x_n) \mapsto x_1, \cdots, x_n$, and $D_1, \cdots, D_n$ the partial derivations in both the standard and distributional sense. Moreover let $\nabla(u, \delta)$ denote the closed ball of radius δ around u. Assume that $I : U \rightarrow \mathbb{R}$ is continuous.

1) If I is positive-homogeneous then

$$A_k := \sum_{l=1}^{n} X_l (D_k D_l I) = 0 \quad \text{for } 1 \leqq k \leqq n.$$

In fact, for $\varphi \in \mathrm{CK}^\infty(U)$ we have in the usual notations

$$< A_k, \varphi > = < I, \sum_{l=1}^{n} D_k D_l (X_l \varphi) > = < I, (n+1) D_k \varphi + \sum_{l=1}^{n} X_l (D_l D_k \varphi) > .$$

For $\mathrm{S}^n \subset \mathbb{R}^n$ the unit sphere and σ^n the surface measure on S^n we obtain

$$< A_k, \varphi >= \int\limits_{\mathrm{S}^n \cap U} \int\limits_{0\leftarrow}^{\rightarrow\infty} t^{n-1} I(ts)\big((n+1)D_k\varphi(ts) + \sum_{l=1}^{n} ts_l D_l D_k \varphi(ts)\big) dt d\sigma^n(s).$$

With $\varphi_s \in \mathrm{CK}^\infty(]0,\infty[)$ for $s \in \mathrm{S}^n$ defined to be $\varphi_s(t) = D_k\varphi(ts)$ it follows that

$$\begin{aligned} < A_k, \varphi > &= \int\limits_{\mathrm{S}^n \cap U} I(s)\Big(\int\limits_{0\leftarrow}^{\rightarrow\infty} \big((n+1)t^n\varphi_s(t) + t^{n+1}\varphi_s'(t)\big)dt\Big) d\sigma^n(s) \\ &= \int\limits_{\mathrm{S}^n \cap U} I(s)\Big(\int\limits_{0\leftarrow}^{\rightarrow\infty} \big(t^{n+1}\varphi_s(t)\big)' dt\Big) d\sigma^n(s) = 0. \end{aligned}$$

2) If I is (sub/super)modular $\vee\wedge$ then

$$D_k D_l I \leqq/\geqq 0 \quad \text{for } 1 \leqq k \neq l \leqq n \text{ in the distributional sense,}$$

that is $< D_k D_l I, \varphi > \leqq/\geqq 0$ for all $0 \leqq \varphi \in \mathrm{CK}^\infty(U)$. For the proof fix $0 \leqq \varphi \in \mathrm{CK}^\infty(U)$ and real $s,t > 0$ so small that $\mathrm{supp}(\varphi) + \nabla(0, s+t) \subset U$. Then in the usual notations

$$\begin{aligned} &\int\limits_U I(x)\big(\varphi(x - se^k - te^l) + \varphi(x) - \varphi(x - se^k) - \varphi(x - te^l)\big) dL(x) \\ =& \int\limits_U \big(I(x + se^k + te^l) + I(x) - I(x + se^k) - I(x + te^l)\big)\varphi(x) dL(x) \\ =& \int\limits_U \Big(I\big((x + se^k) \vee (x + te^l)\big) + I\big((x + se^k) \wedge (x + te^l)\big) \\ & \qquad - I(x + se^k) - I(x + te^l)\Big)\varphi(x) dL(x) \leqq/\geqq 0. \end{aligned}$$

After multiplication with $1/s$ and $s \downarrow 0$ we obtain

$$\int\limits_U I(x)\big(-D_k\varphi(x - te^l) + D_k\varphi(x)\big) dL(x) \leqq/\geqq 0,$$

and then after multiplication with $1/t$ and $t \downarrow 0$ at last

$$< D_k D_l I, \varphi >= \int\limits_U I(x) D_l D_k \varphi(x) dL(x) \leqq/\geqq 0.$$

3) The final step uses the Taylor formula of second degree with remainder term in integral form, which for $f \in \mathrm{CK}^2(U)$ reads

$$f(u+x) = f(u) + \sum_{l=1}^{n} x_l D_l f(u) + \sum_{k,l=1}^{n} x_k x_l \int\limits_0^1 D_k D_l f(u + tx)(1-t) dt$$

$$\text{for } u, u+x \in U.$$

We fix $a,b \in U$ and put $u = \frac{1}{2}(a+b)$, so that $u - a = z$ and $u - b = -z$ with $z = \frac{1}{2}(b-a)$. Also fix $\delta > 0$ with $|a,b| + \nabla(0,\delta) \subset U$, where $|a,b|$ denotes the line segment between a and b. Then for $0 \leqq \varphi \in \mathrm{CK}^\infty(\mathbb{R}^n)$ with $\mathrm{supp}(\varphi) \subset \nabla(u,\delta)$ consider

$$\begin{aligned} A(\varphi) : &= \int\limits_{\nabla(u,\delta)} \Big(I(a-u+x)+I(b-u+x)-2I(x)\Big)\varphi(x)dL(x) \\ &= \int\limits_{U} I(x)\Big(\big(\varphi(x+z)-\varphi(x)\big)+\big(\varphi(x-z)-\varphi(x)\big)\Big)dL(x) \\ &= \int\limits_{U} I(x)\Big(\sum_{k,l=1}^{n} z_k z_l \int\limits_0^1 \big(D_kD_l\varphi(x+sz)+D_kD_l\varphi(x-sz)\big)(1-s)ds\Big)dL(x) \\ &= \int\limits_0^1 (1-s) < \sum_{k,l=1}^{n} z_k z_l (D_kD_lI), \varphi(\cdot+sz)+\varphi(\cdot-sz) > ds. \end{aligned}$$

Now one has as in 0) above the relation

$$\sum_{k,l=1}^{n} z_k z_l (D_kD_lI) = \sum_{k=1}^{n} \frac{z_k^2}{X_k}\big(\sum_{l=1}^{n} X_l (D_kD_lI)\big) - \frac{1}{2}\sum_{k,l=1}^{n} X_k X_l \Big(\frac{z_k}{X_k}-\frac{z_l}{X_l}\Big)^2 (D_kD_lI).$$

For I positive-homogeneous and (sub/super)modular $\vee\wedge$ this relation implies in view of 1)2) that

$$-\sum_{k,l=1}^{n} z_k z_l (D_kD_lI) \leqq/\geqq 0 \quad \text{in the distributional sense.}$$

It follows that $-A(\varphi) \leqq/\geqq 0$ for all functions φ under consideration. Therefore $2I(x) \leqq/\geqq I(a-u+x)+I(b-u+x)$ for all $x \in \nabla(u,\delta)$, since I is continuous. In particular for $x=u$ we have $I(a+b)=2I(u) \leqq/\geqq I(a)+I(b)$. □

An important specialization of 2.1 is the case that I is *increasing* under $\leqq$ (also called *isotone*), because it is the unique one which will reach the *full* situation, but on the other hand will cover all applications known so far.

2.2 Specialization. *Assume that* $I :]0,\infty[^n \to \mathbb{R}$ *is positive-homogeneous and increasing. Then* I *is* $\geqq 0$ *and continuous. Thus* I *fulfils the fundamental implication.*

Proof. 1) For $x \in]0,\infty[^n$ we have $I(x) \leqq I(2x) = 2I(x)$ and hence $I(x) \geqq 0$. 2) If $a \in]0,\infty[^n$ and $0<\varepsilon<1$ then $\{x : (1-\varepsilon)a \leqq x \leqq (1+\varepsilon)a\}$ is a neighbourhood of a on which $(1-\varepsilon)I(a) = I\big((1-\varepsilon)a\big) \leqq I(x) \leqq I\big((1+\varepsilon)a\big) = (1+\varepsilon)I(a)$, that is $|I(x)-I(a)| \leqq \varepsilon I(a)$. □

The Results on Convex Functions. The second proof of the basic step 2.1 will be based on certain facts on convex functions, and will lead to fortified versions. We state and prove the results for convex functions on convex subsets of real vector spaces instead of intervals in $\mathbb{R}$, because this produces no further effort and is the form in which the results will be needed. The first result is a mild fortification of [5] theorem 88, and is included for the sake of completeness. Note that [5] has a different definition of convex functions.

