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Abstract

We study variational problems with integrands of very general structure by introducing certain regularizations leading to particular minimizers. In a second part we apply the method to stationary generalized Newtonian fluids which gives the existence of solutions under weak hypotheses on the dissipative potential.

1 Introduction

Suppose we are given a convex energy density $f\colon \mathbb{R}^{nN} \to [0,\infty)$ satisfying (with positive constants a, $\tilde{a}$, b, $\tilde{b}$) the growth condition

$$aA(|X|) - b \le f(x) \le \tilde{a}|X|^q + \tilde{b} \quad \text{for all } \; X \in \mathbb{R}^{nN} \tag{1.1}$$

for some exponent $q > 1$ and some N-function $A : [0,\infty) \to [0,\infty)$ having the Δ_2-property. For example, we may choose $A(t) = t\ln(1+t)$ or $A(t) = t^p$ with $p \le q$. We then like to consider the problem

$$J[w] = \int_\Omega f(\nabla w)\,\mathrm{d}x \to \min \tag{1.2}$$

among all functions $w\colon \Omega \to \mathbb{R}^N$ such that $w = u_0$ on $\partial\Omega$. Here Ω denotes a bounded Lipschitz domain in $\mathbb{R}^n$, and u_0 is a function of class $W^1_q(\Omega;\mathbb{R}^N)$. To be more precise, we study (1.2) on the energy class

$$\mathcal{C} := \left\{ w \in W^1_A(\Omega;\mathbb{R}^N) : \;\; w - u_0 \in \overset{\circ}{W}{}^1_A(\Omega;\mathbb{R}^N), \;\; J[w] < \infty \right\}, \tag{1.3}$$

where $\overset{\circ}{W}{}^1_A(\Omega;\mathbb{R}^N)$ is the Orlicz-Sobolev space generated by A (see, e.g. [Ad]). From (1.1) we deduce $u_0 \in \mathcal{C}$, and the convexity of f implies that the problem (1.2) admits at least one solution.

If f is a strictly convex function, then the solution u is unique, and in order to study for example the regularity properties of u, the method of (global) regularization of problem (1.2) turned out to be a very powerful tool: for $0 < \delta < 1$ let

$$f_\delta(X) := \delta(1 + |X|^2)^{\frac{q}{2}} + f(X)\,, \qquad X \in \mathbb{R}^{nN}\,,$$

and replace (1.2) by

$$J_\delta[w] := \int_\Omega f_\delta(\nabla w)\,\mathrm{d}x \to \min \qquad \text{in } \; \mathcal{C}' := u_0 + \overset{\circ}{W}{}^1_q(\Omega;\mathbb{R}^N)\,. \tag{1.4}$$

If u_δ denotes the unique solution of (1.4), then $\{u_\delta\}$ forms a minimizing sequence for the problem (1.2) and $u_\delta \rightharpoondown u$ in $W^1_1(\Omega;\mathbb{R}^N)$ as $\delta \to 0$. We refer, for instance, to the papers [Se], [MS], [BFM], [BF1] (and many others, more references are found in [FS] or [Bi]) in

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which mainly the regularity of u is investigated via uniform estimates for the functions u_δ. In the strictly convex case it is also possible to give local variants of the regularization technique leading to corresponding results for local minimizers u of the energy J.

If now f is merely assumed to be just a convex function, then of course problem (1.4) is still well-posed with unique solution u_δ. Moreover, from (1.1) it follows that $\sup_{0<\delta<1} \|u_\delta\|_{W^1_A} < \infty$, hence there is a function $\bar{u} \in W^1_A(\Omega; \mathbb{R}^N)$ having trace u_0 and such that $u_\delta \rightharpoondown \bar{u}$ in $W^1_1(\Omega; \mathbb{R}^N)$ as $\delta \to 0$ at least for a subsequence. Our first result is the observation that $\bar{u}$ is a solution of (1.2) and – as in the case of strict convexity – $\{u_\delta\}$ forms a minimizing sequence, precisely

Theorem 1.1. *With the notation from above we have*

i) $\{u_\delta\}$ *is a minimizing sequence of problem (1.2).*

ii) $J_\delta[u_\delta] \to \inf_{\mathcal{C}} J$ *as* $\delta \to 0$.

iii) *The weak limit* $\bar{u}$ *belongs to the class* $\mathcal{C}$ *and is a solution of the problem (1.2).*

Here $f\colon \mathbb{R}^{nN} \to [0, \infty)$ *is any convex function satisfying (1.1) and in addition*

$$f(\lambda X) \le c(\lambda) f(X) \, , \qquad f(-X) \le c f(X) \tag{1.5}$$

for all $X \in \mathbb{R}^{nN}$ *and* $\lambda \ge 1$ *with some positive constants* c *and* $c(\lambda)$.

