Universität des Saarlandes

Fachbereich 9 – Mathematik

Mathematischer Preprint

Toeplitz Eigenvalues for Radon Measures

E. E. Tyrtyshnikov and N. L. Zamarashkin

Preprint No. 9

Saarbrücken 2000

Universität des Saarlandes

Fachbereich 9 – Mathematik

Toeplitz Eigenvalues for Radon Measures

E. E. Tyrtyshnikov

Institute of Numerical Mathematics
Russian Academy of Sciences
Gubkina 8
Moscow 117333, Russia
E-Mail: tee@inm.ras.ru

N. L. Zamarashkin

Institute of Numerical Mathematics
Russian Academy of Sciences
Gubkina 8
Moscow 117333, Russia
E-Mail: kolya@bach.inm.ras.ru

submitted: February 15, 2000

Preprint No. 9

Saarbrücken 2000

Edited by
FB 9 – Mathematik
Im Stadtwald
D–66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

Abstract

It is well known that for Toeplitz matrices generated by a "sufficiently smooth" real-valued symbol, the eigenvalues behave asymptotically as the values of the symbol on uniform meshes while the singular values, even for complex-valued functions, do as those values in modulus. These facts are expressed analytically by the Szegö and Szegö-like formulas, and, as is proved recently, the "smoothness" assumptions are as mild as those of L_1. In this paper, it is shown that the Szegö-like formulas hold true even for Toeplitz matrices generated by the so-called Radon measures.

Key-words: Toeplitz matrices, eigenvalues, singular values, Szegö formulas, Radon measures.

AMS classification: 15A12, 65F10, 65F15, 65T10.

Acknowledgement: This work was supported by the Russian Fund of Basic Research under Grant 99-01-00017 and completed while the first author was a guest professor at the University of Saarland, Germany, under support of DAAD.

1 Introduction

We consider a sequence of Toeplitz matrices

$$A_n = [a_{kl}], \quad a_{kl} = a_{k-l}, \quad 0 \le k, l \le n-1, \tag{1}$$

constructed from the coefficients of a formal Fourier series

$$f(x) \sim \sum_{k=-\infty}^{\infty} a_k e^{\mathbf{i}kx}, \tag{2}$$

and will be interested in the asymptotic behavior of their eigenvalues $\lambda_i(A_n)$ (in the Hermitian case) and singular values $\sigma_i(A_n)$ (in the non-Hermitian case) as $n \to \infty$. Due to G. Szegö [5] and successive works [1, 6, 8, 9, 11] we enjoy the following beautiful formula:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i(A_n)) \; = \; \frac{1}{2\pi} \int_{-\pi}^{\pi} F(f(x))\, dx, \tag{3}$$

which is valid for any test function $F(x)$ from a suitable set $\mathcal{F}$.
G. Szegö proved (3) for a real-valued $f \in L_\infty$ and $\mathcal{F}$ comprising all continuous functions on the interval [ess inf$f$, ess sup$f$]. For $f \in L_\infty$ this interval contains all $\lambda_i(A_n)$. Since this is not the case for $f \in L_p$ with $p < \infty$, it was proposed in [8] to take up as $\mathcal{F}$ all functions uniformly bounded and uniformly continuous for $-\infty < x < \infty$; a bit more restrictive choice for $\mathcal{F}$ might be all continuous functions with bounded support [8]. For both cases, the same formula (3) holds true for $f \in L_2$ [8, 9] and even for $f \in L_1$ [11].
If f is not necessarily real-valued, under the same "smoothness" assumptions on f and the same $\mathcal{F}$ we have quite a similar formula for the singular values:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\sigma_i(A_n)) \; = \; \frac{1}{2\pi} \int_{-\pi}^{\pi} F(|f(x)|)\, dx. \tag{4}$$

