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A Regularity Result for Stationary Electrorheological Fluids in Two Dimensions

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Abstract

We prove $C^{1,\alpha}$-regularity for the strong solution to a system modeling electrorheological fluids in the stationary case which has been constructed by Ettwein and Ettwein-Růžička in [E] and [ER].

In our short note we like to show that the strong solution to the system of partial differential equations for the velocity field of electrorheological fluids constructed by Ettwein and Růžička (see Theorem 1.1 of [ER]) actually is of class $C^{1,\alpha}$ for any $\alpha \in (0,1)$.

To be more precise, let Ω denote a bounded Lipschitz domain in $\mathbb{R}^2$ and consider the problem of finding a velocity field v: $\Omega \to \mathbb{R}^2$ satisfying

$$\left.\begin{aligned} &-\operatorname{div} S(D(v), E) + [\nabla v]v + \nabla\phi = F \ \text{ in } \ \Omega\,, \\ &\operatorname{div} v = 0 \ \text{ in } \ \Omega\,, \quad v = 0 \ \text{ on } \ \partial\Omega\,. \end{aligned}\right\} \tag{1}$$

An extensive discussion of the problem (1) including the physical point of view can be found for instance in [R], here we just recall the basic terminology used in [ER]. First of all, we note that the tensor valued function S: $\mathbb{R}^{2\times 2}_{\text{sym}} \times \mathbb{R}^2 \to \mathbb{R}^{2\times 2}_{\text{sym}}$, $S = (S_{ij})$, $S_{ij} = S_{ij}(\varepsilon, H)$, i, $j = 1$, 2, defined in formula (3) of [ER] satisfies the growth and ellipticity conditions

$$\left.\begin{aligned} \frac{\partial S_{ij}}{\partial \varepsilon_{\alpha\beta}}(D,E)B_{\alpha\beta}B_{ij} &\geq c(1+|D|^2)^{\frac{p(|E|^2)-2}{2}}|B|^2\,, \\ S_{ij}(D,E)D_{ij} &\geq c(1+|D|^2)^{\frac{p(|E|^2)-2}{2}}|D|^2\,, \\ \Big|\frac{\partial S}{\partial \varepsilon}(D,E)\Big| &\leq C(1+|D|^2)^{\frac{p(|E|^2)-2}{2}}\,, \\ \Big|\frac{\partial S}{\partial H}(D,E)\Big| &\leq C(1+|D|^2)^{\frac{p(|E|^2)-1}{2}}\big(1+\ln(1+|D|^2)\big) \end{aligned}\right\} \tag{2}$$

where here and in the following the sum is taken w.r.t. repeated indices. The inequalities $(2)_1$ – $(2)_4$ are required to hold for all B, $D \in \mathbb{R}^{2\times 2}_{\text{sym}}$ with vanishing trace and for all $E \in \mathbb{R}^2$. Moreover, p is a given material function with $1 < p_\infty \leq p(|E|^2) \leq p_0$ for given numbers p_∞ and p_0. In (1) E denotes the (smooth) electrical field, $D(v)$ is the symmetric gradient, ϕ stands for the a priori unknown pressure function and F is a vector field of class $L^\infty(\Omega;\mathbb{R}^2)$. Finally, we denote by $[\nabla v]v$ the quantity $\big(\frac{\partial v^i}{\partial x_j}v^j\big)$.

The following existence and regularity result has been established by Ettwein [E] and by Ettwein and Růžička (see [ER], Theorem 1.1).

Theorem 1. *Suppose that $E \in C^1(\overline{\Omega};\mathbb{R}^2)$ together with $F \in L^\infty(\Omega;\mathbb{R}^2)$. Further assume that $6/5 < p_\infty \leq p(|E(x)|^2) \leq p_0 < \infty$ with $p \in C^1(\mathbb{R})$. Then, if S satisfies (2), there exists a strong solution v to the problem (1), i.e. a solution v belonging to the space $\bigcap_{0<\varepsilon\leq 1} W^2_{2-\varepsilon,loc}(\Omega;\mathbb{R}^2) \cap E_{p(\cdot)} \cap V_{p_\infty}$.*