Regularity of $\bar{u}$ in turn can be used to obtain information on the behaviour of all solutions to the problem (1.2). We mention the following

Corollary 1.1. *Suppose that the assumptions of Theorem 1.1 hold. Moreover, let* $f = g$ *outside a ball in* $\mathbb{R}^{nN}$ *for a strictly convex function* $g \le f$. *Then, if* $\bar{u}$ *is locally Lipschitz, the same is true for any solution* u *of (1.2) from the energy class* $\mathcal{C}$.

Next we turn our attention to a problem arising in the theory of generalized Newtonian fluids. To be precise, we are looking for a velocity field $u\colon \Omega \to \mathbb{R}^n$ solving the following system of nonlinear partial differential equations

$$\begin{aligned} -\operatorname{div}\big\{T(\varepsilon(u))\big\} + u^k \frac{\partial u}{\partial x_k} + \nabla \pi &= g \quad \text{in } \Omega \, , \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \, , \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.6}$$

in a suitable weak sense. Here π is the a priori unknown pressure function and $g\colon \Omega \to \mathbb{R}^n$ represents a system of volume forces which we assume to be of class $L^\infty(\Omega; \mathbb{R}^n)$. We further assume that the tensor T is the gradient of some convex potential $f\colon \mathbb{S}^n \to [0, \infty)$ of class C^1 which acts on the space $\mathbb{S}^n$ of all symmetric $(n \times n)$-matrices. In (1.6) we take the sum w.r.t. repeated indices, and $\varepsilon(u)$ denotes the symmetric gradient. In case $f(\varepsilon) = |\varepsilon|^2$ (1.6) reduces to the Dirichlet-boundary value problem for the stationary Navier-Stokes system (see [Ga1], [Ga2] or [La]). So-called power-law models are investigated in [KMS]: they assume f to be of class C^2 satisfying for some $1 < p < \infty$

$$\lambda(1 + |\varepsilon|^2)^{\frac{p-2}{2}} |\sigma|^2 \le D^2 f(\varepsilon)(\sigma, \sigma) \le \Lambda(1 + |\varepsilon|^2)^{\frac{p-2}{2}} |\sigma|^2 \quad \text{for all } \varepsilon, \sigma \in \mathbb{S}^n \tag{1.7}$$

with positive constants λ, Λ. Note that (1.7) implies that f is of growth order p, moreover, the first inequality in (1.7) implies strict convexity of f. Then, if $n = 2$, Kaplický, Málek and Stará show that (1.6) admits a globally smooth solution in case $p > 3/2$, whereas for $p > 6/5$ the existence of a solution being smooth in the interior of Ω is established.

In the recent paper [ABF] we replaced (1.7) by the anisotropic condition

$$\lambda(1+|\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2 \le D^2 f(\varepsilon)(\sigma,\sigma) \le \Lambda(1+|\varepsilon|^2)^{\frac{q-2}{2}}|\sigma|^2 \quad \text{for all } \varepsilon,\, \sigma \in \mathbb{S}^n \tag{1.8}$$

with exponents $1 < p \le q < \infty$, $q \ge 2$. Then we proved: if $q < p(1 + 2/n)$ together with

$$p > \begin{cases} \dfrac{6}{5}, & n = 2\,, \\[2mm] \dfrac{9}{5}, & n = 3\,, \end{cases}$$

then (1.6) admits a weak solution $\bar{u}$ whose gradient is locally of class L^{p^*}, where

$$p^* = \begin{cases} 3p & \text{if } \ n = 3\,, \\ \text{any finite number} & \text{if } \ n = 2\,. \end{cases}$$

Moreover, if $q = 2$, then in the two-dimensional case $\bar{u}$ is smooth in the interior of Ω, whereas for $n = 3$ partial regularity holds.