An important and somewhat expected difference is that the eigenvalues behave as the valus of $f(x)$ (when f is real-valued and in some special cases of complex-valued f [10]) while the singular values do as the same values in modulus. Formula (4) was proposed by S. Parter [6] and proved first for a specific subclass of L_∞; then it was extended to the whole of L_∞ [1] and further to L_2 [8, 9] and even to L_1 [11].
However, we have long suspected that L_1 is still not the ultimate extension. For example, let

$$a_k = 1, \quad k = 0, \pm 1, \pm 2, \ldots .$$

It this case $f(x)$ (usually called a *symbol* or *generating function*) is not a function in the classical sense (it is a multiple of the Dirach delta function). Despite this, the eigenvalues of $A_n = A_n(f)$ are easy to find explicitly:

$$\lambda_1 = n; \qquad \lambda_k = 0, \quad k = 2, \dots n.$$

Therefore, the Szegö formula (3) gives the true asymptotic distribution even for this case if only we set $f(x)$ to zero in the integrand.
Thus, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i(A_n)) \;=\; F(0) \tag{5}$$

for any $F \in \mathcal{F}$. From now onwards, let $\mathcal{F}$ be the set of all uniformly bounded and uniformly continuous functions.
If A_n is an arbitrary sequence of matrices satisfying (5), we say that the eigenvalues of A_n have *a cluster* at zero. An equivalent definition reads [8, 9]: zero is a cluster for $\lambda_i(A_n)$ if for any $\varepsilon > 0$ the number $\gamma_n(\varepsilon)$ of those i from 1 to n for which $|\lambda_i(A_n)| > \varepsilon$ is $o(n)$ (that is, $\frac{\gamma_n(\varepsilon)}{n} \to 0$). To denote the fact, we write $\lambda(A_n) \sim 0$. If (5) is fulfilled for the singular values, we write $\sigma(A_n) \sim 0$.
The above observation might suggest that we could have a cluster at zero in all cases when f is *not a function modulo a function* (that is, after subtracting any function from an appropriate space). Of course, it gives just a flavour of where we should look for a rigorous formulation. The purpose of this paper is to propose one by making a step from functions to "non-functions".
Let us assume that the Fourier coefficients are the values of a linear bounded functional $\mathcal{T}(\phi)$ on the space of continuous functions ϕ on the basic closed interval $\Pi = [-\pi,\ \pi]$. Such a functional is called a *Radon measure* [4]. It is well-known that there exists a bounded-variation function μ on Π such that

$$\mathcal{T}(\phi) = \int_{\pi}^{\pi} \phi(x)\ d\mu(x), \tag{6}$$

where the integral is understood in the sense of Stiltijes. Thus, it is μ that can be viewed now as a symbol.
We know that any function μ of bounded variation is a sum of three functions (see, for example, [5])

$$\mu = \mu_a + \mu_s + \mu_j, \tag{7}$$

where μ_a is an *absolutely continuous function*, μ_s is the so-called *singular function* (a continuous function with zero derivative at almost every point), and μ_j is a *function of jumps.* All three components are of bounded variation as well. The derivative $f \equiv \mu_a'$ of μ_a exists almost everywhere in the Lebesgue sense and belongs to L_1. The derivatives of μ_j and μ_s are almost everywhere equal to zero. Consequently, $\mu' = \mu_a'$ almost everywhere. Recall that, by definition, μ_j is a sum of a countable number of *jumps*:

$$\mu_j(x) = \sum_{x<s_k} h_k^- + \sum_{x>s_k} h_k^+,$$

where

$$\sum_{k=1}^{\infty} |h_k^{\pm}| < \infty.$$

(The values at $x = s_k$ do not count.) Note that $f = \mu_a'$ is determined uniquely as a function from L_1.
Our main result is the following theorem.

Theorem 1.1 *Suppose that μ is a function of bounded variation on Π, and $f \equiv \mu' \in L_1$ is its derivative. Let A_n be Toeplitz matrices of the form (1) where*

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\mathbf{i}kx}\, d\mu(x). \tag{8}$$

Then, for any $F \in \mathcal{F}$, the relation (3) holds true, provided that μ is real-valued, and (4) holds true in case μ might be complex-valued. The test-function set $\mathcal{F}$ consists of all uniformly bounded and uniformly continuous functions.

In other words, in the real-valued case the eigenvalues of A_n are distributed as the values of $f(x)$, and in the complex-valued case the singular values of A_n are distributed as the values of $|f(x)|$. Compared to the previous knowledge, a new message is that f in the Szego-like formulas is not a generating function for A_n. It is the derivative of the Radon-measure symbol μ, and it is μ that generates A_n. The Fourier series (2) is not associated with any function in the classical sense. However, at least in the Radon-measure case, it can be juxtaposed to some function from L_1 that describes the spectral distributions precisely by the Szego-like formulas.