2.3 Remark. *Let* $K \subset E$ *be a nonvoid convex subset of the real vector space* E *and* $f : K \to \mathbb{R}$. *Assume that*

i) *for each pair* $u,v \in K$ *there exists* $0<t<1$ *such that*

$$f\big((1-t)u+tv\big) \leqq (1-t)f(u)+tf(v);$$

ii) *for each pair* $u, v \in K$ *the function* $t \mapsto f\big((1-t)u+tv\big)$ *is continuous on* $0<t<1$.

Then f *is convex.*

Proof. Assume not. Then there exist $u, v \in K$ such that the set

$$M := \{0<t<1 : f\big((1-t)u+tv\big) > (1-t)f(u)+tf(v)\} \quad \text{is } \neq \varnothing.$$

It follows from ii) that $M \subset]0,1[$ is open. Let $T \subset M$ be one of its connected components, that is $T =]\alpha, \beta[$ with $0 \leqq \alpha < \beta \leqq 1$. Then

$$\begin{aligned} f\big((1-\alpha)u+\alpha v\big) &\leqq (1-\alpha)f(u)+\alpha f(v), \\ f\big((1-\beta)u+\beta v\big) &\leqq (1-\beta)f(u)+\beta f(v). \end{aligned}$$

In fact, the first assertion is obvious when $\alpha = 0$, and for $0<\alpha<1$ as well since $\alpha \notin M$. Now from i) we have an $0<s<1$ such that

$$\begin{aligned} & f\Big((1-s)\big((1-\alpha)u+\alpha v\big)+s\big((1-\beta)u+\beta v\big)\Big) \\ & \leqq (1-s)f\big((1-\alpha)u+\alpha v\big)+sf\big((1-\beta)u+\beta v\big) \\ & \leqq (1-s)\big((1-\alpha)f(u)+\alpha f(v)\big)+s\big((1-\beta)f(u)+\beta f(v)\big). \end{aligned}$$

This means that for $t := (1-s)\alpha+s\beta$ we have $f\big((1-t)u+tv\big) \leqq (1-t)f(u)+tf(v)$, which contradicts the fact that $\alpha<t<\beta$ and hence $t \in M$. $\square$

The next result is much harder. The author must admit that he does not understand in full what is behind it. The special case $\varphi = \text{const}$ is [5] theorem 111.

2.4 Theorem. *Let* $K \subset E$ *be a nonvoid convex subset of the real vector space* E *and* $f : K \to \mathbb{R}$. *Assume that*

i) *there exists an affine function* $\varphi : K \to]0, \infty[$ *such that*

$$f\Big(\frac{\sqrt{\varphi(v)}u+\sqrt{\varphi(u)}v}{\sqrt{\varphi(v)}+\sqrt{\varphi(u)}}\Big) \leqq \frac{\sqrt{\varphi(v)}f(u)+\sqrt{\varphi(u)}f(v)}{\sqrt{\varphi(v)}+\sqrt{\varphi(u)}} \quad \text{for } u, v \in K;$$

ii) *for each pair* $u, v \in K$ *the function* $t \mapsto f\big((1-t)u+tv\big)$ *is bounded above on some nondegenerate subinterval of* $\{t \in \mathbb{R} : (1-t)u+tv \in K\}$.

Then f *is convex.*

We start with a technical lemma.

2.5 Lemma. *For* $t \in \mathbb{R}$ *define* $h_t : [0,1] \to \mathbb{R}$ *to be*

$$h_t(s) = s \text{ when } t=0 \quad \text{and} \quad h_t(s) = \frac{e^{ts}-1}{e^t-1} \text{ when } t \neq 0.$$

Then 1) $1-h_t(s)+h_t(s)e^t = e^{ts}$.

2) h_t *is* C^1 *with* $e^{-|t|} \leqq h_t'(s) \leqq e^{|t|}$ *and hence strictly increasing. Moreover* $h_t(0)=0$ *and* $h_t(1)=1$, *so that* h_t *is a bijection of* $[0,1]$.

3) *For* $\alpha, \beta \in [0,1]$ *we have*

$$h_t\big(\frac{\alpha+\beta}{2}\big) = \frac{e^{t\beta/2}}{e^{t\beta/2}+e^{t\alpha/2}}h_t(\alpha)+\frac{e^{t\alpha/2}}{e^{t\beta/2}+e^{t\alpha/2}}h_t(\beta).$$

Proof of 2.5. 2) h_t is C^1 with $h_t' = \frac{t}{e^t-1}e^{ts}$ in case $t \neq 0$. The estimation then follows from $x \leqq e^x - 1 \leqq xe^x$ for $x \in \mathbb{R}$. 3) In case $t \neq 0$ we have

$$\begin{aligned} h_t\Big(\frac{\alpha+\beta}{2}\Big)(e^t-1)\big(e^{t\beta/2}+e^{t\alpha/2}\big) &= \big(e^{t\beta/2}+e^{t\alpha/2}\big)\big(e^{t\alpha/2}e^{t\beta/2}-1\big) \\ &= e^{t\beta/2}\big(e^{t\alpha}-1\big)+e^{t\alpha/2}\big(e^{t\beta}-1\big), \end{aligned}$$

and hence the assertion. □

After this we subdivide the proof of 2.4 into four parts. For each part we fix $u, v \in K$ and put $t = \log\frac{\varphi(v)}{\varphi(u)}$, so that $\varphi(v) = e^t\varphi(u)$, and take $h_t : [0,1] \to [0,1]$ as defined in 2.5.

Part 1. Define

$$M := \{s \in [0,1] : f\big((1-h_t(s))u + h_t(s)v\big) \leqq (1-h_t(s))f(u) + h_t(s)f(v)\} \subset [0,1].$$

Thus $0, 1 \in M$. We claim that $\alpha, \beta \in M \Rightarrow \frac{\alpha+\beta}{2} \in M$. Hence M contains all dyadic rationals $s \in [0,1]$.

Proof. Fix $\alpha, \beta \in [0,1]$ and put

$$\begin{aligned} a &= (1-h_t(\alpha))u + h_t(\alpha)v, \\ b &= (1-h_t(\beta))u + h_t(\beta)v, \\ c &= \big(1-h_t\big(\frac{\alpha+\beta}{2}\big)\big)u + h_t\big(\frac{\alpha+\beta}{2}\big)v. \end{aligned}$$

Then 2.5.3) furnishes

$$c = \frac{e^{t\beta/2}}{e^{t\beta/2}+e^{t\alpha/2}}a + \frac{e^{t\alpha/2}}{e^{t\beta/2}+e^{t\alpha/2}}b.$$

On the other hand 2.5.1) implies that $\varphi(a) = e^{t\alpha}\varphi(u)$ and $\varphi(b) = e^{t\beta}\varphi(u)$, and hence

$$\frac{e^{t\beta/2}}{e^{t\beta/2}+e^{t\alpha/2}} = \frac{\sqrt{\varphi(b)}}{\sqrt{\varphi(b)}+\sqrt{\varphi(a)}} \quad \text{and} \quad \frac{e^{t\alpha/2}}{e^{t\beta/2}+e^{t\alpha/2}} = \frac{\sqrt{\varphi(a)}}{\sqrt{\varphi(b)}+\sqrt{\varphi(a)}}.$$

Thus by assumption i) we have

$$f(c) \leqq \frac{e^{t\beta/2}}{e^{t\beta/2}+e^{t\alpha/2}}f(a) + \frac{e^{t\alpha/2}}{e^{t\beta/2}+e^{t\alpha/2}}f(b).$$

Now assume that $\alpha, \beta \in M$, that is

$$\begin{aligned} f(a) &\leqq (1-h_t(\alpha))f(u) + h_t(\alpha)f(v), \\ f(b) &\leqq (1-h_t(\beta))f(u) + h_t(\beta)f(v). \end{aligned}$$