The results of [KMS] and [ABF] are obtained by regularizing problem (1.6) and by proving uniform regularity results for the corresponding solutions which causes a lot of work. We like to describe an easier way leading to the existence of a solution to (1.6) in the anisotropic case which works under less restrictive growth and smoothness assumptions on the potential f. The price we have to pay is that we need a stronger lower bound for the exponent p.

To be precise assume that

$$f\colon\, \mathbb{S}^n \to [0,\infty) \text{ is convex and of class } C^1 \tag{1.9}$$

satisfying with exponents $1 < p < q < \infty$

$$a|\varepsilon|^p - b \le f(\varepsilon) < A|\varepsilon|^q + B \tag{1.10}$$

where a, b, A, B denote positive constants. We define $f_\delta(\varepsilon)$, $0 < \delta < 1$, as before and let u_δ denote a solution of

$$\begin{aligned} &\int_\Omega Df_\delta(\varepsilon(u_\delta)) : \varepsilon(\varphi)\,\mathrm{d}x - \int_\Omega u_\delta \otimes u_\delta : \varepsilon(\varphi)\,\mathrm{d}x = \int_\Omega g\cdot\varphi\,\mathrm{d}x \\ &\text{for all } \ \varphi \in C_0^\infty(\Omega;\mathbb{R}^n)\,, \quad \operatorname{div}\varphi = 0 \end{aligned} \tag{1.6$_\delta$}$$

in the space $\overset{\circ}{W}{}^1_q(\Omega;\mathbb{R}^n) \cap \mathrm{Ker}(\mathrm{div})$. Note that in general we cannot expect unique solvability of (1.6$_\delta$). From

$$\begin{aligned} J_\delta[w] &:= \int_\Omega f_\delta(\varepsilon(w))\,\mathrm{d}x - \int_\Omega u_\delta \otimes u_\delta : \varepsilon(w)\,\mathrm{d}x - \int_\Omega g\cdot w\,\mathrm{d}x\,, \\ J_\delta[u_\delta] &\le J_\delta[0] = f_\delta(0)|\Omega| \end{aligned}$$

and (1.10) it follows by Korn's inequality that

$$\sup_{0<\delta<1} \|u_\delta\|_{W^1_p(\Omega;\mathbb{R}^n)} < \infty\,,$$

where we also made use of the fact that

$$\int_\Omega u_\delta \otimes u_\delta : \varepsilon(u_\delta)\,\mathrm{d}x = 0\,.$$

Thus we find a function $\bar{u} \in \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^n) \cap \mathrm{Ker(div)}$ such that $u_\delta \rightharpoondown \bar{u}$ in $W^1_p(\Omega;\mathbb{R}^n)$ as $\delta \to 0$ at least for a subsequence.

Theorem 1.2. *Let (1.9), (1.10) and the first part of (1.5) hold. Suppose further that* $p > 3n/(n+2)$. *Then, with the notation from above, the limit* $\bar{u}$ *belongs to the energy class*

$$\mathbb{K} := \Big\{ u \in \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^n) :\ \mathrm{div}\, u = 0\,, \int_\Omega f(\varepsilon(u))\,\mathrm{d}x < \infty \Big\}$$

and minimizes

$$J[w] = \int_\Omega f(\varepsilon(w))\,\mathrm{d}x - \int_\Omega \bar{u}\otimes\bar{u} : \varepsilon(w)\,\mathrm{d}x - \int_\Omega g\cdot w\,\mathrm{d}x$$

within $\mathbb{K}$. *If we assume in addition that there is a positive constant* c_0 *such that*

$$|Df(X)| \le c_0\big\{f(X)+1\big\} \quad \textit{for all } X \in \mathbb{S}^n \tag{1.11}$$

then $\bar{u}$ *is a weak solution of (1.6), i.e.*

$$\int_\Omega Df(\varepsilon(u)) : \varepsilon(\varphi)\,\mathrm{d}x - \int_\Omega \bar{u}\otimes\bar{u} : \varepsilon(\varphi)\,\mathrm{d}x = \int_\Omega g\cdot\varphi\,\mathrm{d}x$$

for any $\varphi \in C^\infty_0(\Omega;\mathbb{R}^n)$, $\mathrm{div}\,\varphi = 0$.