2 Preliminaries

Given a matrix sequence A_n, we try to associate it with another sequence B_n for which (3) or (4) is easier to prove and which is close, in a certain sence, to A_n. By definition, two sequences of n-tuples $\{\alpha_i^{(n)}\}_{i=1}^n$ and $\{\beta_i^{(n)}\}_{i=1}^n$ are equally distributed if, for any $F \in \mathcal{F}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(F(\alpha_i^{(n)}) - F(\beta_i^{(n)}) \right) \;=\; 0. \tag{9}$$

We capitalize on the following lemma [9].

Lemma 2.1 *Let $G(x)$ be a continuous, nonnegative, and strictly increasing function for $x \geq 0$, and $G(0) = 0$. Let c_1 and c_2 be positive constants.*
Given two matrix sequences A_n and B_n, assume that for any $\varepsilon > 0$, there exists N such that for all $n \geq N$, the difference between A_n and B_n can be split

$$A_n - B_n \;=\; E_n + R_n \tag{10}$$

so that

$$\sum_{i=1}^{n} G(\sigma_i(E_n)) \;\leq\; c_1 \varepsilon n \tag{11}$$

and

$$\mathrm{rank} R_n \;\leq\; c_2 \varepsilon n. \tag{12}$$

Then the singular values of A_n and B_n are equally distributed.
If A_n and B_n are Hermitian, assume that E_n and R_n are Hermitian and, instead of (11), that

$$\sum_{i=1}^{n} G(|\lambda_i(E_n)|) \;\leq\; c_1 \varepsilon n. \tag{13}$$

Then, the eigenvalues of A_n and B_n are equally distributed as well.

An important example is $G(x) = x^2$; in this case (11) is equivalent to the Frobenius-norm (Schatten 2-norm) estimate

$$||E_n||_F^2 \;\leq\; c_1 \varepsilon n. \tag{14}$$

Another useful example is $G(x) = x$; in this case (11) is equivalent to the Schatten trace-norm estimate (see [2, 7])

$$||E_n||_{tr} \equiv \sum_{i=1}^{n} \sigma_i(E_n) \;\leq\; c_1 \varepsilon n. \tag{15}$$

Once having (14) or (15), from the Weyl inequalities we infer that (13) is also valid (for the respective $G(x)$).

The main vehicle to relate the eigenvalues with the symbol μ is the next observation. Consider the following one-to-one correspondence between vectors and polynomials:

$$p = \begin{bmatrix} p_0 \\ \dots \\ p_{n-1} \end{bmatrix} \quad \leftrightarrow \quad p(x) = \sum_{i=0}^{n-1} p_i x^i.$$

If A_n are Toeplitz matrices with the elements a_k of the form (8), then

$$(A_n p, p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |p(\mathrm{e}^{\mathbf{i}x})|^2 \, d\mu(x). \tag{16}$$

We take advantage of special probe vectors p for which the "kernel" $|p(\mathrm{e}^{\mathbf{i}x})|^2$ can be expressed explicitly. As in [11], these are the columns of the Descrete Fourier Transform matrix:

$$p_k^{(n)} = \frac{1}{\sqrt{n}} \begin{bmatrix} \mathrm{e}^{-\mathbf{i}\frac{2\pi}{n}k\cdot 0} \\ \dots \\ \mathrm{e}^{-\mathbf{i}\frac{2\pi}{n}k\cdot(n-1)} \end{bmatrix}, \quad k = 0, \dots, n-1. \tag{17}$$

On having made this choice, we obtain

$$(A_n p_k^{(n)}, p_k^{(n)}) \;=\; \int_{-\pi}^{\pi} \Phi_n(k,x)\, d\mu(x), \quad \Phi_n(k,x) \equiv \frac{1}{2\pi} \, |p_k^{(n)}(\mathrm{e}^{\mathbf{i}x})|^2. \tag{18}$$

A direct calculation yields [11]

$$\Phi_n(k,x) \;=\; \frac{\sin^2(H_n(k,x)n)}{2\pi n \sin^2 H_n(k,x)}, \tag{19}$$

where

$$H_n(k,x) = \frac{2\pi k + xn}{2n}.$$

We use this formula to prove an important lemma which all the constructions hinge on. This is a touch-up of the result from [11].