Then these inequalities combine with 2.5.3) to furnish

$$f(c) \leqq \big(1-h_t\big(\frac{\alpha+\beta}{2}\big)\big)f(u) + h_t\big(\frac{\alpha+\beta}{2}\big)f(v),$$

which means that $\frac{\alpha+\beta}{2} \in M$. □

Part 2. Fix $0 \leqq \alpha < \beta \leqq 1$ and form as before

$$\begin{aligned} a &= (1-h_t(\alpha))u + h_t(\alpha)v, \\ b &= (1-h_t(\beta))u + h_t(\beta)v. \end{aligned}$$

For these $a,b\in K$ put $\tau=\log\frac{\varphi(b)}{\varphi(a)}$, so that $\varphi(b)=e^\tau\varphi(a)$, and take h_τ after 2.5. Then 1) $\tau=t(\beta-\alpha)$. 2) For $\alpha\leqq s\leqq\beta$ we have

$$(1-h_t(s))u+h_t(s)v=(1-h_\tau(\sigma))a+h_\tau(\sigma)b \quad \text{with } \sigma=\frac{s-\alpha}{\beta-\alpha}.$$

Proof. 1) follows from $\varphi(a)=e^{t\alpha}\varphi(u)$ and $\varphi(b)=e^{t\beta}\varphi(u)$. 2) From 1) we have $\tau\sigma=t(s-\alpha)$. Then

$$\begin{aligned} & (1-h_\tau(\sigma))a+h_\tau(\sigma)b \\ = \; & (1-h_\tau(\sigma))\big((1-h_t(\alpha))u+h_t(\alpha)v\big)+h_\tau(\sigma)\big((1-h_t(\beta))u+h_t(\beta)v\big) \\ = \; & (1-\zeta)u+\zeta v \quad \text{with} \quad \zeta=(1-h_\tau(\sigma))h_t(\alpha)+h_\tau(\sigma)h_t(\beta). \end{aligned}$$

In case $t=0$ (and hence $\tau=0$) we have

$$\zeta=(1-\sigma)\alpha+\sigma\beta=\frac{\beta-s}{\beta-\alpha}\alpha+\frac{s-\alpha}{\beta-\alpha}\beta=s,$$

and in case $t\neq 0$ (and hence $\tau\neq 0$) we have

$$\begin{aligned} (e^t-1)(e^\tau-1)\zeta &= (e^\tau-e^{\tau\sigma})(e^{t\alpha}-1)+(e^{\tau\sigma}-1)(e^{t\beta}-1) \\ &= \big(e^{t(\beta-\alpha)}-e^{t(s-\alpha)}\big)(e^{t\alpha}-1)+\big(e^{t(s-\alpha)}-1\big)(e^{t\beta}-1) \\ &= (e^{ts}-1)\big(e^{t(\beta-\alpha)}-1\big)=(e^{ts}-1)(e^\tau-1), \end{aligned}$$

that is $\zeta=h_t(s)$. □

Part 3. The function $\lambda\mapsto f\big((1-\lambda)u+\lambda v\big)$ is bounded above on $0\leqq\lambda\leqq 1$.

Proof. 1) By assumption ii) we have for fixed $u,v\in K$ real $\alpha<\beta$ such that

$$(1-t)u+tv\in K \text{ and } f\big((1-t)u+tv\big)\leqq c<\infty \quad \text{for } \alpha\leqq t\leqq\beta.$$

We first reduce the situation to the special case $0\leqq\alpha<\beta\leqq 1$. In fact, for $\xi:=0\wedge\alpha$ and $\eta:=1\vee\beta$ we have $a:=(1-\xi)u+\xi v\in K$ and $b:=(1-\eta)u+\eta v\in K$, and

$$(1-s)a+sb=(1-t)u+tv \quad \text{with} \quad t=(1-s)\xi+s\eta.$$

It follows that both $\{(1-t)u+tv:0\leqq t\leqq 1\}$ and $\{(1-t)u+tv:\alpha\leqq t\leqq\beta\}$ are contained in $\{(1-t)u+tv:\xi\leqq t\leqq\eta\}=\{(1-s)a+sb:0\leqq s\leqq 1\}$. Thus for $a,b\in K$ we are in the special case, and the assertion for a,b implies that for u,v.

2) For $u,v\in K$ in the special case there are $0<\alpha<\beta<1$ such that

$$f\big((1-h_t(s))u+h_t(s)v\big)\leqq \text{ some real } c \text{ for } \alpha\leqq s\leqq\beta \text{ and } s=0,1.$$

To be shown is that this relation holds true for all $0\leqq s\leqq 1$, that is remains true for $0<s<\alpha$ and $\beta<s<1$. We shall restrict ourselves to the case $0<s<\alpha$, and put $x=(1-h_t(s))u+h_t(s)v$. We choose $n\in\mathbb{N}$ such that $2^ns\frac{\beta-\alpha}{\alpha\beta}=2^n\frac{s}{\alpha}-2^n\frac{s}{\beta}>1$. Then there exists $k\in\mathbb{Z}$ with $2^n\frac{s}{\beta}<k<2^n\frac{s}{\alpha}$, so that $1\leqq k<2^n$. We put $\delta:=2^n\frac{s}{k}$ and obtain $0<s<\alpha<\delta<\beta<1$. Now we apply Part 2 to the pair $0<\delta$ with $u=(1-h_t(0))u+h_t(0)v$ and $w:=(1-h_t(\delta))u+h_t(\delta)v$, and to $0<s<\delta$ with $x=(1-h_t(s))u+h_t(s)v$. It follows that $x=(1-h_\tau(\sigma))u+h_\tau(\sigma)w$ with $\sigma=\frac{s-0}{\delta-0}=\frac{s}{\delta}=k2^{-n}$, which thus is dyadic rational. From $f(u),f(v)\leqq c$ and Part 1 we therefore obtain $f(x)\leqq c$. □

Part 4. We have

$$f\big((1-h_t(s))u+h_t(s)v\big)\leqq(1-h_t(s))f(u)+h_t(s)f(v) \quad \text{for } 0\leqq s\leqq 1.$$

In view of 2.5.2) this is the final assertion.

Proof. 0) Assume that the assertion is not true. Then there exists $0 < s < 1$ such that

$$x = (1 - h_t(s))u + h_t(s)v \quad \text{fulfils} \quad f(x) > (1 - h_t(s))f(u) + h_t(s)f(v) =: R.$$

1) In view of Part 1 then s is not dyadic rational. Thus for each $n \in \mathbb{N}$ there exists $p(n) \in \mathbb{Z}$ such that $0 \leqq 2^{-n}(p(n) - 1) < s < 2^{-n}p(n) \leqq 1$; hence $1 \leqq p(n) \leqq 2^n$. We put $\alpha_n = 2^{-n}(p(n) - 1)$, so that

$$0 \leqq \alpha_n < s < \alpha_n + 2^{-n} \leqq 1 \quad \text{and hence} \quad 0 < s - \alpha_n < 2^{-n}.$$

Then let $a_n := (1 - h_t(\alpha_n))u + h_t(\alpha_n)v$. From Part 1 we have

$$f(a_n) \leqq (1 - h_t(\alpha_n))f(u) + h_t(\alpha_n)f(v).$$

2) From $s < 1$ we see that $\frac{1-\alpha_n}{s-\alpha_n} > 1$. Now $\frac{1-\alpha_n}{s-\alpha_n}$ cannot be $= 2^q$ for the $q \in \mathbb{N}$, because then

$$2^q s = 1 + (2^q - 1)\alpha_n = 1 + 2^{-n}(2^q - 1)(p(n) - 1) \quad \text{and hence} \quad 2^{n+q}s \in \mathbb{N},$$

so that s were dyadic rational. Thus there exists an integer $q(n) \geqq 0$ such that $2^{q(n)} < \frac{1-\alpha_n}{s-\alpha_n} < 2^{q(n)+1}$. From 1) it follows that

$$2^{q(n)+1} > \frac{1-s}{s-\alpha_n} > 2^n(1-s) \quad \text{and hence} \quad 2^{q(n)} > 2^{n-1}(1-s).$$

Now we put $\beta_n = \alpha_n + 2^{q(n)}(s - \alpha_n)$, so that

$$2^{q(n)}s = \beta_n + \big(2^{q(n)} - 1\big)\alpha_n \quad \text{or} \quad s = \big(1 - 2^{-q(n)}\big)\alpha_n + 2^{-q(n)}\beta_n.$$

It follows that $0 \leqq \alpha_n < s \leqq \beta_n < 1$. We also form $b_n := (1 - h_t(\beta_n))u + h_t(\beta_n)v$.