Remark 1.1. i) *Let us first remark that Theorem 1.2 gives the existence of a weak solution* $\bar{u}$ *to problem (1.6) for the anisotropic case under much weaker conditions on the potential* f *than in [ABF]:* f *is just* C^1 *and no growth condition on* $D^2 f$ *is imposed. We do not even require strict convexity of* f.

ii) *The approach given here is much easierin comparison with [ABF], in particular we do not need involved a priori estimates in order to prove the above existence result. As a consequence, our arguments are not restricted to the particular models discussed in this short note. The solution also turns out to be a global minimizer of a variational problem in its natural energy class. On the other hand, the assumptions concerning the exponents* p *and* q *are slightly stronger compared to [ABF].*

iii) *It should be noted that the condition (1.11) is just used to get the Euler equation from the minimizing property of* $\bar{u}$. *If we assume that there is a positive constant* c_0' *such that*

$$|Df(X)| \le c_0'\big\{Df(X) : X + 1\big\} \quad \textit{for all } X \in \mathbb{S}^n\,,$$

then we have (1.11) by the convexity of f. *If we assume that* $q \le p+1$, *then we also have (1.11). In fact, the r.h.s. of (1.10) gives*

$$|Df(X)| \le c\big\{|X|^{q-1}+1\big\}$$

(compare [Da], p. 156, Lemma 2.2). This, together with the l.h.s. of (1.10) implies (1.11).

iv) *In the recent paper [FMS], the isotropic situation is studied. Given a uniform ellipticity condition, the authors use a Lipschitz truncation method to handle even the case $p > 2n/(n+2)$. Moreover, T is not assumed to be the gradient of some potential. It would be interesting to know, whether this method works in the non-uniformly elliptic situation.*

2 Proofs of Theorem 1.1 and Corollary 1.1

For technical simplicity we assume that Ω is star-shaped w.r.t. the origin, the general case follows from a covering argument (see [FS], Appendix A). Consider $w \in \mathcal{C}$, extend u_0 to a function (denoted also by u_0) in the space $W^1_q(\Omega^*;\mathbb{R}^N)$, where Ω^* is a domain such that $\Omega \Subset \Omega^*$. Let $w := u_0$ on $\Omega^* - \Omega$. For $\rho > 1$ sufficiently close to 1 we let

$$w_\rho(x) := (w - u_0)(\rho x)\,.$$

Clearly $\operatorname{spt} w_\rho$ is compact in Ω so that the mollification

$$w_\rho^\gamma := [w_\rho]^\gamma$$

is a function in the space $C_0^\infty(\Omega;\mathbb{R}^N)$ provided $\gamma < \gamma(\rho)$. Here the symbol $[\cdot]^\gamma$ denotes the mollification of a function with radius γ. Since u_δ is the solution of (1.4), we get

$$J_\delta[u_\delta] \le J_\delta[u_0 + w_\rho^\gamma]\,. \tag{2.1}$$

The l.h.s of (2.1) is bounded from below by $J[u_\delta]$, weak lower-semicontinuity of J implies

$$J[\bar u] \le \liminf_{\delta\to 0} J[u_\delta]\,,$$

and since

$$\delta \int_\Omega \big(1 + |\nabla(u_0 + w_\rho^\gamma)|^2\big)^{\frac{q}{2}}\, dx \to 0 \quad \text{as } \delta \to 0\,,$$

we deduce from (2.1)

$$J[\bar u] \le J[u_0 + w_\rho^\gamma] \tag{2.2}$$

being valid for all $\rho > 1$ close to 1 and all $0 < \gamma < \gamma(\rho)$. Let us fix such a number ρ. We have a.e.

$$\begin{aligned} f(\nabla u_0 + \nabla w_\rho^\gamma) &= f\Big(\frac{1}{2}2\nabla u_0 + \frac{1}{2}2\nabla w_\rho^\gamma\Big) \le \frac{1}{2}f(2\nabla u_0) + \frac{1}{2}f(2\nabla w_\rho^\gamma)\\ &\le c\big[f(\nabla u_0) + f(\nabla w_\rho^\gamma)\big]\,, \end{aligned}$$

where we used the convexity of f as well as the condition (1.5). Jensen's inequality implies

$$f(\nabla w_\rho^\gamma) = f\big([\nabla w_\rho]^\gamma\big) \le \big[f(\nabla w_\rho)\big]^\gamma\,,$$

thus

$$\tilde f_\gamma(x) := f(\nabla u_0(x) + \nabla w_\rho^\gamma(x)) \le c\big\{f(\nabla u_0(x)) + \big[f(\nabla w_\rho)\big]^\gamma(x)\big\} =: g_\gamma(x)\,. \tag{2.3}$$