Lemma 2.2 *Let $0 < \delta < \pi$. Then, for any n,*

$$\max_{-\delta \le x \le \delta} \Phi_n(k, x) \;\le\; \frac{c_1(\delta)}{n}, \quad c_1(\delta) = \frac{1}{2\pi \sin^2 \frac{\delta}{2}}, \tag{20}$$

for all $k \in \{0, \ldots n-1\}$ except for at most $c_2 \delta\, n + 1$ indices with $c_2 = 2/\pi$.

Proof. Denote by ν_n the number of $k \in \{0, \ldots, n-1\}$ for which (20) does not hold, and let τ_n be the number of those k for which the denominator in (19) is strictly less than $n/c_1(\delta)$. That means that

$$\min_{-\delta < x < \delta} \sin H_n(k, x) \;<\; \sin \frac{\delta}{2}. \tag{21}$$

Since (20) takes place whenever (21) does not, we conclude that $\nu_n \le \tau_n$.
To estimate τ_n, assume by the moment that $\delta \le \pi/2$. Then (21) amounts to the claim that

$$\pi m - \frac{\delta}{2} \;<\; -\frac{\pi k}{n} + \frac{x}{2} \;<\; \frac{\delta}{2} + \pi m$$

for some integer m and $x \in [-\delta, \delta]$. The latter implies that

$$\pi m - \delta \;<\; -\frac{\pi k}{n} \;<\; \delta + \pi m.$$

Since $0 \le k \le n-1$, it is possible only when $m = 0$ or $m = -1$. Thus, we can estimate τ_n by counting how many indices k satisfy

$$1 - \frac{\delta}{\pi} \;<\; \frac{k}{n} \;<\; 1 \quad \text{or} \quad 0 \le \frac{k}{n} \;<\; \frac{\delta}{\pi}.$$

Thus, $\tau_n < \frac{2\delta}{\pi} n + 1$. The same estimate stands also when $\pi/2 < \delta \le \pi$. $\square$

3 Main results

We call a Radon measure *nonnegative* if the corresponding symbol μ is a monotone nondecreasing function. The general case can be reduced to those because an arbitrary function of bounded variation is a difference of two monotone nondecreasing functions.
For a Radon measure, a point is called *essential* if the full variation in any its neibourhood is nonzero. The closure of the set of all essential points is said to be a *support* of this measure. We are going to show that a "small" support for a nonnegative measure means that the eigenvalues of the corresponding Toeplitz matrices are "almost clustered" at zero.

Lemma 3.1 *Consider a nonnegative Radon measure with symbol μ, and assume that it is supported on a closed interval of length δ. Then the Toeplitz matrices $A_n = A_n(\mu)$ generated by μ can be split*

$$A_n = A_{1n} + A_{2n} \tag{22}$$

so that

$$\sigma(A_{1n}) \sim 0 \tag{23}$$

and, for some $c > 0$ independent of δ and n,

$$\text{rank} A_{2n} \leq c\,\delta\,n \tag{24}$$

for all sufficiently large n.

Proof. Assume, first, that the interval of length δ is inside $[-\delta, \delta]$. Set $P_n = [P_{1n}, P_{2n}]$, where P_{1n} contains all the columns $p_k^{(n)}$ for which (20) is fulfilled, all other $p_k^{(n)}$ being relegated to P_{2n}. Then

$$A_{1n} = P_n \begin{bmatrix} P_{1n}^* A_n P_{1n} & 0 \\ 0 & 0 \end{bmatrix} P_n^*, \quad A_{2n} = P_n \begin{bmatrix} 0 & * \\ * & * \end{bmatrix} P_n^*.$$

¿From (16) and thanks to the nonnegativeness of the Radon measure, A_n are Hermitian nonnegative matrices. Obviously, A_{1n} is also a Hermitian nonnegative matrix. Hence,

$$\sum_{k=1}^{n} \sigma_k(A_{1n}) = \text{trace}\, A_{1n} = \text{trace}\, P_{1n}^* A_n P_{1n},$$

and by Lemma 2.2,

$$\text{trace}\, P_{1n}^* A_n P_{1n} \leq c_1(\delta) \int_{-\pi}^{\pi} d\mu = o(n).$$