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Remark 2. i) *Here $W^k_{q,loc}(\ldots)$ is the standard Sobolev space or its local variant (see [A]), the spaces $E_{p(\cdot)}$ and V_{p_∞} are defined in [KR] and [R].*

ii) *In [ER] the assumptions on F are less restrictive.*

In the papers [BF] and [ABF] we discussed generalized Newtonian fluids with an anisotropic dissipative potential and showed interior $C^{1,\alpha}$-regularity of the velocity field for stationary flows in two dimensions, whereas for $\Omega \subset \mathbb{R}^3$ interior partial $C^{1,\alpha}$-regularity was established. In the two-dimensional case we used a technique due to Frehse and Seregin (see [FS]) which can be adjusted to prove

Theorem 3. *Let the hypotheses of Theorem 1 hold and consider the solution v constructed in Theorem 1. Then, if $p_0 = 2$, v is of class $C^{1,\alpha}(\Omega;\mathbb{R}^2)$ for any $0 < \alpha < 1$.*

Remark 4. *It should be noted that recently Acerbi and Mingione (compare [AM]) proved partial regularity results for stationary electrorheological fluids in the three-dimensional case.*

Proof of Theorem 3.

From Theorem 1 and Sobolev's embedding theorem we get

$$\nabla v \in L^r_{loc}(\Omega;\mathbb{R}^{2\times 2}) \quad \text{for all } 1 \le r < \infty \tag{3}$$

and

$$v \in C^{0,\alpha}(\Omega;\mathbb{R}^2) \quad \text{for all } 0 < \alpha < 1\,. \tag{4}$$

Moreover, the weak form of (1) reads

$$\int_\Omega S(D(v),E) : D(\varphi)\,\mathrm{d}x + \int_\Omega [\nabla v]v \cdot \varphi\,\mathrm{d}x - \int_\Omega \phi \operatorname{div}\varphi\,\mathrm{d}x = \int_\Omega F\cdot\varphi\,\mathrm{d}x \tag{5}$$

for $\varphi \in C^\infty_0(\Omega;\mathbb{R}^2)$. Consider $\eta \in C^\infty_0(\Omega), 0 \le \eta \le 1$, and let Δ_h denote the difference quotient in direction $e_k, k = 1,2$, for some real number $h \neq 0, |h|$ sufficiently small. For $Q \in \mathbb{R}^{2\times 2}$ we then let $\varphi = \Delta_{-h}\{\eta^2\Delta_h(v - Qx)\}$ and obtain from (5)

$$\begin{aligned}
&\int_\Omega \Delta_h S(D(v),E) : D(\eta^2\Delta_h[v-Qx])\,\mathrm{d}x \\
&= \int_\Omega [\nabla v]v \cdot \Delta_{-h}(\eta^2\Delta_h[v-Qx])\,\mathrm{d}x - \int_\Omega \phi \operatorname{div}(\Delta_{-h}(\eta^2\Delta_h[v-Qx]))\,\mathrm{d}x \\
&\quad - \int_\Omega F\cdot\Delta_{-h}\big(\eta^2\Delta_h[v-Qx]\big)\,\mathrm{d}x \\
&=: I_1 - I_2 - I_3\,.
\end{aligned} \tag{6}$$

For the terms I_i on the r.h.s. of (6) we observe

$$\begin{aligned}
I_1 &\underset{h\to 0}{\longrightarrow} \int_\Omega [\nabla v]v\cdot\partial_k\big(\eta^2\partial_k[v-Qx]\big)\,\mathrm{d}x\,,\\
-I_2 &= \int_\Omega \Delta_h\phi \operatorname{div}\big(\eta^2\Delta_h[v-Qx]\big) \underset{h\to 0}{\longrightarrow} \int_\Omega \partial_k\phi \operatorname{div}\big(\eta^2\partial_k[v-Qx]\big)\,\mathrm{d}x\\
&= \int_\Omega \partial_k\phi\nabla\eta^2\cdot\partial_k[v-Qx]\,\mathrm{d}x\,,\\
I_3 &\underset{h\to 0}{\longrightarrow} \int_\Omega F\cdot\partial_k\big(\eta^2\partial_k[v-Qx]\big)\,\mathrm{d}x\,,
\end{aligned}$$

which is an immediate consequence of (3), (4) and the integrability properties of $\nabla^2 v$, for example, we may quote Vitali's variant of dominated convergence (compare, e.g. [AFP], p. 38). Note at this stage that we have to work with difference quotients just for the reason that integrability of the quantity $\partial_k S\big(D(v),E\big) : D(\partial_k v)\eta^2$ is not immediate but will be a consequence of the following calculations. We write