3) After this we apply Part 2 to the pair $\alpha_n < \beta_n$ with a_n and b_n, and to $\alpha_n < s \leqq \beta_n$ with x. It follows that

$$x = \big(1 - h_{\tau(n)}(\sigma(n))\big)a_n + h_{\tau(n)}(\sigma(n))b_n,$$

where $\tau(n) = t(\beta_n - \alpha_n)$ and hence $|\tau(n)| \leqq |t|$, and $\sigma(n) = \frac{s-\alpha_n}{\beta_n-\alpha_n} = 2^{-q(n)}$. Thus from Part 1 we obtain

$$f(x) \leqq \big(1 - h_{\tau(n)}(\sigma(n))\big)f(a_n) + h_{\tau(n)}(\sigma(n))f(b_n).$$

4) For the final step we recall the last formulas of 0)1)3) above. Moreover we see from 2.5.2) combined with 1)2)3) that

$$\begin{aligned} 0 < h_t(s) - h_t(\alpha_n) &\leqq e^{|t|}(s - \alpha_n) < 2^{-n}e^{|t|}, \\ 0 < h_{\tau(n)}(\sigma(n)) &\leqq e^{|\tau(n)|}\sigma(n) \leqq 2^{-q(n)}e^{|t|} < 2^{-n}\frac{2}{1-s}e^{|t|}. \end{aligned}$$

It follows that

$$\begin{aligned} &\big(1 - h_{\tau(n)}(\sigma(n))\big)(f(x) - R) \\ =\ & f(x) - h_{\tau(n)}(\sigma(n))f(x) - \big(1 - h_{\tau(n)}(\sigma(n))\big)(R - f(a_n)) \\ &\qquad\qquad - \big(1 - h_{\tau(n)}(\sigma(n))\big)f(a_n) \\ \leqq\ & h_{\tau(n)}(\sigma(n))f(b_n) - h_{\tau(n)}(\sigma(n))f(x) \\ &\qquad\qquad - \big(1 - h_{\tau(n)}(\sigma(n))\big)(h_t(s) - h_t(\alpha_n))(f(v) - f(u)) \\ \leqq\ & h_{\tau(n)}(\sigma(n))c + h_{\tau(n)}(\sigma(n))|f(x)| + (h_t(s) - h_t(\alpha_n))|f(v) - f(u)| \\ \leqq\ & 2^{-n}e^{|t|}\Big(\frac{2}{1-s}\big(c + |f(x)|\big) + |f(v) - f(u)|\Big) \quad \text{with } 0 < c < \infty \text{ from Part 3.} \end{aligned}$$

For $n \to \infty$ we obtain $f(x) \leqq R$, which contradicts the assumption. This completes the proof of 2.4. $\square$

The Fortified Basic Step and the Second Proof. The fortified versions of the basic step 2.1 which result from 2.3 and 2.4 read as follows.

2.6 PROPOSITION. *Assume that I $:]0,\infty[^n \to \mathbb{R}$ is positive-homogeneous. The further assumptions are*

1) *for each pair $u,v \in]0,\infty[^n$ the function $t \mapsto I\big((1-t)u+tv\big)$ is continuous on $0<t<1$;*

2) *for each pair $u,v \in]0,\infty[^n$ the function $t \mapsto I\big((1-t)u+tv\big)$ is bounded (above/below) on some nondegenerate subinterval of $\{t \in \mathbb{R} : (1-t)u+tv > 0\}$.*

Of course 1) $\Longrightarrow$ 2). *Each of these assumptions implies that I fulfils the fundamental implication.*

The point is that the efforts in order to obtain the fundamental implication from the two assumptions 1)2) are so different: Under 1) we shall use 2.3, while under 2) we have to invoke 2.4. This time the proof is via induction. The case $n=1$ is obvious. The induction step $1 \leqq n \Rightarrow n+1$ will be based on the lemma which follows.

2.7 LEMMA. *Assume that I $:]0,\infty[^{n+1} \to \mathbb{R}$ is positive-homogeneous and (sub/ super)modular $\vee\wedge$, and that for each $0<c<\infty$ the function $]0,\infty[^n \to \mathbb{R} : x \mapsto I(cx_1,x)$ is (sub/super)additive. Then the function f $:]0,\infty[^n \to \mathbb{R}$ defined to be $f(x)=I(1,x)$ fulfils*

$$f\Big(\frac{\sqrt{v_1}u+\sqrt{u_1}v}{\sqrt{v_1}+\sqrt{u_1}}\Big) \leqq/\geqq \frac{\sqrt{v_1}f(u)+\sqrt{u_1}f(v)}{\sqrt{v_1}+\sqrt{u_1}} \quad \textit{for } u,v \in]0,\infty[^n.$$

Proof of 2.7. In the sequel we shall represent the points $x=(x_0,x_1,\cdots,x_n) \in \mathbb{R}^{n+1}$ in the form $x=(x_0,x')$ with $x'=(x_1,\cdots,x_n) \in \mathbb{R}^n$. Now fix $u,v \in]0,\infty[^n$ and define $a,b \in]0,\infty[^{n+1}$ to be

$$\begin{aligned} a &= \sqrt{v_1}(1,u) \vee \sqrt{u_1}(1,v), \\ b &= \sqrt{v_1}(1,u) \wedge \sqrt{u_1}(1,v). \end{aligned}$$

Then

$$\begin{aligned} a_0 &= \sqrt{v_1} \vee \sqrt{u_1} \quad \text{and} \quad a_1 = \sqrt{u_1 v_1}(\sqrt{u_1} \vee \sqrt{v_1}) = \sqrt{u_1 v_1}\, a_0, \\ b_0 &= \sqrt{v_1} \wedge \sqrt{u_1} \quad \text{and} \quad b_1 = \sqrt{u_1 v_1}(\sqrt{u_1} \wedge \sqrt{v_1}) = \sqrt{u_1 v_1}\, b_0, \end{aligned}$$

and hence $a_0/a_1 = b_0/b_1 =: c$. Thus the assumption furnishes on the one hand

$$\begin{aligned} I(a+b) &= I(ca_1+cb_1, a'+b') = I\big((ca_1,a')+(cb_1,b')\big) \\ &\leqq/\geqq I(ca_1,a')+I(cb_1,b') = I(a)+I(b), \end{aligned}$$

and on the other hand

$$I(a)+I(b) \leqq/\geqq I\big(\sqrt{v_1}(1,u)\big)+I\big(\sqrt{u_1}(1,v)\big) = \sqrt{v_1}f(u)+\sqrt{u_1}f(v).$$

Moreover we have

$$\begin{aligned} a+b &= \sqrt{v_1}(1,u)+\sqrt{u_1}(1,v) = (\sqrt{v_1}+\sqrt{u_1})\Big(1, \frac{\sqrt{v_1}u+\sqrt{u_1}v}{\sqrt{v_1}+\sqrt{u_1}}\Big), \\ I(a+b) &= (\sqrt{v_1}+\sqrt{u_1}) f\Big(\frac{\sqrt{v_1}u+\sqrt{u_1}v}{\sqrt{v_1}+\sqrt{u_1}}\Big). \end{aligned}$$

These formulas combine to furnish the assertion. □

Proof of the induction step in 2.6. Let $I\ :]0,\infty[^{n+1}$ be as assumed in 2.6 and (sub/super)modular $\vee\wedge$. For fixed $0<c<\infty$ define $J\ :]0,\infty[^n\to\mathbb{R}$ to be $J(x)=I(cx_1,x)$. It is obvious that J inherits all assumptions from I, and hence by the induction hypothesis is (sub/super)additive. Thus 2.7 ensures that the results 2.3 and 2.4 can be applied, each in the respective case, to the function $\pm f$ for $f\ :\ ]0,\infty[^n\to\mathbb{R}$ defined $f(x)=I(1,x)$. Therefore f is convex/concave. Then we see for $u,v\in]0,\infty[^{n+1}$ that

$$\begin{aligned} I(u+v) &= (u_0+v_0)I\Big(1,\frac{u'+v'}{u_0+v_0}\Big)=(u_0+v_0)f\big(\frac{u_0}{u_0+v_0}\big(\frac{u'}{u_0}\big)+\frac{v_0}{u_0+v_0}\big(\frac{v'}{v_0}\big)\big)\\ &\leqq/\geqq\ u_0f\big(\frac{u'}{u_0}\big)+v_0f\big(\frac{v'}{v_0}\big)=u_0I\big(1,\frac{u'}{u_0}\big)+v_0I\big(1,\frac{v'}{v_0}\big)=I(u)+I(v). \end{aligned}$$

This is the assertion. □

The Final Results in the Finite Situation. The basic step 2.1 was for the open positive cone $]0,\infty[^n$, but it implies at once the identical result for the full positive cone $[0,\infty[^n$ of $\mathbb{R}^n$.