Obviously it holds

$$\left.\begin{array}{rcl} \tilde{f}_\gamma(x) & \stackrel{\gamma\to 0}{\longrightarrow} & f(\nabla u_0(x)+\nabla w_\rho(x))\,, \\ g_\gamma(x) & \stackrel{\gamma\to 0}{\longrightarrow} & g(x) := c\big\{f(\nabla u_0(x)) + f(\nabla w_\rho(x))\big\} \end{array}\right\} \tag{2.4}$$

for almost all $x \in \Omega$. We claim (w.l.o.g. assuming $f(0)=0$)

$$g \in L^1(\Omega)\,, \qquad \text{i.e. } f(\nabla w_\rho) \in L^1(\Omega) \text{ with compact support} \tag{2.5}$$

so that the general properties of $[\cdot]^\gamma$ will imply

$$\big[f(\nabla w_\rho)\big]^\gamma \stackrel{\gamma\to 0}{\longrightarrow} f(\nabla w_\rho) \text{ in } L^1(\Omega)\,,$$

hence

$$g_\gamma \stackrel{\gamma\to 0}{\longrightarrow} g \text{ in } L^1(\Omega)\,. \tag{2.6}$$

Note that on account of (2.3), (2.4), (2.6) the variant of the dominated convergence theorem given in [EG], Theorem 4, p. 21, implies

$$\tilde{f}_\gamma \stackrel{\gamma\to 0}{\longrightarrow} f(\nabla u_0 + \nabla w_\rho) \text{ in } L^1(\Omega)\,. \tag{2.7}$$

We discuss (2.5): by definition we have for a.a. $x \in \Omega$

$$\begin{aligned} f(\nabla w_\rho)(x) &= f(\rho\nabla w(\rho x) - \rho\nabla u_0(\rho x)) = f\Big(\frac{1}{2}\,2\rho\nabla w(\rho x) + \frac{1}{2}\,(-2\rho)\nabla u_0(\rho x)\Big) \\ &\le \frac{1}{2} f(2\rho\nabla w(\rho x)) + \frac{1}{2} f(-2\rho\nabla u_0(\rho x)) \\ &\le \frac{1}{2} c(2\rho) f(\nabla w(\rho x)) + \frac{1}{2} c(2\rho) f(-\nabla u_0(\rho x)) \\ &\le \frac{1}{2} c(2\rho)\big(f(\nabla w(\rho x)) + c f(\nabla u_0(\rho x))\big)\,, \end{aligned}$$

where we used the convexity of f together with the condition (1.5) (recall that u_0 and w have been extended to a domain Ω^* and ρ is such that $\rho x \in \Omega^*$ for $x \in \Omega$). Now we observe ($f \ge 0$)

$$\begin{aligned} \int_\Omega f(\nabla w(\rho x))\,\mathrm{d}x &= \rho^{-n}\int_{\rho\Omega} f(\nabla w)\,\mathrm{d}x \\ &= \rho^{-n}\bigg\{\int_\Omega f(\nabla w)\,\mathrm{d}x + \int_{\rho\Omega-\Omega} f(\nabla u_0)\,\mathrm{d}x\bigg\} < \infty \end{aligned}$$

since w should belong to the energy class $\mathcal{C}$. This proves (2.5), and we deduce (2.7). Recalling (2.2) we obtain

$$J[\bar{u}] \le J[u_0 + w_\rho]\,, \tag{2.8}$$

and it remains to discuss the r.h.s. of (2.8) in the limit $\rho \to 1$. We have (on account of $\nabla w_\rho \to \nabla w - \nabla u_0$ in L^1)

$$m_\rho(x) := f(\nabla u_0(x) + \nabla w_\rho(x)) \stackrel{\rho\to 1}{\longrightarrow} m(x) := f(\nabla w(x))$$

a.e. (at least for a subsequence) and as before

$$\begin{aligned} 0 \;&\le\; m_\rho(x) \;\le\; \frac{1}{2} f(2\nabla u_0(x)) + \frac{1}{2} f(2\nabla w_\rho(x))\,, \\ f(2\nabla w_\rho(x)) \;&=\; f(2\rho\nabla(w-u_0)(\rho x)) \;\le\; c(2\rho) f(\nabla(w-u_0)(\rho x))\,, \end{aligned}$$

so that

$$m_\rho(x) \le M_\rho(x) := K\big\{f(\nabla u_0(x)) + f(\nabla(w-u_0)(\rho x)\big\}\,.$$