$$\begin{aligned}
&\Delta_h S(D(v),E) : D(\Delta_h v) \\
= \quad & \int_0^1 \frac{1}{h}\frac{d}{dt} S_{ij}\big(D(v)(x) + t\big[Dv(x+he_k) - D(v)(x)\big], E(x) + t\big[E(x+he_k) - E(x)\big]\big) \\
& \cdot D_{ij}(\Delta_h v)dt \\
= \quad & \int_0^1 \frac{\partial S_{ij}}{\partial \varepsilon_{\alpha\beta}}(\ldots,\ldots) D_{\alpha\beta}(\Delta_h v) D_{ij}(\Delta_h v) dt + \int_0^1 \frac{\partial S_{ij}}{\partial E}(\ldots,\ldots)\cdot \Delta_h E\, D_{ij}(\Delta_h v)dt \\
=: \quad & A_h + B_h
\end{aligned}$$

and observe $A_h \geq 0$ on account of $(2)_1$. Moreover

$$A_h \to \frac{\partial S_{ij}}{\partial \varepsilon_{\alpha\beta}}\big(D(v),E\big) D_{\alpha\beta}(\partial_k v) D_{ij}(\partial_k v)$$

a.e. so that by Fatou's Lemma

$$\int_\Omega \eta^2 \frac{\partial S_{ij}}{\partial \varepsilon_{\alpha\beta}}\big(D(v),E\big) D_{\alpha\beta}(\partial_k v) D_{ij}(\partial_k v)\,\mathrm{d}x \leq \liminf_{h\to 0} \int_\Omega A_h \eta^2\,\mathrm{d}x\,,$$

where the r.h.s. is uniformly bounded if so is $\int_\Omega B_h \eta^2\,\mathrm{d}x$. From $(2)_4$ and the integrability properties of ∇v and $\nabla^2 v$ we deduce

$$B_h \underset{h\to 0}{\longrightarrow} \frac{\partial S_{ij}}{\partial E}\big(D(v),E\big)\cdot \partial_k E D_{ij}(\partial_k v)$$

in $L^1_{loc}(\Omega)$ and a.e. Finally we observe that

$$\int_\Omega \Delta_h S_{ij}\big(D(v),E\big)\partial_i \eta^2 \Delta_h [v - Qx]^j\,\mathrm{d}x \underset{h\to 0}{\longrightarrow} \int_\Omega \partial_k S_{ij}\big(D(v),E\big)\partial_i \eta^2 \partial_k [v - Qx]^j\,\mathrm{d}x$$

which follows from (2) and (3). Let us introduce the tensor $\sigma = S(D(v),E)$. Then we deduce from (6) and the subsequent discussion

$$\begin{aligned}
\int_\Omega \eta^2 \partial_k \sigma : D(\partial_k v)\,\mathrm{d}x \quad \leq \quad & -2\int_\Omega \eta \partial_k \sigma_{ij} \partial_i \eta \partial_k [v - Qx]^j\,\mathrm{d}x + \int_\Omega [\nabla v] v \cdot \partial_k\big(\eta^2 \partial_k [v - Qx]\big)\,\mathrm{d}x \\
& - \int_\Omega F \cdot \partial_k\big(\eta^2 \partial_k [v - Qx]\big)\,\mathrm{d}x + 2\int_\Omega \partial_k \phi \eta \nabla \eta \cdot \partial_k [v - Qx]\,\mathrm{d}x\,. \quad (7)
\end{aligned}$$

Moreover, the integral on the l.h.s. is well defined, i.e. $\partial_k \sigma : D(\partial_k v) \in L^1_{loc}(\Omega)$. The last term on the r.h.s. of (7) is handled with the help of the inequality

$$|\nabla \phi| \leq |\nabla \sigma| + |\nabla v||v| + |F|$$

which follows from (1). In order to put (7) in an easier form we now fix a subdomain $\Omega' \Subset \Omega$ and consider a disc $B_{2r}(\overline{x}) \subset \Omega'$. We abbreviate $T_r(\overline{x}) = B_{2r}(\overline{x}) - B_r(\overline{x})$ and choose η in such a way that $\eta = 1$ on $B_r(\overline{x})$, $\operatorname{spt}\eta \subset B_{2r}(\overline{x})$, $|\nabla\eta| \leq 2/r$. Moreover, we observe $v \in L^\infty(\Omega';\mathbb{R}^2)$ and denote by the symbol c a constant depending also on $\operatorname{dist}(\Omega', \partial\Omega)$ whose value may vary from line to line. Then the r.h.s. of (7) is bounded from above by