Consequently, $\sigma(A_{1n}) \sim 0$ and, from Lemma 2.2, the rank of A_{2n} does not exceed $(c_2 + 1)\delta n$ for all sufficiently large n.
If $\mathcal{I}$ is an arbitarily located interval of length δ, then we choose a shift s so that $s + \mathcal{I} \subset [-\delta, \delta]$. Thus, the said-above splitting is taken for granted for Toeplitz matrices $\tilde{A}_n$ generated by $\mu(s + x)$. As is readily seen from (8),

$$\tilde{A}_n = D_n^* A_n D_n, \qquad \text{where} \qquad D_n = \begin{bmatrix} e^{\mathbf{i}s\cdot 0} & & \\ & \ddots & \\ & & e^{\mathbf{i}s\cdot(n-1)} \end{bmatrix}$$

is a unitary diagonal matrix. Having had $\tilde{A}_n = \tilde{A}_{1n} + \tilde{A}_{2n}$, now we set

$$A_{1n} = D_n \tilde{A}_{1n} D_n^*, \quad A_{2n} = D_n \tilde{A}_{2n} D_n^*,$$

which complets the proof. $\square$

Lemma 3.2 *Assume that Toeplitz matrices A_n are generated by a nonnegative Radon measure with a compact support of the Lebesgue measure δ. Then $A_n = A_{1n} + A_{2n}$ so that (23) and (24) are valid.*

Proof. Since the support of the Radon measure is a compact set, it can be covered by finitely many (say, m) open intervals (a_i, b_i) so that

$$\sum_{i=1}^{m} (b_i - a_i) \ < \ 2\delta.$$

Let $\mu_i = \mu$ on $[a_i, b_i]$ and an appropriate constant elsewhere so that $\mu = \sum_{i=1}^{m} \mu_i$. Now we obtain

$$A_n(\mu) = \sum_{i=1}^{n} A_n(\mu_i)$$

and apply Lemma 3.1 to every $A_n(\mu_i)$. The claim follows immediately. $\square$

Denote by $\operatorname{var} \mu$ the full variation of μ. By $\operatorname{meas} \operatorname{supp} \mu$, it is meant the Lebesgue measure of the support of μ. The next lemma is a rather well-known assertion [5] (we give a bit more straightforward proof).

Lemma 3.3 *Let μ be a singular function or function of jumps coupled with a nonnegative Radon measure. Then for any $\varepsilon > 0$, μ can be split*

$$\mu = \mu_1 + \mu_2 \tag{25}$$

so that

$$\operatorname{meas} \operatorname{supp} \mu_1 \leq \varepsilon \tag{26}$$

and

$$\operatorname{var} \mu_2 \leq \varepsilon. \tag{27}$$

Moreover, the support of μ_1 is a union of finitely many closed intervals.

Proof. We know that $\mu' = 0$ almost everywhere. Therefore, the set of those x where $\mu'(x) > \varepsilon/2$ or does not exist is of zero Lebesgue measure. Thus, for any $\delta > 0$, it can be covered by a union of countably many non-intersecting open intervals (a_i, b_i) such that

$$\sum_{i=1}^{\infty}(b_i - a_i) \;<\; \delta.$$

Denote by $\operatorname{var}(\mu; a_i, b_i)$ the full vatiation on the interval $[a_i, b_i]$. Since

$$\sum_{i=1}^{\infty} \operatorname{var}(\mu; a_i, b_i) \le \operatorname{var}\mu \;<\; +\infty,$$

for a sufficiently large $m = m(\varepsilon)$ we obtain $\sum\limits_{i=m+1}^{\infty} \operatorname{var}(\mu; a_i, b_i) \le \varepsilon/2$. Set $E = \bigcup\limits_{i=1}^{m} [a_i, b_i]$ and write $\mu = \mu_1 + \mu_2$ so that μ_1 is supported within E and $\mu_1 = \mu$ on E. It is clear that meas supp $\mu_1 \le \delta$ and, also,

$$\operatorname{var}\mu_2 \le \sum_{i=m+1}^{\infty} \operatorname{var}(\mu; a_i, b_i) + \frac{1}{2\pi} \int\limits_{[-\pi,\pi]\setminus E} \mu'(x)dx \le \varepsilon.$$