Here K is a constant independent of ρ which follows from the fact that $c(2\rho) \le c(4)$ for ρ close to 1. Obviously

$$M_\rho(x) \overset{\rho\to 1}{\longrightarrow} K\big\{f(\nabla u_0(x)) + f(\nabla(w-u_0)(x))\big\} \ \text{ a.e.}$$

(note: as $\rho \to 0$ we have $\nabla(w-u_0)(\rho\cdot) \to \nabla w - \nabla u_0$ in $L^1(\Omega;\mathbb{R}^{nN})$, so that $\nabla(w-u_0)(\rho x) \to \nabla w(x) - \nabla u_0(x)$ a.e. at least for a subsequence) and

$$\begin{aligned} \int_\Omega M_\rho \,\mathrm{d}x \;&=\; K\left[\int_\Omega f(\nabla u_0)\,\mathrm{d}x + \rho^{-n}\int_{\rho\Omega} f(\nabla(w-u_0))\,\mathrm{d}x\right] \\ &\overset{\rho\to 1}{\longrightarrow}\; K\left[\int_\Omega f(\nabla u_0)\,\mathrm{d}x + \int_\Omega f(\nabla(w-u_0))\,\mathrm{d}x\right] < \infty\,, \end{aligned}$$

the finiteness of $\int_\Omega f(\nabla(w-u_0))\,\mathrm{d}x$ being a consequence of $w \in \mathcal{C}$ and (1.5). The variant of Lebesgue's theorem on dominated convergence implies

$$\int_\Omega m_\rho \,\mathrm{d}x \overset{\rho\to 1}{\longrightarrow} \int_\Omega m\,\mathrm{d}x\,,$$

together with (2.8) this yields

$$J[\bar u] \le J[w]\,. \tag{2.9}$$

Now, (2.9) is exactly the statement that $\bar u$ is a minimizer in the energy class $\mathcal{C}$.

Finally, we establish ii) of Theorem 1.1, which will also imply i). From (2.9) we get

$$\alpha := \inf_{\mathcal{C}} J = J[\bar u] \le J[u_\delta] \le J_\delta[u_\delta]\,,$$

hence

$$\alpha \le \liminf_{\delta\to 0} J_\delta[u_\delta] \le \limsup_{\delta\to 0} J_\delta[u_\delta] \overset{(2.1)}{\le} \limsup_{\delta\to 0} J_\delta[u_0 + w_\rho^\gamma] = J[u_0 + w_\rho^\gamma]$$

being valid for all $\rho \in (1, 1+\varepsilon)$ and all $0 < \gamma < \gamma(\rho)$. Recall that the calculations following (2.2) actually show that

$$\lim_{\rho\to 1}\Big(\lim_{\gamma\to 0} J[u_0 + w_\rho^\gamma]\Big) = J[w]$$

holds for any $w \in \mathcal{C}$. If we therefore make the particular choice $w = \bar u$, pass to the limit $\gamma \to 0$ and then to the limit $\rho \to 1$ in the above inequality, we obtain the desired equation

$$\alpha = \liminf_{\delta\to 0} J_\delta[u_\delta] = \limsup_{\delta\to 0} J_\delta[u_\delta]\,,$$

which completes the proof of Theorem 1.1. $\square$

Let us now assume that the hypotheses of Corollary 1.1 are valid. Hence there exists $M > 0$ such that $f(X) = g(X)$ for all $X \in \mathbb{R}^{nN}$, $|X| \geq M$. We fix $\Omega' \Subset \Omega$ and a number $K = K(\Omega')$ s.t. $|\nabla \bar{u}| \leq K$ on Ω'. Suppose that some minimizer u satisfies $|\nabla u| \geq l$ on a subset A of Ω' with positive measure, where $l := 2M + K$. If we let $w := \frac{1}{2}\, u + \frac{1}{2}\, \bar{u}$, we obtain