$$\begin{aligned} c\Big\{ &\int_{T_r(\overline{x})} \eta|\nabla\sigma||\nabla\eta||\nabla v - Q|\,\mathrm{d}x + \int_{T_r(\overline{x})} \eta|\nabla\eta||\nabla v||\nabla v - Q|\,\mathrm{d}x \\ &+ \int_{B_{2r}(\overline{x})} \eta^2|\nabla v||\nabla^2 v|\,\mathrm{d}x + \int_{T_r(\overline{x})} |F|\eta|\nabla\eta||\nabla v - Q|\,\mathrm{d}x \\ &+ \int_{B_{2r}(\overline{x})} |F|\eta^2|\nabla^2 v|\,\mathrm{d}x \Big\} =: c\big\{I_1 + \ldots + I_5\big\}\,. \end{aligned} \tag{8}$$

Consider the functions

$$H = \Big(\frac{\partial S_{ij}}{\partial\varepsilon_{\alpha\beta}}\big(D(v), E\big) D_{ij}(\partial_k v) D_{\alpha\beta}(\partial_k v)\Big)^{\frac{1}{2}}$$

and

$$h = \big(1 + |D(v)|^2\big)^{\frac{p_\infty}{4}}\,.$$

The calculations concerning the l.h.s. of (7) show $H \in L^2(\Omega')$, moreover, we have

$$|\nabla h| \leq c\big(1 + |D(v)|^2\big)^{\frac{p_\infty - 2}{4}} |\nabla D(v)|\,,$$

hence (recall $(2)_1$)

$$|\nabla h|^2 \leq c\big(1 + |D(v)|^2\big)^{\frac{p_\infty - 2}{2}} |\nabla D(v)|^2 \leq cH^2\,,$$

therefore $h \in W^1_2(\Omega')$. Finally, we note that clearly

$$hH \geq c|\nabla D(v)|$$

is valid. In order to estimate I_j, $j = 1, \ldots, 5$, we choose $Q = \fint_{T_r(\overline{x})} \nabla v\,\mathrm{d}x$ and recall the inequality $|\nabla^2 v| \leq c|\nabla D(v)|$, i.e. any second derivative of v is bounded in terms of first derivatives of the symmetric gradient. With these observations we get (using the Sobolev-Poincaré - inequality)

$$\begin{aligned} I_1 &\leq cr^{-1}\Big[\int_{T_r(\overline{x})} |\nabla\sigma|^2\,\mathrm{d}x\Big]^{\frac{1}{2}} \Big[\int_{T_r(\overline{x})} |\nabla v - Q|^2\,\mathrm{d}x\Big]^{\frac{1}{2}} \\ &\leq cr^{-1}\Big[\int_{T_r(\overline{x})} |\nabla\sigma|^2\,\mathrm{d}x\Big]^{\frac{1}{2}} \int_{T_r(\overline{x})} |\nabla^2 v|\,\mathrm{d}x \\ &\leq cr^{-1}\Big[\int_{T_r(\overline{x})} |\nabla\sigma|^2\,\mathrm{d}x\Big]^{\frac{1}{2}} \int_{T_r(\overline{x})} hH\,\mathrm{d}x\,, \end{aligned}$$

$$I_2 \leq \frac{c}{r}\Big[\int_{T_r(\overline{x})} |\nabla v|^2\,\mathrm{d}x\Big]^{\frac{1}{2}} \Big[\int_{T_r(\overline{x})} |\nabla v - Q|^2\,\mathrm{d}x\Big]^{\frac{1}{2}} \leq cr^{-1}r^\alpha \int_{T_r(\overline{x})} hH\,\mathrm{d}x\,,$$