The choice $\delta = \varepsilon$ completes the proof. $\square$

Lemma 3.4 *Let μ be a symbol of a nonnegative Radon measure. Then*

$$\frac{1}{n}\sum_{k=1}^{n} \sigma_k(A_n) \le \frac{1}{2\pi} \operatorname{var}\mu. \tag{28}$$

Proof. We take into account that $A_n = A_n^* \ge 0$. Hence, the singular values coincide with the eigenvalues, and their sum is equal to trace A_n. Since A_n is a Toeplitz matrix, it is sufficient to show that $a_0 \le \frac{1}{2\pi}\operatorname{var}\mu$. This trivially emanates from (8). $\square$

Proof of Theorem 1.1. Assume, first, that μ is a monotone non-decreasing function. Then $\mu = \mu_a + \mu_s + \mu_j$, where μ_a is an absolutely continuous function, μ_s is a singular function, μ_j is a function of jumps, and all three are also monotone non-decreasing functions. Apart from $A_n = A_n(\mu)$, consider Toeplitz matrices B_n generated by μ_a. We intend to show that A_n and B_n enjoy the premises of Lemma 2.1.

Take an arbitrary $\varepsilon > 0$. Using Lemma 3.3, we can write $\mu_s + \mu_j = \mu_1 + \mu_2$ so that (26) and (27) are fulfilled. Denote by T_n and U_n the Toeplitz matrices generated by μ_1 and μ_2, respectively.
Due to Lemma 3.2, we have $T_n = T_{1n} + T_{2n}$ with $\text{trace}\, T_{1n} = o(n)$ and $\text{rank} T_{2n} \le c_2 \varepsilon n$. By Lemma 3.4, $\text{trace}\, U_n \le \frac{1}{2\pi} \varepsilon n$. Thus, setting up $E_n = U_n + T_{1n}$ and $R_n = T_{2n}$, we obtain, for some $c > 0$,

$$||E_n||_{tr} \le c\varepsilon n \quad \text{and} \quad \text{rank} R_n \le c\varepsilon n$$

for all sufficiently large n. As Lemma 2.1 states, A_n and B_n are bound to have equally distributed singular values (and eigenvalues).
In the general case, we write $\mu = \mu_+ - \mu_-$, where μ_+ and μ_- are monotone non-decreasing functions. Then, we consider the above splittings and make use of the triangular inequality for the trace norm and that the rank of a sum does not exceed the sum of ranks. The Szegö-like formulas for Toeplitz matrices generated by the absolutely continuous component of μ were proved in [11]. $\square$

References

[1] F. Avram, On bilinear forms on Gaussian random variables and Toeplitz matrices, *Probab. Theory Related Fields* 79: 37–45 (1988).

[2] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1991.

[3] A. Bötcher and B. Silbermann, *Introduction to Large Truncated Toeplitz Matrices*, Springer-Verlag, New York, 1998.

[4] R. E. Edwards, *Fourier Series. A Modern Introduction. Volumes 1 and 2*, Springer-Verlag, New York, 1979; 1982.

[5] V. Grenander and G. Szegö, *Toeplitz Forms and Their Applications,* Univ. of California Press, Berkeley, 1958.

[6] S. V. Parter, On the distribution of the singular values of Toeplitz matrices, *Linear Algebra Appl.* 80: 115–130 (1986).

[7] G. W. Stewart and J. Sun, *Matrix Perturbation Theory*, Academic Press, Inc., San Diego, 1990.

[8] E. E. Tyrtyshnikov, New theorems on the distribution of eigen and singular values of multilevel Toeplitz matrices, *Dokl. RAN* 333, no. 3: 300–302 (1993). (In Russian.)

[9] E. E. Tyrtyshnikov, A unifying approach to some old and new theorems on distribution and clustering, *Linear Algebra Appl.* 232: 1–43 (1996).

[10] E. E. Tyrtyshnikov and N. L. Zamarashkin, Thin structure of eigenvalue clusters for non-Hermitian Toeplitz matrices, *Linear Algebra Appl.* 292 (1999) 297–310.

[11] N. L. Zamarashkin and E. E. Tyrtyshnikov, Distribution of eigen and singular values under relaxed requirements to a generating function, Math. Sbornik 188 (8): 83–92 (1997). (In Russian.)