$$\begin{aligned}
\int_\Omega f(\nabla w)\,\mathrm{d}x &= \int_{\Omega-\Omega'} f(\nabla w)\,\mathrm{d}x + \int_{\Omega'-A} f(\nabla w)\,\mathrm{d}x + \int_A f(\nabla w)\,\mathrm{d}x \\
&\leq \frac{1}{2}\int_{\Omega-\Omega'} f(\nabla u)\,\mathrm{d}x + \frac{1}{2}\int_{\Omega-\Omega'} f(\nabla \bar{u})\,\mathrm{d}x + \frac{1}{2}\int_{\Omega'-A} f(\nabla u)\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\Omega'-A} f(\nabla \bar{u})\,\mathrm{d}x + \int_A f(\nabla w)\,\mathrm{d}x\,,
\end{aligned}$$

where we used the convexity of f. On the set A we have

$$|\nabla w| \geq \frac{1}{2}|\nabla u| - \frac{1}{2}|\nabla \bar{u}| \geq \frac{1}{2}l - \frac{1}{2}K = M\,,$$

hence

$$\int_A f(\nabla w)\,\mathrm{d}x = \int_A g(\nabla w)\,\mathrm{d}x < \frac{1}{2}\int_A g(\nabla u)\,\mathrm{d}x + \frac{1}{2}\int_A g(\nabla \bar{u})\,\mathrm{d}x$$

and we arrive at the contradiction (observe $g \leq f$)

$$\int_\Omega f(\nabla w)\,\mathrm{d}x < \frac{1}{2}\int_\Omega f(\nabla u)\,\mathrm{d}x + \frac{1}{2}\int_\Omega f(\nabla \bar{u})\,\mathrm{d}x = \inf_{\mathcal{C}} J\,.$$

This proves $|\nabla u| \leq l$ a.e. on Ω'. $\square$

3 Proof of Theorem 1.2

The proof of Theorem 1.2 is a modification of the ideas given in the last section. Again we assume that Ω is star-shaped w.r.t. the origin and we identify in the following a function w of Sobolev class $\overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^n)$ with its extension $\tilde{w}$ to $\mathbb{R}^n$,

$$\tilde{w}(x) := \begin{cases} w(x) & \text{if} \quad x \in \Omega\,, \\ 0 & \text{if} \quad x \in \mathbb{R}^n - \Omega\,. \end{cases}$$

Again, for any $1 < \rho$, $0 < \gamma$ and for any w as above we let

$$w_\rho^\gamma := \Big[w(\rho x)\Big]^\gamma\,.$$

If $1 < \rho$ is fixed and if $0 < \gamma < \gamma(\rho)$ is sufficiently small, then $w \in \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^n)$ implies that w_ρ^γ is compactly supported in Ω. Moreover, note that $\operatorname{div} w = 0$ gives $\operatorname{div} w_\rho^\gamma = 0$. With these preliminaries a sequence $\{u_\delta\}$ of solutions to the problem (1.6_δ) is fixed s.t. $u_\delta \rightharpoondown: \bar{u}$

in $W^1_p(\Omega;\mathbb{R}^n)$. Lower semicontinuity w.r.t. weak W^1_p-convergence implies together with continuity of the convective part (recall $p > 3n/(n+2)$)

$$J[\bar{u}] \le \liminf_{\delta\to 0} \left\{ \int_\Omega f_\delta(\varepsilon(u_\delta))\,\mathrm{d}x - \int_\Omega u_\delta \otimes u_\delta : \varepsilon(u_\delta)\,\mathrm{d}x - \int_\Omega g\cdot u_\delta\,\mathrm{d}x \right\} = \liminf_{\delta\to 0} J_\delta[u_\delta]\,.$$

Now we consider an element w of the natural energy class $\mathbb{K}$. With the above notation, w^γ_ρ is admissible in J_δ and the minimality of u_δ implies passing to the limit $\delta \to 0$

$$J[\bar{u}] \le \int_\Omega f(\varepsilon(w^\gamma_\rho))\,\mathrm{d}x - \int_\Omega \bar{u}\otimes\bar{u} : \varepsilon(w^\gamma_\rho)\,\mathrm{d}x - \int_\Omega g\cdot w^\gamma_\rho\,\mathrm{d}x\,.$$