for some (any) exponent $0<\alpha<1$ which follows from (3). We further have

$$\begin{aligned}
I_3 &\leq c\int_{B_{2r}(\overline{x})}\eta^2|\nabla D(v)||\nabla v|\,\mathrm{d}x\\
&= c\int_{B_{2r}(\overline{x})}\eta^2\big(1+|D(v)|^2\big)^{\frac{p_\infty-2}{4}}|\nabla D(v)||\nabla v|\big(1+|D(v)|^2\big)^{\frac{2-p_\infty}{4}}\,\mathrm{d}x\\
&\overset{(2)_1}{\leq} \rho\int_{B_{2r}(\overline{x})}\eta^2H^2\,\mathrm{d}x+c(\rho)\int_{B_{2r}(\overline{x})}|\nabla v|^2\big(1+|D(v)|^2\big)^{\frac{2-p_\infty}{2}}\,\mathrm{d}x\\
&\leq \rho\int_{B_{2r}(\overline{x})}\eta^2H^2\,\mathrm{d}x+c(\rho)r^{2\beta}\,,
\end{aligned}$$

where $\rho\in(0,1)$ is fixed later and $\beta\in(0,1)$ is arbitrary. Again we made use of (3). Next we observe the boundedness of F and get

$$\begin{aligned}
I_4 &\leq \frac{c}{r}\int_{T_r(\overline{x})}|\nabla v-Q|\,\mathrm{d}x \;\leq\; c\int_{T_r(\overline{x})}|\nabla^2 v|\,\mathrm{d}x \;\leq\; c\int_{T_r(\overline{x})}hH\,\mathrm{d}x\,,\\
I_5 &\leq c\int_{B_{2r}(\overline{x})}\eta^2|\nabla^2 v|\,\mathrm{d}x \overset{(\text{compare } I_3)}{\leq} \rho\int_{B_{2r}(\overline{x})}\eta^2H^2\,\mathrm{d}x+c(\rho)r^{2\beta}\,.
\end{aligned}$$

Putting together (7), (8) we deduce

$$\begin{aligned}
\int_{B_{2r}(\overline{x})}\eta^2\partial_k\sigma:D(\partial_k v)\,\mathrm{d}x \;&\leq\; cr^{-1}\left[r^{2\alpha}+\int_{T_r(\overline{x})}|\nabla\sigma|^2\,\mathrm{d}x\right]^{\frac{1}{2}}\int_{T_r(\overline{x})}hH\,\mathrm{d}x\\
&\quad+\rho\int_{B_{2r}(\overline{x})}\eta^2H^2\,\mathrm{d}x+c(\rho)r^{2\beta}\,. \qquad (9)
\end{aligned}$$

In a next step we express $\nabla\sigma$ in terms of the function H on both sides of (9). The definition of σ implies

$$\begin{aligned}
\partial_k\sigma:D(\partial_k v) &= H^2+\frac{\partial S_{ij}}{\partial E}\big(D(v),E\big)\cdot\partial_k ED_{ij}(\partial_k v)\\
&\overset{(2)_4}{\geq} H^2-c\big(1+|D(v)|^2\big)^{\frac{1}{2}}\big(1+\ln(1+|D(v)|^2)\big)|\nabla D(v)|\\
&\geq H^2-c\big(1+|D(v)|^2\big)^{t/2}|\nabla D(v)|\,,
\end{aligned}$$

t denoting some exponent slightly larger than 1. From Young's inequality we deduce

$$\begin{aligned}
\big(1+|D(v)|^2\big)^{\frac{t}{2}}|\nabla D(v)| &\leq \varepsilon\big(1+|D(v)|^2\big)^{\frac{p_\infty-2}{2}}|\nabla D(v)|^2+c(\varepsilon)\big(1+|D(v)|^2\big)^{t-\frac{p_\infty-2}{2}}\\
&\leq \varepsilon H^2+c(\varepsilon)\big(1+|D(v)|^2\big)^{t+\frac{2-p_\infty}{2}}
\end{aligned}$$

for any $\varepsilon>0$, thus ($\tilde{t}=2t+2-p_\infty$)

$$\partial_k\sigma:D(\partial_k v)\geq\frac{1}{2}H^2-c\big(1+|D(v)|^2\big)^{\frac{\tilde{t}}{2}}\,.$$