2.8 Remark. *Assume that* $I:[0,\infty[^n\to\mathbb{R}$ *is positive-homogeneous and continuous. Then it fulfils the fundamental implication.*

However, this assertion is kind of a dead end, because it carries an unnatural proper restriction: For $I\ :\ [0,\infty[^n\to\ \mathbb{R}$ positive-homogeneous the properties (sub/super)additive mean *convex/concave*, and it is well-known that these functions need not be continuous at the boundaries of their domains. An example is the function

$$I:[0,\infty[^2\to[0,\infty[\quad\text{with } I(x)=x_1 \text{ for } x_2>0 \text{ and } I(x)=0 \text{ for } x_2=0,$$

which is positive-homogeneous, supermodular and superadditive, and moreover increasing. Thus we continue to extend the specialization 2.2 and the fortified version 2.6, which will not produce such restrictions but will require a bit of further work. Moreover we want to admit that the function I attains the value ∞.

2.9 Theorem. *Assume that* $I\ :\ [0,\infty[^n\to\ [0,\infty]$ *is positive-homogeneous and increasing. Then* I *fulfils the fundamental implication.*

2.10 Theorem. *Assume that* $I:[0,\infty[^n\to]-\infty,\infty]$ *is positive-homogeneous, and for the pairs* $u\leqq v$ *in* $[0,\infty[^n$ *fulfils* $I(u)=\infty\Rightarrow I(v)=\infty$. *The further assumptions are*

1) *for each pair* $u,v\in[0,\infty[^n$ *with* $I(u+v)<\infty$ *the function* $t\mapsto I\big((1-t)u+tv\big)$ *is continuous on* $0<t<1$*;*
2) *for each pair* $u,v\in[0,\infty[^n$ *with* $I(u+v)<\infty$ *the function* $t\mapsto I\big((1-t)u+tv\big)$ *is bounded* (*above/below*) *on some nondegenerate subinterval of* $\{t\in\mathbb{R}:(1-t)u+tv\geqq0\}$.

Of course 1) $\Longrightarrow$ 2). *Each of these assumptions implies that* I *fulfils the fundamental implication.*

Theorem 2.9 will be the foundation for the treatment of the *full* situation in section 3, while theorem 2.10 is our final result in the finite situation. We want to obtain the two theorems with a simultaneous proof.

Proof of 2.9 and 2.10. First of all $I(0) = I(t0) = tI(0)$ for $0 < t < \infty$ implies that $I(0) = 0$ or $I(0) = \infty$, and from $I(0) = \infty$ we have $I \equiv \infty$. Thus we can assume that $I(0) = 0$. This in particular settles the case $n = 1$: We have $I(x) = cx$ for $x \in]0, \infty[$ with $c = I(1) \in] - \infty, \infty]$, and hence I is additive. Thus it remains to perform the induction step $1 \leqq n \Rightarrow n + 1$.

For this purpose we assume $I : [0, \infty[^{n+1} \rightarrow] - \infty, \infty]$ to be positive-homogeneous with $I(0) = 0$ and to fulfil the further assumptions of one fixed situation out of the three ones under consideration, and moreover to be (sub/super)modular $\vee\wedge$. We define $J : [0, \infty[^{n} \rightarrow] - \infty, \infty]$ to be $J(x) = I(0, x)$. It is obvious that J inherits all assumptions from I, and hence by the induction hypothesis is (sub/super)additive.

Then in all situations we have the alternative between the two cases

$$(\star)\ I < \infty \quad \text{and} \quad (\star\star)\ I|]0, \infty[^{n+1} \equiv \infty.$$

In fact, if $I(a) < \infty$ for some $a \in]0, \infty[^{n+1}$ then for each $x \in [0, \infty[^{n+1}$ the relation $x \leqq ta$ with some $0 < t < \infty$ enforces $I(x) < \infty$. Thus we have to prove that I is (sub/super)additive both under $(\star)$ and $(\star\star)$.

First assume that $(\star)$ $I < \infty$. In case 2.9 it follows from 2.2 that $I|]0, \infty[^{n+1}$ is (sub/super)additive. In case 2.10 the same follows from 2.6. It remains to consider a pair of $u, v \in [0, \infty[^{n+1}$ which are not both in $]0, \infty[^{n+1}$. We can assume that $u_0 = 0$, and use the notation $x = (x_0, x')$ as in the proof of 2.7 above. We have

$$u + v = (0, u' + v') \vee v \quad \text{and} \quad (0, v') = (0, u' + v') \wedge v,$$

which implies that

$$\begin{aligned} I(u + v) + I(0, v') &\leqq/\geqq I(0, u' + v') + I(v) = J(u' + v') + I(v) \\ \leqq/\geqq\ J(u') + J(v') + I(v) &= I(0, u') + I(0, v') + I(v) = I(u) + I(v) + I(0, v'), \end{aligned}$$

and hence $I(u + v) \leqq/\geqq I(u) + I(v)$ since $I(0, v') < \infty$.

Then assume that $(\star\star)$ $I|]0, \infty[^{n+1} \equiv \infty$ and fix $u, v \in [0, \infty[^{n+1}$. There are the two cases

i) there exists $l \in \{0, 1, \cdots, n\}$ with $u_l = v_l = 0$, and
ii) for all $l \in \{0, 1, \cdots, n\}$ we have $u_l + v_l \geqq u_l \vee v_l > 0$.

In case i) we can assume that $u_0 = v_0 = 0$. Then $I(u + v) = J(u' + v') \leqq/\geqq J(u') + J(v') = I(u) + I(v)$. In case ii) we have $u + v, u \vee v \in]0, \infty[^{n+1}$ and hence $I(u + v) = I(u \vee v) = \infty$. The relation $I(u \vee v) = \infty$ settles the sub case because it enforces $I(u) + I(v) = \infty$, and the relation $I(u + v) = \infty$ settles the super case. $\square$

At last we transfer the results into the versions which follow. Let X be a nonvoid set, and define $F(X) \subset [0, \infty[^X$ to consist of the functions $f \in [0, \infty[^X$ with finitely many values. $F(X)$ is a convex cone with $0 \in F(X)$ and stable under the pointwise lattice operations $\vee\wedge$.

2.11 Theorem. *Assume that $I : F(X) \rightarrow [0, \infty]$ is positive-homogeneous and increasing. Then I fulfils the fundamental implication.*

2.12 Theorem. *Assume that $I : F(X) \rightarrow] - \infty, \infty]$ is positive-homogeneous, and for the pairs $u \leqq v$ in $F(X)$ fulfils $I(u) = \infty \Rightarrow I(v) = \infty$. The further assumptions are*

1) *for each pair $u, v \in F(X)$ with $I(u + v) < \infty$ the function $t \mapsto I\big((1 - t)u + tv\big)$ is continuous on $0 < t < 1$;*

2) *for each pair* $u, v \in F(X)$ *with* $I(u+v) < \infty$ *the function* $t \mapsto I\big((1-t)u + tv\big)$ *is bounded* (*above/below*) *on some nondegenerate subinterval of* $\{t \in \mathbb{R} : (1-t)u + tv \geqq 0\}$.