Here we used the fact that $\limsup_{\delta\to 0}\delta\int_\Omega(1+|\varepsilon(w^\gamma_\rho)|^2)^{q/2}\,\mathrm{d}x = 0$ since ρ and γ are fixed and since w^γ_ρ is by definition a smooth function. Since the convergence of the convective term as $\gamma \to 0$ and as $\rho \to 1$ is clear, it remains to show (at least for a subsequence)

$$\lim_{\rho\to 1}\lim_{\gamma\to 0}\int_\Omega f(\varepsilon(w^\gamma_\rho))\,\mathrm{d}x \le \int_\Omega f(\varepsilon(w))\,\mathrm{d}x\,.$$

This however is proved with the same arguments as outlined in the last section (note that on account of $u_0 = 0$ we just need the first part of (1.5)). Thus we have for any $w \in \mathbb{K}$

$$J[\bar{u}] \le \int_\Omega f(\varepsilon(w))\,\mathrm{d}x - \int_\Omega \bar{u}\otimes\bar{u} : \varepsilon(w)\,\mathrm{d}x - \int_\Omega g\cdot w\,\mathrm{d}x\,,$$

which is the J-minimality of $\bar{u}$ in the class $\mathbb{K}$. For $\varphi \in C^\infty_0(\Omega;\mathbb{R}^n)$, $\operatorname{div}\varphi = 0$, we have

$$\begin{aligned}\int_\Omega f(\varepsilon(\bar{u}+\varphi))\,\mathrm{d}x &= \int_\Omega f\Big(2\Big[\frac{1}{2}\varepsilon(\bar{u}) + \frac{1}{2}\varepsilon(\varphi)\Big]\Big)\,\mathrm{d}x \;\le\; c(2)\int_\Omega f\Big(\frac{1}{2}\varepsilon(\bar{u}) + \frac{1}{2}\varepsilon(\varphi)\Big)\,\mathrm{d}x\\ &\le \frac{1}{2}c(2)\left[\int_\Omega f(\varepsilon(\bar{u}))\,\mathrm{d}x + \int_\Omega f(\varepsilon(\varphi))\,\mathrm{d}x\right] < \infty\,,\end{aligned}$$

so that $u + t\varphi \in \mathbb{K}$ for any φ as above and any real parameter t. Clearly we have

$$\frac{1}{t}\big\{f(\varepsilon(\bar{u}) + t\varepsilon(\varphi)) - f(\varepsilon(\bar{u}))\big\} =: \Delta_t \overset{t\to 0}{\longrightarrow} Df(\varepsilon(\bar{u})) : \varepsilon(\varphi) \quad \text{a.e.} \tag{3.1}$$

Now we make use of (1.11) to obtain

$$\int_\Omega |Df(\varepsilon(\bar{u}))|\,\mathrm{d}x \le c\int_\Omega \big(f(\varepsilon(\bar{u})) + 1\big)\,\mathrm{d}x < \infty\,,$$

hence we have integrability of the r.h.s. of (3.1). By (1.11) we also know that

$$\begin{aligned}|\Delta_t| &= \left|\frac{1}{t}\int_0^t Df(\varepsilon(\bar{u}) + \lambda\varepsilon(\varphi)) : \varepsilon(\varphi)\,\mathrm{d}\lambda\right|\\ &\le \int_0^1 |Df(\varepsilon(\bar{u}) + st\varepsilon(\varphi))||\varepsilon(\varphi)|\,\mathrm{d}s\\ &\le c_0\int_0^1 \Big[f(\varepsilon(\bar{u}) + st\varepsilon(\varphi)) + 1\Big]|\varepsilon(\varphi)|\,\mathrm{d}s\,.\end{aligned}$$

Observing that we have as above

$$f(\varepsilon(\bar{u}) + st\varepsilon(\varphi)) \leq c\Big[f(\varepsilon(\bar{u})) + f(\varepsilon(\varphi))\Big]\,,$$

the desired weak form of (1.6) follows from dominated convergence and from $J[\bar{u}] \leq J[\bar{u} + t\varphi]$. □

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