Observing

$$\int_{B_{2r}(\overline{x})}\big(1+|D(v)|^2\big)^{\frac{\tilde{t}}{2}}\,\mathrm{d}x\leq cr^{2\beta}$$

we see that on the l.h.s. of (9) the quantity $\partial_k\sigma : D(\partial_k v)$ can be replaced by H^2. Next observe that

$$\begin{aligned}
|\nabla\sigma|^2 &= \partial_k\sigma : \partial_k\sigma \;=\; \frac{\partial S_{ij}}{\partial\varepsilon_{\alpha\beta}}\big(D(v),E\big)D_{\alpha\beta}(\partial_k v)\partial_k\sigma_{ij} + \frac{\partial S_{ij}}{\partial E}\big(D(v),E\big)\cdot\partial_k E\,\partial_k\sigma_{ij} \\
&=: A+B\,,
\end{aligned}$$

$$\begin{aligned}
|A| &\le c\Big|\frac{\partial S}{\partial\varepsilon}\big(D(v),E\big)\Big||\nabla D(v)||\nabla\sigma| \\
&\overset{(2)_3}{\le} c\big(1+|D(v)|^2\big)^{\frac{p(|E|^2)-2}{2}}|\nabla D(v)||\nabla\sigma| \\
&= c\big(1+|D(v)|^2\big)^{\frac{p(|E|^2)-2}{4}}|\nabla D(v)|\big(1+|D(v)|^2\big)^{\frac{p(|E|^2)-2}{4}}|\nabla\sigma| \\
&\le c(\varepsilon)\big(1+|D(v)|^2\big)^{\frac{p(|E|^2)-2}{2}}|\nabla D(v)|^2+\varepsilon|\nabla\sigma|^2 \\
&\overset{(2)_1}{\le} c(\varepsilon)H^2+\varepsilon|\nabla\sigma|^2,
\end{aligned}$$

so that for $\varepsilon = 1/2$ we get

$$|\nabla\sigma|^2 \le cH^2 + c|B|. \tag{10}$$

Note that in the above calculation we made essential use of $p\le 2$. For estimating B we recall $(2)_4$, hence

$$\begin{aligned}
|B| &\le c\big(1+|D(v)|^2\big)^{\frac{p(|E|^2)-1}{2}}\big(1+\ln(1+|D(v)|^2)\big)|\nabla\sigma| \\
&\le c\big(1+|D(v)|^2\big)^{\frac{t}{2}}|\nabla\sigma| \;\le\; \varepsilon|\nabla\sigma|^2 + c(\varepsilon)\big(1+|D(v)|^2\big)^t,
\end{aligned}$$

and (10) turns into

$$|\nabla\sigma|^2 \le cH^2 + c\big(1+|D(v)|^2\big)^t. \tag{11}$$

Inserting (11) on the r.h.s. of (9), choosing ρ small enough and taking into account that $\int_{T_r(\overline{x})}(1+|D(v)|^2)^t\,\mathrm{d}x$ can be bounded by $cr^{2\alpha}$, we arrive at the starting inequality of the Frehse-Seregin lemma (see [FS], Lemma 4.1)

$$\int_{B_r(\overline{x})} H^2\,\mathrm{d}x \le cr^{-1}\left[r^{2\alpha}+\int_{T_r(\overline{x})}H^2\,\mathrm{d}x\right]^{\frac{1}{2}}\int_{T_r(\overline{x})}hH\,\mathrm{d}x + cr^{2\beta}. \tag{12}$$

To be more precise, it is easy to check that with (12) the estimate (A 3.6) of [FS] now takes the form

$$\int_{B_r(\overline{x})} H^2\,\mathrm{d}x \le c\left[\sqrt{\log_2\frac{2R}{r}}\int_{T_r(\overline{x})}H^2\,\mathrm{d}x + r^\alpha\sqrt{\log_2\frac{2R}{r}}\right] + cr^\beta\,,$$

where R is some fixed radius and $r\le R$ together with $B_{2R}(\overline{x})\subset\Omega'$. If we choose $\beta>\alpha$, then clearly the latter inequality reduces to (A 3.6) of [FS], hence we have the statement of [FS], Lemma 4.1, which means that for any $q>1$ there is a local constant such that

$$\int_{B_r(\overline{x})} H^2\,\mathrm{d}x \le K(q)|\ln r|^{-q}\,. \tag{13}$$

Combining (11) and (13) and observing again

$$\int_{B_r(\overline{x})} \left(1 + |D(v)|^2\right)^t \, \mathrm{d}x \le c r^{2\gamma}$$

for any exponent $\gamma < 1$, we see

$$\int_{B_r(\overline{x})} |\nabla \sigma|^2 \, \mathrm{d}x \le K(q) |\ln r|^{-q},$$

and the version of the Dirichlet-Growth Theorem given in [F], p. 287, implies the continuity of σ.