Of course 1) $\Longrightarrow$ 2). *Each of these assumptions implies that I fulfils the fundamental implication.*

Proof of 2.11 and 2.12. Assume $I : F(X) \to]-\infty, \infty]$ to be positive-homogeneous and to fulfil the further assumptions of one of our three situations, and moreover to be (sub/super)modular $\vee\wedge$. Fix $u, v \in F(X)$ and let $X = X(1) \cup \cdots \cup X(n)$ be a decomposition of X such that $u|X(l) = a_l$ and $v|X(l) = b_l$ for $1 \leqq l \leqq n$, and form $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ in $[0, \infty[^n$. Define $\vartheta : [0, \infty[^n \to F(X)$ to be

$$x = (x_1, \cdots, x_n) \mapsto \vartheta(x) = \sum_{l=1}^{n} x_l \chi_{X(l)},$$

so that $\vartheta(a) = u$ and $\vartheta(b) = v$. The map ϑ is positive-linear and increasing under $\leqq$, and commutes with the pointwise $\vee\wedge$. Then define $J = I \circ \vartheta : [0, \infty[^n \to]-\infty, \infty]$. It is obvious that J inherits all assumptions from I, and hence is (sub/super)additive by 2.9 and 2.10. It follows that $I(u+v) = I(\vartheta(a+b)) = J(a+b) \leqq/\geqq J(a) + J(b) = I(\vartheta(a)) + I(\vartheta(b)) = I(u) + I(v)$. $\square$

3. **The Full Situation**

Return to the Choquet Integral. We follow the model of the Choquet integral and have to recall some further of its properties. As above we refer to [7] section 11 and [8] section 2.

We need a few terms on nonvoid function systems $S \subset [0, \infty]^X$ and functionals $I : S \to [0, \infty]$. We define S to be *Stonean* iff $f \in S \Rightarrow f \wedge t, (f-t)^+ \in S$ for $0 < t < \infty$; note that $f = f \wedge t + (f-t)^+$. In this case I is called *Stonean* iff

$$I(f) = I(f \wedge t) + I((f-t)^+) \quad \text{for all } f \in S \text{ and } 0 < t < \infty.$$

Moreover an *increasing* I is called *truncable* iff

$$I(f) = \sup\{I((f-a)^+ \wedge (b-a)) : 0 < a < b < \infty\} \quad \text{for all } f \in S.$$

We note that this relation holds true when f on its $[f > 0]$ fulfils $\alpha \leqq f \leqq \beta$ for some constants $0 < \alpha < \beta < \infty$, because then $f \leqq (\alpha/\alpha - a)((f-a)^+ \wedge (b-a))$ for $0 < a < \alpha < \beta < b < \infty$. Thus to be truncable is a mild continuity condition on I.

Next assume that $0 \in S$ and that I is increasing with $I(0) = 0$. Then we define the *envelopes* $I^\star, I_\star : [0, \infty]^X \to [0, \infty]$ to be

$$\begin{aligned} I^\star(f) &= \inf\{I(u) : u \in S \text{ with } u \geqq f\} \quad \text{with } \inf \varnothing := \infty, \text{ and} \\ I_\star(f) &= \sup\{I(u) : u \in S \text{ with } u \leqq f\}. \end{aligned}$$

Thus $I_\star \leqq I^\star$ and $I^\star|S = I_\star|S = I$. Moreover $I^\star$ and $I_\star$ are increasing. When S is stable under $\vee\wedge$ then to be *submodular* $\vee\wedge$ carries over from I to $I^\star$, and to be *supermodular* $\vee\wedge$ carries over from I to $I_\star$.

One then notes the properties which follow. The subsequent representation theorem is in essence due to Greco [4]. Let $\mathfrak{S}$ be a lattice of subsets with $\varnothing \in \mathfrak{S}$ in X.

3.1 Properties. i) $\mathrm{UM}(\mathfrak{S})$ *and* $\mathrm{LM}(\mathfrak{S})$ *are Stonean.*

ii) *For an increasing* $\varphi : \mathfrak{S} \to [0,\infty]$ *with* $\varphi(\varnothing) = 0$ *the Choquet integral* $I : I(f) = \oint f d\varphi$ *on* $\mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ *is Stonean and truncable.*

3.2 THEOREM. *Assume that* $S \subset \mathrm{UM}(\mathfrak{S})/\mathrm{LM}(\mathfrak{S})$ *is positive-homogeneous with* $0 \in S$ *and Stonean, and that* $I : S \to [0,\infty]$ *with* $I(0) = 0$ *is increasing. Then*

there exist increasing set functions $\varphi : \mathfrak{S} \to [0,\infty]$ *with* $\varphi(\varnothing) = 0$
which represent $I : I(f) = \oint f d\varphi$ *for all* $f \in S$

iff I *is Stonean and truncable. In this case an increasing* $\varphi : \mathfrak{S} \to [0,\infty]$ *with* $\varphi(\varnothing) = 0$ *represents* I *iff* $I_\star(\chi_A) \leqq \varphi(A) \leqq I^\star(\chi_A)$ *for all* $A \in \mathfrak{S}$.

At this point we return to the final aim of the present enterprise. We observe that the above representation theorem 3.2 allows to formulate the decisive additive behaviour of the Choquet integral $I : I(f) = \oint f d\varphi$, that is the implication $\Longrightarrow$ in (A), *in exclusive terms of the functional* I *without reference to the set function* φ.

3.3 REMARK. *On a nonvoid set* X *the following are equivalent.* i) *For each lattice* $\mathfrak{S}$ *with* $\varnothing \in \mathfrak{S}$ *in* X *and each increasing* $\varphi : \mathfrak{S} \to [0,\infty]$ *with* $\varphi(\varnothing) = 0$ *one has the implication* $\Longrightarrow$ *in* (A).

ii) *For each positive-homogeneous* $S \subset [0,\infty]^X$ *with* $0 \in S$ *which is stable under* $\vee\wedge$ *and Stonean, and for each positive-homogeneous* $I : S \to [0,\infty]$ *with* $I(0) = 0$ *which is increasing, Stonean and truncable, one has the fundamental implication.*

Proof. One obtains ii)$\Longrightarrow$i) as an immediate consequence of 1.2 and 3.1 combined with $\Longrightarrow$ in (M). To see i)$\Longrightarrow$ii) one applies 3.2 to I and $\mathfrak{S} = \mathfrak{P}(X)$, and takes

$$\begin{aligned} \varphi : \varphi(A) &= I^\star(\chi_A) \text{ for } A \subset X \text{ when } I \text{ is submodular } \vee\wedge, \\ \varphi(A) &= I_\star(\chi_A) \text{ for } A \subset X \text{ when } I \text{ is supermodular } \vee\wedge. \end{aligned}$$

Then $\varphi : \mathfrak{P}(X) \to [0,\infty]$ represents I, and is (sub/super)modular in view of the noted properties of the envelopes. Thus i) asserts that $f \mapsto \oint f d\varphi$ on $[0,\infty]^X$ is (sub/super)additive, and hence that I is (sub/super)additive in the sense of our definition. $\square$

The new formulation 3.3.ii) of the implication $\Longrightarrow$ in (A) looks in fact like the *theorem on the fundamental implication* we are in search of. However, there are several additional conditions: Besides the almost familiar condition that I be *increasing* these are the conditions that I be *Stonean* and *truncable* (with the prerequisite one that S be *Stonean*). The condition to be *truncable* can be dismissed as a mild continuity assumption. However, the condition that I be *Stonean* is a critical one, because it expresses that I be *additive* in a certain partial sense, and thus collides with the conclusion. In fact, there are situations where the prospective theorem will be invoked *in order to conclude that* I *is Stonean.* A case in point will be described below. Therefore it is imperative that in a comprehensive version of the theorem like the desired one the assumption that I be *Stonean* does not occur.

Now the fundamental fact is that the above reformulation 3.3.ii) holds true *without the assumption that* I *be Stonean.* This will be the main and final result of the present work. It is much more comprehensive than 3.3.ii).