Now we are going to prove continuity of $D(v)$. Let $\Gamma_{(x)}$: $\mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$, $\Gamma_{(x)}(\varepsilon) = S(\varepsilon, E(x))$. From $(2)_1$ it follows that $\Gamma_{(x)}$ is one-to-one, the coercivity condition $(2)_2$ implies (see [D], Satz 2, p. 44) that S is onto. Moreover, $\Gamma_{(x)}$ is an open mapping (compare [D], Satz 3, p. 52), thus a continuous inverse exists, and we obtain $D(v)(x) = \Gamma^{-1}_{(x)}(\sigma(x))$. It therefore remains to discuss that $\Gamma^{-1}_{(x)}$ depends continuously on the parameter $x \in \Omega$.

Alternatively, we may use the implicit-function-theorem (see, e.g. [D], Satz 2, p. 17): let $x_0 \in \Omega$ and $\varepsilon_0 = \Gamma^{-1}_{(x_0)}(\sigma(x_0))$. Let further

$$F(x, \varepsilon) = S(\varepsilon, E(x)) - \sigma(x) = \Gamma_{(x)}(\varepsilon) - \sigma(x)\,.$$

Then we have $F(x_0, \varepsilon_0) = 0$, moreover, $\frac{\partial F}{\partial \varepsilon}(x_0, \varepsilon_0)$ is an isomorphism. Thus there exists a neighborhood U of x_0 in Ω and a continuous function g: $U \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ such that $g(x_0) = \varepsilon_0$ and $F(x, g(x)) = 0$ on U. Therefore we have

$$\sigma(x) = S(g(x), E(x))$$

and

$$\sigma(x) = S(D(v)(x), E(x))$$

on U, hence $g(x) = D(v)(x)$ by the injectivity of $\Gamma_{(x)}$. This proves the continuity of $D(v)$ and we may proceed as in [ABF], proof of Corollary 5.1, or as in [BF] to get $v \in C^{1,\alpha}(\Omega; \mathbb{R}^2)$ for any $0 < \alpha < 1$: to this purpose we observe that (1) implies after an integration by parts

$$\begin{aligned}
&\int_\Omega \frac{\partial S_{ij}}{\partial \varepsilon_{\alpha\beta}}(D(v), E) D_{\alpha\beta}(\bar v) D_{ij}(\varphi)\, \mathrm{d}x \\
&\quad = \; - \int_\Omega \frac{\partial S_{ij}}{\partial E}(D(v), E) \cdot \partial_k E \, D_{ij}(\varphi)\, \mathrm{d}x + \int_\Omega \partial_k (v^i v^j) \partial_j \varphi^i \, \mathrm{d}x + \int_\Omega F \cdot \partial_k \varphi \, \mathrm{d}x \quad (14)
\end{aligned}$$

being valid for any $\varphi \in C^\infty_0(\Omega; \mathbb{R}^2)$ s.t. $\operatorname{div} \varphi = 0$. Here we have set $\bar v = \partial_k v$. By the continuity of $D(v)$, (14) can be seen as a linear elliptic system for the function $\bar v$ with continuous coefficients $A^{ij}_{\alpha\beta}(x) = \frac{\partial S_{ij}}{\partial \varepsilon_{\alpha\beta}}(D(v)(x), E(x))$. We fix a disc $B_R(x_0) \Subset \Omega$ and consider the solution $v_0 \in W^1_2(B_R(x_0); \mathbb{R}^2)$ to the problem

$$\left.\begin{aligned}
&\int_{B_R(x_0)} A^{ij}_{\alpha\beta}(x_0) D_{\alpha\beta}(v_0) D_{ij}(\varphi)\, \mathrm{d}x = 0 \\
&\text{for all } \; \varphi \in C^\infty_0(B_R(x_0); \mathbb{R}^2)\,, \qquad \operatorname{div}\varphi = 0\,, \\
&v_0 = \bar v \quad \text{on } \partial B_R(x_0)\,, \qquad \operatorname{div} v_0 = 0\,.
\end{aligned}\right\}$$

Then the comparison arguments outlined after (5.3) in [ABF] give the claim. □

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