The Main Theorem. 3.4 THEOREM. *Assume that the positive-homogeneous* $S \subset [0,\infty]^X$ *with* $0 \in S$ *is stable under* $\vee\wedge$ *and Stonean, and that the positive-homogeneous* $I : S \to [0,\infty]$ *with* $I(0) = 0$ *is increasing and truncable. Then* I *fulfils the fundamental implication.*

The proof starts from the result 2.11 in the finite situation and proceeds via certain approximations which make essential use of the assumption that I be *increasing*. An important intermediate step is the specialization $S = [0,\infty]^X$.

3.5 SPECIALIZATION. *Assume that the positive-homogeneous* $I : [0,\infty]^X \to [0,\infty]$ *with* $I(0) = 0$ *is increasing and truncable. Then* I *fulfils the fundamental implication.*

The above specialization is the first result 1998 of the author in the present context [8] theorem 1.1. It is, aside from the theories of measure and integration developed in [7], the basic pillar which carries the comprehensive Daniell-Stone and Riesz type representation theorems [8] 5.3 = [9] 6.3 and [8] 5.8 = [9] 6.6. These theorems are the other important application of the new (sub/super)additivity theorem. They will be described in the final subsection below. We note that the applications of [8] 1.1 which served to obtain these theorems were in the proof of [8] 3.10 and had in fact the aim to prove that the functionals under consideration were *Stonean*. For the details we have to refer to that paper.

Proof of 3.5. 0) We first recall from the author's textbook [7] the basic estimation 11.6. We use the version $\geqq$, but the version $>$ would do as well. For $f : X \to \overline{\mathbb{R}}$ and real $a = t(0) < t(1) < \cdots < t(r) = b$ the estimation reads

$$\sum_{l=1}^{r}(t(l)-t(l-1))\chi_{[f\geqq t(l)]} \leqq (f-a)^+ \wedge (b-a) \leqq \sum_{l=1}^{r}(t(l)-t(l-1))\chi_{[f\geqq t(l-1)]}.$$

We also note for $\delta := \max\{t(l)-t(l-1) : 1 \leqq l \leqq r\}$ the estimation

$$\begin{aligned}\sum_{l=1}^{r}(t(l)-t(l-1))\chi_{[f\geqq t(l-1)]} &= \sum_{l=1}^{r}(t(l)-t(l-1))\Big(\chi_{[f\geqq t(l)]} + \chi_{[t(l-1)\leqq f<t(l)]}\Big)\\ &\leqq \sum_{l=1}^{r}(t(l)-t(l-1))\chi_{[f\geqq t(l)]} + \delta\chi_{[a\leqq f<b]}.\end{aligned}$$

1) We prove the sub implication. Assume that I is submodular $\vee\wedge$ and fix $u,v \in [0,\infty]^X$. We can assume that $I(u), I(v) < \infty$. For $0 < a, c < \infty$ let $b = a + c$. For $a = t(0) < t(1) < \cdots < t(r) = b$ we see from 0) that

$$\begin{aligned}((u+v)-2a)^+ \wedge c &= \big((u-a)+(v-a)\big)^+ \wedge c\\ &\leqq \big((u-a)^+ + (v-a)^+\big) \wedge c \leqq (u-a)^+ \wedge c + (v-a)^+ \wedge c\\ &\leqq \sum_{l=1}^{r}(t(l)-t(l-1))\chi_{[u\geqq t(l)]} + \delta\chi_{[a\leqq u<b]}\\ &+ \sum_{l=1}^{r}(t(l)-t(l-1))\chi_{[v\geqq t(l)]} + \delta\chi_{[a\leqq v<b]}.\end{aligned}$$

We know from 2.11 that $I|F(X)$ is subadditive. From this fact and once more from the first estimation in 0) we obtain

$$\begin{aligned}I\big(((u+v)-2a)^+ \wedge c\big) &\leqq I\Big(\sum_{l=1}^{r}(t(l)-t(l-1))\chi_{[u\geqq t(l)]}\Big) + \delta I(\chi_{[a\leqq u<b]})\\ &+ I\Big(\sum_{l=1}^{r}(t(l)-t(l-1))\chi_{[v\geqq t(l)]}\Big) + \delta I(\chi_{[a\leqq v<b]})\\ &\leqq \big(1+\frac{\delta}{a}\big)I(u) + \big(1+\frac{\delta}{a}\big)I(v).\end{aligned}$$

It follows that

$$I\big(((u+v)-2a)^+ \wedge c\big) \leqq I(u) + I(v) \quad \text{for all } 0 < a, c < \infty,$$

and hence $I(u+v) \leqq I(u)+I(v)$ since I is truncable. $\square$

2) We prove the super implication. Assume that I is supermodular $\vee\wedge$ and fix $u,v \in [0,\infty]^X$. We can assume that $I(u+v)<\infty$. For $0<a<b<\infty$ and $a=t(0)<t(1)<\cdots<t(r)=b$ we see from 0) that

$$\begin{aligned}\big(1+\frac{\delta}{a}\big)(u+v) &\geqq (u-a)^+\wedge(b-a)+\delta\chi_{[a\leqq u<b]}\\ &+ (v-a)^+\wedge(b-a)+\delta\chi_{[a\leqq v<b]}\\ &\geqq \sum_{l=1}^{r}(t(l)-t(l-1))\chi_{[u\geqq t(l-1)]}\\ &+ \sum_{l=1}^{r}(t(l)-t(l-1))\chi_{[v\geqq t(l-1)]}.\end{aligned}$$

As before we know from 2.11 that $I|F(X)$ is superadditive. ¿From this fact and once more from the first estimation in 0) we obtain

$$\begin{aligned}\big(1+\frac{\delta}{a}\big)I(u+v) &\geqq I\big(\sum_{l=1}^{r}(t(l)-t(l-1))\chi_{[u\geqq t(l-1)]}\big)\\ &+ I\big(\sum_{l=1}^{r}(t(l)-t(l-1))\chi_{[v\geqq t(l-1)]}\big)\\ &\geqq I\big((u-a)^+\wedge(b-a)\big)+I\big((v-a)^+\wedge(b-a)\big).\end{aligned}$$

It follows that

$$\begin{aligned}I(u+v) &\geqq I\big((u-a)^+\wedge(b-a)\big)+I\big((v-a)^+\wedge(b-a)\big)\\ &\quad\text{for all } 0<a<b<\infty,\end{aligned}$$

and hence $I(u+v)\geqq I(u)+I(v)$ since I is truncable. $\square$

Proof of 3.4. 1) The super implication is a simple application of 3.5. Assume that I is supermodular $\vee\wedge$.

1.1) We know that the envelope $I_\star:[0,\infty]^X\to[0,\infty]$ is positive-homogeneous and increasing and supermodular $\vee\wedge$. We claim that $I_\star$ is truncable as well. To see this fix $f\in[0,\infty]^X$ and $c<I_\star(f)$, and then $u\in S$ with $u\leqq f$ and $c<I(u)$. Since S is Stonean and I is truncable there exist $0<a<b<\infty$ such that $c<I((u-a)^+\wedge(b-a))$. It follows that $c<I_\star((f-a)^+\wedge(b-a))$ and hence the present claim.

1.2) We conclude from 3.5 that $I_\star$ is superadditive. In view of our definition therefore $I_\star|S=I$ is superadditive as well. $\square$

2) The sub implication is more involved. Assume that I is submodular $\vee\wedge$.

2.1) We know that the envelope $I^\star:[0,\infty]^X\to[0,\infty]$ is positive-homogeneous and increasing and submodular $\vee\wedge$. But we cannot assert that $I^\star$ is truncable.

2.2) In order to proceed we consider for fixed $0<a<b<\infty$ the map $[0,\infty]\to[0,\infty]$ defined to be $x\mapsto x_a^b:=(x-a)^+\wedge(b-a)$. Its relevant properties are as follows.

i) $(tx)_{ta}^{tb}=t(x_a^b)$ for $0<t<\infty$.
ii) The map $x\mapsto x_a^b$ is increasing.
iii) x_a^b is decreasing in a and increasing in b.
iv) $(u\vee v)_a^b=(u_a^b)\vee(v_a^b)$ and $(u\wedge v)_a^b=(u_a^b)\wedge(v_a^b)$.
v) For $0<s<a<b<\infty$ we have $x_a^b=(x_s^b)_{a-s}^{b-s}$.

Here i)ii)iii) are obvious, and iv) is an immediate consequence of ii). To see v) one distinguishes the three cases $x \in [0,a], [a,b], [b,\infty]$.

2.3) After this we define the functional $J : [0,\infty]^X \to [0,\infty]$ to be

$$J(f) = \sup\{I^\star(f_a^b) : 0 < a < b < \infty\} \quad \text{with} \quad f_a^b := (f-a)^+ \wedge (b-a).$$

Then first of all $J|S = I$, because S is Stonean and $I^\star|S = I$ and I is truncable. Next one concludes from i) that J is positive-homogeneous, from ii) that J is increasing, and from iii)iv) that J is submodular $\vee\wedge$. We claim that J is truncable as well. To see this fix $f \in [0,\infty]^X$ and $c < J(f)$, and then $0 < a < b < \infty$ such that $c < I^\star(f_a^b)$. For $0 < s < a < b < \infty$ we obtain from v) that $c < I^\star\big((f_s^b)_{a-s}^{b-s}\big) \leqq J(f_s^b)$ and hence the present claim.

2.4) We conclude from 3.5 that J is subadditive. In view of our definition therefore $J|S = I$ is subadditive as well. This finishes the proof of 3.4. □

Another Example. We want to add one more example, in order to show what can happen when the functional I is not increasing.

3.6 Example. Let $X \subset \mathbb{R}$ be an interval with $\sup X = \infty$. Define $P \subset [0,\infty[^X$ to consist of the functions $f : X \to [0,\infty[$ which are constant *near* ∞, that means on some upward unbounded subinterval of X, and $Q \subset [0,\infty[^X$ to consist of the functions $f : X \to [0,\infty[$ which are strictly decreasing near ∞. Then $P \cap Q = \varnothing$, and $S := P \cup Q \subset [0,\infty[^X$ is a convex cone with $0 \in S$ which is stable under $\vee\wedge$. Define $I : S \to [0,\infty[$ to be $I(f) = 0$ for $f \in P$ and $I(f) = \lim\limits_{t\to\infty} f(t)$ for $f \in Q$. Thus I is positive-homogeneous with $I(0) = 0$, but of course not increasing. One verifies that I is modular $\vee\wedge$. But I is not additive, since for $u \in P$ with $u = c > 0$ near ∞ and $v \in Q$ one has $u + v \in Q$ with $I(u+v) = c + I(v) > I(v) = I(u) + I(v)$. We note that I has certain continuity properties. Thus S is Stonean, and I is *truncable* in the sense that

$$I\big((f-a)^+ \wedge (b-a)\big) \uparrow I(f) \text{ under } a \downarrow 0 \text{ and } b \uparrow \infty \quad \text{for all } f \in S.$$

Also for each pair $u, v \in S$ the function $t \mapsto I\big((1-t)u + tv\big)$ is continuous on $0 < t < 1$ (but need not be continuous on $0 \leqq t \leqq 1$).

The Daniell-Stone-Riesz representation theorems. The representation theorems of the present subsection are quite different from the former representation theorem 3.2: The aim is to represent particular classes of functionals in terms of certain classes of distinguished set functions, like in the classical Riesz representation theorem, but in a much more extended frame. The basis are the extension theories in measure and integration developed in the author's textbook [7] and in subsequent articles like [8], and summarized in [9]. We recall that there are parallel inner and outer extension theories, and also parallel sequential and nonsequential versions (as usual labelled as σ and τ versions). We also recall the most basic notions: For an increasing set function $\varphi : \mathfrak{S} \to [0,\infty]$ on a set system $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ and for $\bullet = \sigma\tau$ one forms the *envelopes*

$$\varphi_\bullet, \varphi^\bullet : \mathfrak{P}(X) \to [0,\infty] \text{ with the } \textit{satellites } \varphi_\bullet^B : \mathfrak{P}(X) \to [0,\infty[\text{ with } B \in \mathfrak{S},$$

and on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ one defines the *inner* and *outer* $\bullet$ *premeasures* φ. Likewise for an increasing functional $I : S \to [0,\infty]$ on a function class $S \subset [0,\infty]^X$

with $0 \in S$ and $I(0) = 0$ and for $\bullet = \sigma\tau$ one forms the *envelopes*

$$I_\bullet, I^\bullet : [0,\infty]^X \to [0,\infty] \text{ with the } \textit{satellites } I_\bullet^v : [0,\infty]^X \to [0,\infty] \text{ with } v \in S;$$

the notions of *inner* and *outer* $\bullet$ *preintegrals* I will appear below.

For the sequel we assume a positive-homogeneous function class

$S \subset [0,\infty[^X$ in the inner situation,
$S \subset [0,\infty]^X$ in the outer situation,

with $0 \in S$ which is stable under $\vee\wedge$ and *Stonean*, and a functional

$I : S \to [0,\infty[$ in the inner situation,
$I : S \to [0,\infty]$ in the outer situation,

with $I(0) = 0$ which is increasing. These are the assumptions made in [8] [9], while those in [7] were much narrower. We form the set systems

$\mathrm{um}(S) = \{[f \geqq t] : f \in S \text{ and } 0 < t < \infty\}$ for the inner situation,
$\mathrm{lm}(S) = \{[f > t] : f \in S \text{ and } 0 < t < \infty\}$ for the outer situation,

which are lattices with $\varnothing$. Then we define

the *inner sources* of I to be those increasing set functions $\varphi : \mathrm{um}(S) \to [0,\infty[$,
the *outer sources* of I to be those increasing set functions $\varphi : \mathrm{lm}(S) \to [0,\infty]$,

which have $\varphi(\varnothing) = 0$ and which represent $I : I(f) = ⨍ f d\varphi$ for all $f \in S$. The representation theorem 3.2 tells us that such inner/outer sources of I exist iff I is *Stonean* and *truncable*. In this case their characterization is $I_\star(\chi_A) \leqq \varphi(A) \leqq I^\star(\chi_A)$ for all $A \in \mathrm{um}(S)/\mathrm{lm}(S)$, so that as a rule one must expect a lot of inner and outer sources of I.

After this we define for $\bullet = \sigma\tau$ the functional I to be an *inner/outer* $\bullet$ *preintegral* iff it admits at least one inner/outer source which is an inner/outer $\bullet$ premeasure. Then the fundamental results quoted above are the theorems on the inner and outer $\bullet$ preintegrals which follow.

3.7 Inner Theorem ($\bullet = \sigma\tau$). *The functional I is an inner $\bullet$ preintegral iff*

1) *I is supermodular and Stonean and downward $\bullet$ continuous at $\varnothing$,*
2) *$I(v) \leqq I(u) + I_\bullet^v(v-u)$ for all $u \leqq v$ in S.*

In this case $\varphi := I^\star(\chi_\cdot)|\mathrm{um}(S)$ is the unique inner source of I which is an inner $\bullet$ premeasure. It fulfils $I_\bullet(f) = ⨍ f d\varphi_\bullet$ for all $f \in [0,\infty]^X$.

3.8 Outer Theorem ($\bullet = \sigma\tau$). *The functional I is an outer $\bullet$ preintegral iff*

1) *I is submodular and Stonean and upward $\bullet$ continuous,*
2) *$I(v) \geqq I(u) + I^\bullet(v-u)$ for all $u \leqq v$ in S with $u < \infty$,*
3) *moreover for $\bullet = \tau$* (while this is automatic for $\bullet = \sigma$)

$$I^\bullet(f) = \sup\{I^\bullet(f \wedge u) : u \in [I < \infty]\} \quad \textit{for all } f \in [I^\bullet < \infty].$$

In this case $\varphi := I_\star(\chi_\cdot)|\mathrm{lm}(S)$ is the unique outer source of I which is an outer $\bullet$ premeasure. It fulfils $I^\bullet(f) = ⨍ f d\varphi^\bullet$ for all $f \in [0,\infty]^X$.

We refer to the cited papers for the collection of more or less familiar special cases. Thus the classical Riesz representation theorem and its extension to arbitrary Hausdorff topological spaces are immediate consequences of the inner τ theorem, whereas the conventional Daniell-Stone theorem falls under the outer σ theorem.

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